Concavity Properties of Numbers Satisfying the Binomial Recurrence

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Abstract. We consider square arrays of numbers $\{a(n,k)\}$, generalizing the binomial coefficients: $a(n,0) = c_n$ where the c_n are non-negative real numbers; $a(0,k) = c_0$, and if $n, k > 0$, then $a(n,k) = a(n,k-1) + a(n-1,k)$. We give generating functions and arithmetical relations for these numbers. We show that every row of such an array is eventually log concave, and give a few sufficient conditions for columns to be eventually log concave. We also give a necessary condition for a column to be eventually log concave, and give examples to show that there exist such arrays in which no column is eventually log concave.

Introduction

Pascal originally considered the binomial coefficients as arranged in a square (the Pascal square), with the element in the n^{th} row and k^{th} column being the binomial coefficient $C(n+k,k)$ $(n,k \geq 0)$. For each integer $r \geq 0$, the r^{th} diagonal of this square, $\{C(n+k,k) \mid n+k = r\}$ is the r^{th} row of the usual Pascal triangle. (For a historical discussion, see [2].)

Various generalizations of the Pascal square have appeared (e.g. [1], [3]). In this paper we consider a generalization which does not appear to have been much studied: we generalize the initial conditions, and keep the same recurrence relation as in the Pascal square.

Our original motivation came from the study of linear Diophantine equations. Thus, let $j \geq 1, c_1, c_2, \ldots, c_j$, be fixed integers; for $n, k \geq 0$, define $a(n,k)$ to be the number of non-negative integral solutions of the equation

$$c_1x_1 + c_2x_2 + \cdots + c_jx_j + y_1 + y_2 + \cdots + y_k = n.$$

Then $a(0,k) = 1$, $a(n,0)$ is a non-negative integer, and $\{a(n,k)\}$ satisfies the recurrence of the Pascal square.

The square arrays we consider will have $a(0,k) = c_0 > 0$, and $a(n,0) \geq 0$. For these arrays, generating functions and inductive relations, analogous to those of the Pascal square, are found in a straightforward way. Most of this paper is concerned with the more difficult question of the logarithmic concavity of the rows or columns of such a square array.

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In Section 1, we discuss the straightforward properties of these arrays, and prove some basic lemmas which are most useful in the study of log concavity. From the generating functions, it is easy to prove that a necessary condition for some column to be eventually log concave, is that the initial column should be bounded by an exponential, and hence, to give examples of arrays in which no column is eventually log concave.

In Section 2, we show that for each of these squares, $A = \{a(n,k)\}$, every row of A is eventually log concave, and if any column of A is log concave, then so are all the columns to the right of it.

The "duals" of these statements (switching rows and columns) are not true. The behavior of log concavity in columns is more complicated than for rows; this reflects the fact that the top row of A is constant (which makes log concavity behave tamely), while the initial column is variable.

In Section 3, we give some sufficient conditions on a column which will ensure that some column to the right of it will be log concave. One such condition is, that only finitely many of the initial values c_i are non-zero (Theorem 3.2); another requires that the column in question should be bounded above and below by polynomials which are "close together" (Theorem 3.4).

In view of the fact that the $(k+1)^{st}$ column is formed from the k^{th} column by the finite-difference analogue of integration, we have the following conjecture:

If there exists a polynomial $P(x)$ such that $a(n,0) \leq P(n)$ for almost all n, then some column of $\{a(n,k)\}$ is log concave.

In our computer calculations, we have yet to find a square in which the initial column is exponentially bounded, which does not have an eventually log concave column.

1. Preliminaries

In this section, we define square arrays of integers, satisfying the binomial recurrence, and give some of their elementary properties. We also give some of the basic definitions and properties for logarithmically concave sequences, which we will use later.

Definition 1.1. *Let $\{c_0, c_1, \ldots\}$ be a sequence of non-negative integers, with $c_0 > 0$. Define integers $a(n,k)$ recursively by*

$$\begin{aligned} a(n,k) &= c_n \quad \text{for } n \geq 0 \\ a(0,k) &= c_0 \quad \text{for } k \geq 0 \\ a(n,k) &= a(n-1,k) + a(n,k-1) \quad \text{for } n,k > 0. \end{aligned}$$

Then we say that the set $\{a(n,k) \mid n,k = 0,1,\ldots\}$ is a *BR-set*, with *initial sequence* $\{c_0, c_1, \ldots\}$. We consider this as a square array, with row index n and column index k. The top row is constant, and the left-most column is the initial sequence.

Example: If the initial sequence is $\{1, 0, 0, 0, \dots\}$, then $a(n, k)$ is the binomial coefficient $C(n+k-1, k-1)$. If the initial sequence is $\{1, 1, 1, \dots\}$, then $a(n, k)$ is the binomial coefficient $C(n+k, k)$.

Generating functions, and a closed-form formula, are found in a routine way.

Theorem 1.2. *Let $A = \{a(n, k)\}$ be a BR-set, with initial sequence $\{c_i\}$. Put*

$$S_k(x) = \sum_{n=0}^{\infty} a(n, k) x^n; \qquad F(x, y) \sum_{k=0}^{\infty} S_k(x) y^k.$$

Then $F(x, y) = (1-x)S_0(x)/(1-x-y)$, and a closed-form formula for the coefficients is

$$a(n, k) = \sum_{j=0}^{n} C(n+k-j-1, k-1) c_j. \tag{1.3}$$

Definition 1.4. *Let $\{x_i \mid i = 0, 1, 2, \dots\}$ be a sequence of positive real numbers. If $x_i^2 \geq x_{i-1}x_{i+1}$ for $i = 1, 2, \dots$ (or, equivalently, the sequence $\{x_{i+1}/x_i\}$ is non-increasing) then we say that $\{x_i\}$ is* logarithmically concave (LC). *If $\{x_i \mid i = N, N+1, \dots\}$ is* LC *for some $N > 0$, then we say that $\{x_i\}$ is* eventually LC, *or* ELC. *If a single member of the sequence, say x_i, satisfies the inequality $x_i^2 \geq x_{i-1}x_{i+1}$, then we say that the element x_i is* locally LC (LLC) *in the sequence $\{x_i\}$, or just* LLC *if the sequence is understood.*

Remark 1.5. *Since the radius of convergence R_k of the series $S_k(x)$ is given by*

$$\frac{1}{R_k} = \lim\sup \{a(n+1, k)/a(n, k)\},$$

it is clear that if any column $\{a(n, k) \mid n = 0, 1, \dots\}$ is ELC, *then $1/R_k$ is the upper limit of a non-increasing positive sequence, and hence not zero. On the other hand, from the recurrence, $S_k(x) = (1-x)S_{k+1}(x)$, and hence if any S_k has radius of convergence $R_k = 0$, then they all do. Thus, if $R_0 = 0$, then none of the columns of $\{a(n, k)\}$ is* ELC.

Example: If $c_n = n!$, then no column of $\{a(n, k)\}$ is ELC.

2. Basic properties of BR-sets

It is not surprising that BR-sets have many of the familiar properties of the binomial coefficients. As far as logarithmic concavity is concerned, we will be interested mainly in the rows $\{a(n, k) \mid k = 0, 1, 2, \dots\}$ and the columns $\{a(n, k) \mid n = 0, 1, 2, \dots\}$. (It is easy to see that if $k \geq 1$, then $a(n, k) > 0$.)

We begin with some additive properties, which are easily proved by induction.

Lemma 2.1. *Let $A = \{a(n,k)\}$ be a BR-set, with initial sequence $\{c_i\}$. Then*

(a) $a(n,k) = \sum_{j=o}^{n} a(j,k-1) \qquad (n,k = 1,2,\ldots);$

(b) $a(n,k) = a(n,0) + \sum_{j=1}^{k} a(n-1,j) \qquad (n,k = 1,2,\ldots);$

(c) $\sum_{n+k=r} a(n,k) = c_r + \sum_{j=0}^{r-1} 2^{r-j-1} c_j;$

(d) $\sum_{n+k=r} (-1)^k a(n,k) = c_r - c_{r-1}.$

The next result concerns the LLC property. If one element $a(n,k)$ has the LLC property in the row (or column, or diagonal) in which it occurs, then some of its neighbors must also have the LLC property in either the row or column or diagonal in which they occur.

Theorem 2.2. *Let $A = \{a(n,k)\}$ be a BR-set, and let $n, k \geq 0$ be fixed. Define*

$$b_{ij} = a(n-1+i, k-1+j) \qquad (i = 1,2,3,4;\ j = 1,2,3)$$

so that $\{b_{ij}\}$ is a 4×3 block *in A.*

(a) (The Triangle Lemma)

$$(b_{23})^2 - b_{13}b_{33} = (b_{22})^2 - b_{13}b_{31} = (b_{32})^2 - b_{33}b_{31}.$$

(b) (i) $(b_{22})^2 - b_{12}b_{32} = b_{22}b_{21} - b_{31}b_{12};$

(ii) $(b_{22})^2 - b_{21}b_{23} = b_{22}b_{12} - b_{13}b_{21};$

(iii) $(b_{22})^2 - b_{13}b_{31} = b_{23}b_{22} - b_{32}b_{13}.$

(c) If $k > 0$, and if $(b_{23})^2 \geq b_{13}b_{33}$ and $(b_{32})^2 \geq b_{22}b_{42}$, then $(b_{33})^2 \geq b_{23}b_{43}$.

Proof: (a) Consider the 2×2 matrices

$$X = \begin{pmatrix} b_{23} & b_{13} \\ b_{33} & b_{23} \end{pmatrix}, \quad Y = \begin{pmatrix} b_{22} & b_{13} \\ b_{31} & b_{22} \end{pmatrix}, \quad Z = \begin{pmatrix} b_{32} & b_{31} \\ b_{33} & b_{32} \end{pmatrix}.$$

Each of these can be transformed into either of the others by row/column operations of the type: replace a row (column) with the sum or difference of the rows (columns). Thus they all have the same determinant, and (a) follows.

(b) We prove (i); the others are similar. By the recurrence, $b_{32} = b_{22} + b_{31}$ and $b_{22} = b_{12} + b_{21}$. Then

$$(b_{22})^2 - b_{32}b_{12} = b_{22}(b_{12} + b_{21}) - (b_{22} + b_{31})b_{12} = b_{22}b_{21} - b_{31}b_{12}.$$

(c) It is well known that if a, b, c, d are positive real numbers, then $a/b \leq c/d$ if and only if $a/b \leq (a+c)/(b+d) \leq c/d$. Since we assumed $k > 0$, then all the numbers b_{ij} are positive, and (by part (b)) the assumptions imply that

$$\frac{b_{32}}{b_{22}} \geq \frac{b_{41}}{b_{31}} \quad \text{and} \quad \frac{b_{23}}{b_{13}} \geq \frac{b_{22}}{b_{13}}.$$

Then

$$\frac{b_{41}}{b_{31}} \leq \frac{b_{41}+b_{32}}{b_{31}+b_{22}} = \frac{b_{42}}{b_{32}} \leq \frac{b_{32}}{b_{22}}$$
$$\frac{b_{32}}{b_{22}} \leq \frac{b_{23}+b_{32}}{b_{22}+b_{13}} = \frac{b_{33}}{b_{23}} \leq \frac{b_{23}}{b_{13}}$$

From this, $b_{42}/b_{32} \leq b_{33}/b_{23}$, and from part (b), $(b_{33})^2 \geq b_{23}b_{43}$. ∎

Corollary 2.3. *Suppose that A is a BR-set such that for some fixed k, and some* $N \geq 1$*, the set* $\{a(n,k) \mid n = N, N+1, \ldots\}$ *is LC. Suppose that for some* $J \geq N-1$ *we have a* $(J, k+1)2 \geq a(J-1, k+1)a(J+1, k+1)$*. Then the set* $\{a(n,k) \mid n = J, J+1, \ldots\}$ *is LC.*

Proof: This follows by induction, using Theorem 2.2(c). ∎

Theorem 2.4. *Suppose that A is a BR-set with initial sequence* $\{c_i\}$*, such that for some fixed k the column* $\{a(n,k) \mid n = 0,1,2,\ldots\}$ *is LC. Then for every* $j \geq k$*, the column* $\{a(n,j) \mid n = 0,1,2,\ldots\}$ *is LC.*

Proof: Put $x_n = a(n,k)$ and $y_n = a(n,k+1)$. Then $x_0 = y_0 = c_0$ which is positive by assumption. For all i, we have $(x_i)^2 \geq x_{i-1}x_{i+1}$. When $i = 0$, we have

$$\frac{x_1+x_0}{x_0} = \frac{x_1+y_0}{y_0} = \frac{y_1}{y_0} \geq \frac{x_1}{x_0} \geq \frac{x_2}{x_1}.$$

By Theorem 2.3(b), this implies that $(y_1)^2 \geq y_2/y_0$, and the result follows by induction, using Corollary 2.4. ∎

We now consider the behavior of the rows of a BR-set. The next Theorem shows that all but finitely many elements of a row will have the LLC property in the column in which they occur.

Theorem 2.5. *Suppose that* $A = \{a(n,k)\}$ *is a BR-set with initial sequence* $\{c_i\}$*. Let* $n \geq 1$ *be fixed. There exists an integer K such that if* $k \geq K$*, then* $a(n,k)^2 \geq a(n+1,k)a(n-1,k)$*.*

Proof: We use the formula (1.3) for the numbers $a(n,k)$. Consider the binomial coefficient $C(n+k-1-j, k-1)$ as a polynomial in k. It has degree $n-j$ and leading coefficient $1/(n-j)!$ and so, considering the numbers $a(n,k)$ also as polynomials in k, we have

$$\begin{aligned} P(k) &= a(n,k)^2 - a(n-1,k)a(n+1,k) \\ &= \left\{ \sum_{j=0}^{n} C(n+k-j-1,k-1)c_j \right\}^2 \\ &\quad - \left\{ \sum_{j=0}^{n} C(n+k-j-2,k-1)c_j \right\} \left\{ \sum_{j=0}^{n} C(n+k-j,k-1)c_j \right\} \end{aligned}$$

In $a(n,k)$, the term $c_0 C(n+k-1,k-1)$ has degree n in k, and all the other terms have lower degree; it follows that the coefficient of k_{2n} in $P(k)$ is

$$c_0^2 \left\{ \left(\frac{1}{n!}\right)^2 - \left(\frac{1}{(n-1)!}\right)\left(\frac{1}{(n+1)!}\right) \right\}$$

which is positive. Since the leading coefficient of $P(k)$ is positive, then for all sufficiently large k, the values $P(k)$ will be positive. ∎

Corollary 2.6. *If $A = \{a(n,k)\}$ is a BR-set, then every row of A is ELC.*

Proof: The set $\{a(0,k)\}$ is a constant (positive) sequence, and therefore ELC. From the definition, $a(1,k) = c_1 + kc_0$, and (for $k \geq 1$) it is easy to compute that $a(1,k)^2 - a(1,k-1)a(1,k+1) = c_0^2$. If $n \geq 2$, then from Theorem 2.5, for all suitably large k, we have

$$a(n-1,k+1)^2 \geq a(n,k+1)a(n-2,k+1)$$

and from this, by the Triangle Lemma, $a(n,k)^2 \geq a(n,k-1)a(n,k+1)$. ∎

Unfortunately, this kind of argument does not apply to the columns of A, since every binomial coefficient appearing in Formula 1.3 has the same degree $(k-1)$ when regarded as a polynomial in n. This lack of duality reflects the fact that the top row of A is constant, while the initial column of A is variable.

3. Log concavity in columns.

The question of eventual log concavity in a column of a BR-set A, seems to be in general quite difficult. In this section, we prove that if the initial column of A has only finitely many non-zero elements, then all but finitely many columns of A are LC, and all columns except possibly the initial column, are ELC (Theorem 3.2). In Theorem 3.3 we give a sufficient condition, in terms of polynomial bounds, for A to have an ELC column.

Lemma 3.1. *Let* $A = \{a(n,k)\}$ *be a BR-set with initial sequence* $\{ci\}$, *and define the quantities* $M(n,k), X_{ij}(n,k), B_{ij}(n,k), A_{ij}(n,k)$ *by:*

$$M(n,k) = a(n,k)^2 - a(n+1,k)a(n-1,k) \quad (1)$$

$$M(n,k) = \sum X_{ij}(n,k)c_ic_j \qquad (i,j = 0,...,n+1) \quad (2)$$

and for $0 \leq i,j \leq n+1$ *and* $k \geq 1$,

$$B_{ij}(n,k) = \frac{(n+k-i-2)!(n+k-j-2)!}{(k-1)!(k-1)!(n-i+1)!(n-j+1)!} \quad (3)$$

$$A_{ij}(n,k) = (k-1)\{2n+2-i-j-(i-j)^2\} + \{2n^2+2n(1-i-j)+i^2+j^2-i-j\}. \quad (4)$$

Then (a) $X_{nn}(n,k) = X_{n-1,n+1}(n,k) = 1$ *for all* k;

(b) $X_{ij}(n,k) = 0$ *unless* $(i,j) = (n,n)$ *or* $(n-1,n+1)$;

(c) $X_{ij}(n,1) = \begin{cases} 1 & \text{if } j = n, 0 \leq i \leq n, \\ -1 & \text{if } j = n+1, 0 \leq i \leq n-1, \\ 0 & \text{otherwise;} \end{cases}$

(d) $X_{ij}(n,2) = \begin{cases} 2 & \text{if } j = n, 0 \leq i \leq n-1, \\ -2(n-i) & \text{if } j = n+1, 0 \leq i \leq n-1, \\ 1 & \text{if } (i,j) = (n,n), \\ -1 & \text{if } (i,j) = (n-1,n+1), \\ 0 & \text{otherwise;} \end{cases}$

(e) $(1+\delta_{ij})X_{ij}(n,k) = (k-1)B_{ij}(n,k)A_{ij}(n,k)$ *if* $k \geq 3$, *where* δ_{ij} *is the Kronecker delta.*

Proof: Substitute the expressions from Formula 1.3 into $M(n,k)$; it has the form (2). The values of $X_{ij}(n,k)$ are easily computed directly for $k = 0,1,2$. For $k \geq 3$, and $0 \leq i,j \geq n+1$,

$$\begin{aligned}(1+\delta_{ij})X_{ij}(n,k) = &2C(n+k-j-1,k-1)C(n+k-i-1,k-1)\\ &- C(n+k-j,k-1)C(n+k-i-2,k-1)\\ &- C(n+k-i,k-1)C(n+k-j-2,k-1).\end{aligned}$$

Rewriting the binomial coefficients as quotients of factorials and factoring out $B_{ij}(n,k)$, we get

$$\begin{aligned}\frac{(1+\delta_{ij})X_{ij}(n,k)}{B_{ij}(n,k)} = &2(n+k-j-1)(n+k-i-1)(n-i+1)(n-j+1)\\ &- (n+k-j-1)(n+k-j)(n-i)(n-i+1)\\ &- (n+k-i-1)(n+k-i)(n-j)(n-j+1)\end{aligned}$$

which reduces to $(k-1)A_{ij}(n,k)$. This proves (e). ∎

An obvious sufficient (but not necessary) condition for $M(n,k)$ to be non-negative, is that all the coefficients $X_{ij}(n,k)$ should be non-negative. We use this observation to show that if only finitely many of the c_i are non-zero, then every column of A (except possibly the initial column), is ELC, and all but finitely many columns are LC.

Theorem 3.2. *Let $A=\{a(n,k)\}$ be a BR-set, with initial sequence $\{c_i\}$, and suppose that for some $N \geq 0$, $c_N > 0$, and $c_i = 0$ for all $i > N$.*

(a) For every $k \geq 1$, there exists an integer $J = J(k)$ such that the sequence $\{a(n,k) \mid n = J, J+1, \ldots\}$ is LC.

(b) There exists an integer K such that for all $k > K$, the sequence $\{a(n,k) \mid n = 0,1,2,\ldots\}$ is LC.

Proof: In the expression (2) for $M(n,k)$ we need only consider terms with $0 \leq i,j \leq N$. From Lemma 3.1 (c) and (d), it suffices to take $J(1) = J(2) = N$. Now suppose that $k \geq 3$, and n is fixed. Since $(k-1)B_{ij}(n,k)$ is positive, we need only consider the factor $A_{ij}(n,k)$. Putting $r = n-i+1$ and $s = n-j+1$, we have a simpler expression for $A_{ij}(n,k)$

$$A_{ij}(n,k) = (k-1)\{r+s-(r-s)^2\} + r^2 + s^2 - r - s.$$

Since r and s are integers, r^2+s^2-r-s is non-negative. If $r+s \geq (r-s)^2$, then $A_{ij}(n,k)$ is non-negative.

Since $r-s = j-i$, and $0 \leq i,j \leq N$, then we have $(r-s)^2 \leq N^2$, and $i+j \geq 2N$. Then, solving the inequality

$$r+s = 2n-(i+j)+2 \geq 2n-2N+2$$

it follows that $r+s \geq N^2 \geq (r-s)^2$ provided that $n \geq \{(N+1)^2-3\}/2 = J(k)$. For this value $J(k)$, $M(n,k)$ is a sum of non-negative terms, and this proves (a).

Now put $J = \max(N, \{(N+1)^2-3\}/2)$. Then for all $k \geq 1$, the sequence $\{a(n,k) \mid n = J, J+1, \ldots\}$ is LC. By Corollary 2.6, each of the first J rows is ELC, and so there exists an integer L such that if $0 \leq n \leq J$, the sequence $\{a(n,k) \mid k = L, L+1, \ldots\}$ is LC. Then, by Theorem 2.2(a), and part (a) above, the column $\{a(n,L) \mid n = 0,1,2,\ldots\}$ is LC. Then (b) follows from Theorem 2.4. ∎

The next result shows that suitable polynomial bounds for one column imply that the adjacent column on the right is ELC.

Theorem 3.3. *Let $A=\{a(n,k)\}$ be a BR-set, and for some fixed k, put $x_i = a(i,k)$ and $y_i = a(i,k+1)$ $(i = 0,1,\ldots)$. Suppose there exist polynomials $P(x)$ and $Q(x)$ of the same degree d, with respective positive leading* *coefficients p and q, such that for all but finitely many i we have*

$$P(i) \leq x_{i+1} - xi \leq Q(i) \qquad (i > 0). \qquad (*)$$

If $(d+2)p^2 \geq (d+1)q^2$, *then the column* $\{y_i\}$ *is ELC.*

Proof: From $(*)$, it follows that for all $n \geq 0$,

$$x_n \geq x_0 + \sum_{j=1}^{n} P(j)$$

$$x_n x_0 + \sum_{j=1}^{n} Q(j)$$

that is, $x_n - x_0$ is an upper sum for $\int_0^n P(x)$, and is a lower sum for $\int_o^{n+1} Q(x)$.

From the recurrence, $x_{n+1} - x_n = a(n+1, k-1)$, and then by Theorem 2.3 (a), we have $(y_n)^2 - y_{n-1}\, y_{n+1} = (x_n)^2 - y_{n-1}(x_{n+1} - x_n)$. From $(*)$,

$$(x_n)^2 \geq \left\{ x_0 + \int_o^n P(x) \right\}^2 ,$$

and the high-order term (in the variable n) on the right is

$$\frac{p^2 n^{2d+2}}{(d+1)^2}.$$

The term $y_{n-1}(x_{n+1} - xn)$ is more complicated. Put $Q^*(i) = \int_0^{i+1} Q(x)$; then for $i \geq 1, x_i \leq x_0 + Q^*(i)$. Since $y_{n-1} = x_0 + x_1 + \cdots + x_{n-1}$, then we also have

$$y_{n-1} \leq nx_0 + \sum_{i=1}^{n-1} Q^*(i) \leq nx_0 + \int_0^n Q^*(x).$$

and then

$$y_{n-1}(x_{n+1} - xn) \leq Q(n+1)\left\{ nx_0 + \int_0^n Q^*(x) \right\}.$$

After integrating, the high-order term on the right is

$$\frac{q^2 n^{2d+2}}{(d+1)}(d+2).$$

The quantity $(x_n)^2 - y_{n-1}(x_{n+1} - x_n)$ exceeds the difference of these two estimates, and for sufficiently large n, this difference will depend entirely on the high-order terms; it will be positive eventually, provided

$$\frac{q^2 n^{2d+2}}{(d+1)}(d+2) \leq \frac{p^2 n^{2d+2}}{(d+1)^2}.$$

Simplifying this inequality gives the statement of the theorem. ∎

The next results have a similar flavor, and are useful in many special cases.

Theorem 3.4. *Let* $P(x) = \sum a_i x^i$ *and* $Q(x) = \sum b_i x^i$ *be two polynomials with positive leading coefficients, and degrees* ≥ 3. *Suppose that for all but finitely many positive integers* n,

$$P(n) \leq Q(n) \quad \text{and} \quad P(n)^2 \geq Q(n+1)Q(n-1). \qquad (**)$$

Then $P(x)$ *and* $Q(x)$ *have the same degree* d, *and*

$$a_d = b_d, \quad a_{d-1} = b_{d-1}, \quad \text{and} \quad db_d \geq 2(b_{d-2} - ad - 2).$$

Conversely, if $P(x)$ and $Q(x)$ satisfy these conditions on the degree and coefficients, then (**) is true for all but finitely many integers n.

Proof: Multiply out $P(x)^2$ and $Q(x+1)Q(x-1)$ and compare coefficients. ∎

Theorem 3.5. *Let* $P(x)$ *and* $Q(x)$ *be polynomials with positive leading coefficients, and* $\{x_i\}$ *a positive sequence. If for all but finitely many* n,

$$P(n) \leq x_n \leq Q(n) \quad \text{and} \quad P(n)^2 \geq Q(n+1)Q(n-1),$$

then the sequence $\{x_i\}$ *is ELC.*

Proof: For sufficiently large n, we have

$$\frac{P(n)}{Q(n-1)} \leq \frac{x_n}{x_{n-1}} \leq \frac{Q(n)}{P(n-1)}$$
$$\frac{P(n+1)}{Q(n)} \leq \frac{x_{n+1}}{x_n} \leq \frac{Q(n+1)}{P(n)}$$
$$\frac{Q(n+1)}{P(n)} \leq \frac{P(n)}{Q(n-1)}.$$

Then (for sufficiently large n) $x_{n+1}/x_n \leq x_n/x_{n+1}$, and $\{x_n\}$ is ELC. ∎

Corollary 3.6. *If* $P(x)$ *is a polynomial with positive leading coefficient, then there is an integer* $N \geq 0$ *such that the sequence* $\{P(n) \mid n = N, N+1, \dots\}$ *is LC.*

Proof: If $P(x)$ has degree $d \geq 3$, this follows from Theorem 3.5, with $Q(x) = P(x)$. If $d < 3$, the statement is easily checked directly. ∎

References

1. C. Cadogan, *Some generalizations of the Pascal triangle*, Mathematics Magazine May (1972).
2. A.W.F. Edwards, "Pascal's Arithmetical Triangle", Charles Griffin and Co., Ltd., London, 1987.
3. D.C. Kurtz, *A note on the concavity property of triangular arrays of numbers*, J. Comb. Theory (A) **13** (1972).