# Non-isomorphic Extremal Graphs without Three-Cycles or Four-Cycles

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**Abstract.** We define an *extremal graph* on $v$ vertices to be a graph that has the maximum number of edges on $v$ vertices, and that contains neither 3-cycles nor 4-cycles. We establish that every vertex of degree at least 3, in an extremal graph of at least 7 vertices is in a 5-cycle; we enumerate all of the extremal graphs on 21 or fewer vertices; and we determine the size of extremal graphs of orders 25, 26, and 27.

## 1. Introduction

In 1975, P. Erdös mentioned the problem of determining the values of $f(v)$, the maximum number of edges in a graph of order $v$ and girth at least 5 ([3]). We will begin with some basic definitions, and then describe what is known about this problem.

For simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$, $|V(G)| = v$ is the *order* of $G$, and $|E(G)| = e$ is the *size* of $G$. For any $x$ in $V(G)$, $d(x)$ is the degree of vertex $x$ and $d_k(G)$ is the number of vertices of degree $k$ in $G$. $\delta(G)$ and $\Delta(G)$ are the minimum and maximum degrees respectively of vertices in $G$. $N(x)$ (the *neighborhood* of $x$) is the set of vertices in $V(G)$ that are adjacent to $x$; if $S \subseteq V(G)$, then $N(S) = \cup_{x \in S} N(x)$. We denote the induced subgraph on $S \subseteq V(G)$ by $\langle S \rangle$. $g(G)$, the *girth* of graph $G$, is the number of vertices in the shortest cycle in $G$. $C_n$ is the cycle on $n$ vertices, which we call an *n-cycle*. $P_n$ is the path on $n$ vertices. For sets $S$ and $T$, we use $S/T$ to denote the elements in $S$ that are not in $T$. As we use the above notations in the paper, we will leave out the argument $G$ when doing so does not lead to ambiguities.

We define $f(v)$ to be the maximum number of edges in a graph of order $v$ and girth at least 5. Note that any graph with girth at least 5 does not have a *triangle* or *quadrilateral* ($C_3$ or $C_4$). We say that a graph of girth at least 5 is $\{C_3, C_4\}$*-free*. A $\{C_3, C_4\}$-free graph $G$ is *extremal* if $e = f(v)$. $\mathcal{G}_v$ is the set of all extremal graphs of order $v$, and $F(v) = |\mathcal{G}_v|$.

The authors of [4] provided the following bounds on $f(v)$:

**Theorem 1.1.** *$f(v)$, the size of any extremal graph with order $v$, has the following bound:*

$$\frac{1}{2\sqrt{2}} \leq \lim_{v \to \infty} \sup \frac{f(v)}{v^{3/2}} \leq \frac{1}{2}.$$

As well, they determined the following values of $f(v)$ and $F(v)$:

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**Theorem 1.2.**

| $v$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(v)$ | 0 | 1 | 2 | 3 | 5 | 6 | 8 | 10 | 12 | 15 | 16 | 18 | 21 | 23 | 26 |
| $F(v)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | | | | | |

| $v$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|
| $f(v)$ | 28 | 31 | 34 | 38 | 41 | 44 | 47 | 50 | 54 | 175 |
| $F(v)$ | | | | 1 | 1 | | | | | 1 |

*The values of $f(50)$ and $F(50)$ are derived from the existence of the Moore graph on 50 vertices (known as the Hoffman-Singleton graph [6]), the unique extremal graph of order 50.*

In Section 2, we provide some results pertaining to the connectivity and girth of extremal graphs, and in Section 3 we enumerate all extremal graphs for the orders up to 21. Section 3 also describes a graph invariant that is easy and fast to compute, and, for extremal graphs, appears to be highly effective; we have yet to find non-isomorphic extremal graphs that are indistinguishable under the invariant. In Section 4 we determine the values of $f(v)$ for $25 \leq v \leq 27$. The concluding section provides some questions for further research.

## 2. Results on the structure of extremal graphs

We present some theoretical results concerning the structure of extremal graphs. First we cite the following result from [4]:

**Proposition 2.1.** *For all extremal graphs $G$:*

1. *The diameter of $G$ is at most 3, and therefore $G$ is connected.*
2. *If $x \in V(G)$ has degree 1, then $G - x$ has diameter at most 2.*

The authors of [4] also determined that for any $\{C_3, C_4\}$-free graph $G$ with $v > 1$, $\delta \geq e - f(v-1)$. We generalize this result to a lower bound on the number of edges that can be incident on any set of $k$ vertices in a $\{C_3, C_4\}$-free graph.

**Proposition 2.2.** *For any $k$ vertices $x_1, x_2, \ldots, x_k$ in $\{C_3, C_4\}$-free $G$ with $v > 1$,*

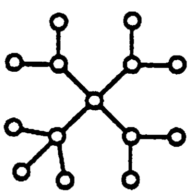
$$\sum_{i=1}^{k} d(x_i) - |E(\langle x_1, x_2, \ldots, x_k \rangle)| \geq e - f(v-k).$$

Proof: Let $N$ be the induced subgraph on the $k$ vertices $x_1, x_2, \ldots, x_k$. Since summing the degrees of the $k$ vertices will include each edge that extends from $H$ to $G - H$, and will include twice each of the edges in $E(H)$, then $e' = \sum_{i=1}^{k} d(x_i) - |E(H)|$ is the number of edges in $G$ that are incident on vertices in $V(H)$. If we assume that $e' < e - f(v - k)$, then the $\{C_3, C_4\}$-free graph $G - H$ will have $v - k$ vertices, but more than $f(v - k)$ edges. Since this is a contradiction, then $e' \geq e - f(v - k)$. ∎

The result from [4] is a special case of Proposition 2.2 where $k = 1$, and $d(x_1) = \delta$. We will use Proposition 2.2 in two ways: to determine the lower bound on $\delta$, and to show the non-existence of certain subgraphs in extremal graphs. We will also make use of the fact that the average degree in a graph is greater than or equal to the minimum degree, and less than or equal to the maximum degree. We state this in the following proposition.

**Proposition 2.3.** *For all graphs $G$, $\delta \leq 2e/v \leq \Delta$.*

The authors of [4] noted the presence of $(m, n)$-stars in extremal graphs, where the $(m, n)$-star $S_{m,n}$ is a tree where the root has degree $m$, and each of the $m$ neighbors of the root have $n$ additional independent neighbors. If $r$ is the root vertex of an $(m, n)$-star, $S$, then a *branch* $i$ of $S$ consists of a *parent*, labeled $b_i$, in $N(r)$, and the $n$ *children* in $N(b_i) - r$. The pendent vertices in $S$ are also called *leaves*. We will label the leaves of $S_{m,n}$ as $b_{i,j}$, $1 \leq i \leq m$, $1 \leq j \leq n$ where such a label denotes leaf $j$ in branch $i$. Sometimes we will augment a branch by adding leaves to it. Thus we can refer to an $(m, n)$-star which has a $k$-branch containing $k \geq n$ vertices. We will refer to a leaf of degree $k$ as a *k-leaf*. Figure 1a shows $S_{4,2}$, where the fourth branch has been augmented with a leaf. Thus it has three 3-branches, and one 4-branch. Figure 1b shows the same star with a typical labeling of the leaves; to reduce clutter in depictions of complex graphs, often we will leave out the root and its neighbors.

a) 

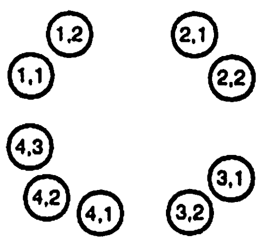
b) 


**Figure 1:** An augmented $(4, 2)$-star and a labeling of the leaves

The authors of [4] noted the following facts and proposition with respect to $(m, n)$-stars in extremal graphs.

1. $|V(S_{m,n})| = 1 + m + mn$ and $|E(S_{m,n})| = m + mn$.
2. Every $\{C_3, C_4\}$-free graph $G$ with at least 5 vertices contains $S_{\Delta,\delta-1}$.
3. For any nonadjacent vertices $t$ and $u$ in $S_{m,n}$ that are not both leaves, $S_{m,n} + tu$ contains a $C_3$ or $C_4$.

4. In any $\{C_3, C_4\}$-free graph $G$ containing an $(m,n)$-star $S$, no vertex in $G - S$ can be adjacent to two siblings in $S$. In other words, every set of siblings has a unique common neighbor, namely, their parent.

The following proposition is derived from the number of vertices in $S$ which must be contained in every $\{C_3, C_4\}$-free graph with minimum degree $\delta$ and maximum degree $\Delta$.

**Proposition 2.4.** *For all $\{C_3, C_4\}$-free graphs $G$, $v \geq 1 + \Delta\delta \geq 1 + \delta^2$.*

A special case of this proposition is when $\delta = \Delta$; that is, the graph is $r$-regular. In this case $v \geq r^2 + 1$, which provides a lower bound on the order of a $(r,5)$-*cage*, an $r$-regular graph with girth 5 and of minimal order (see [1], [8]). Cages will play a role in our proof of the theorem in Section 3. General results on cages are surveyed in [9].

If $G$ is an extremal graph, then the definition provides that $g(G) \geq 5$; we will prove that, if $v \geq 7$, then $g(G) = 5$. First, we will prove a stronger result concerning the order of the smallest cycle containing any vertex in $V(G)$. To do so, we derive some basic facts about the existence and lengths of cycles in extremal graphs.

**Lemma 2.5.** *For all extremal graphs $G$ with $v \geq 5$, if $x \in V(G)$ and $d(x) \geq 2$, then $x$ is in a cycle in $G$.*

Proof: By inspection of $\mathcal{G}_5$, this is true for $v = 5$. For $v > 5$, we first establish that for every vertex $x$ in any extremal graph $G$, if $d(x) \geq 2$, then $N(x)$ contains at least two vertices of degree at least 2. Assume that $x$ has at most one neighbor of degree at least 2. Let $s$ be a degree 1 vertex in $N(x)$, and let $H = G - s$; since clearly $f(v) > f(v-1)$, $H$ is extremal. Further, by Proposition 2.1, the diameter of $H$ is at most 2. Note that $H$ contains a degree 1 vertex, $t$; if $d(x)$ in $G$ is 2, then let $t = x$ otherwise let $t$ be a degree one vertex in $N(x) - s$. Since no vertex in $H$ is further than distance 2 from pendent vertex $t$, and $H$ is $\{C_3, C_4\}$-free then $H$ is a tree with two levels, rooted at the neighbor of $t$. Thus, $|E(H)| < |V(H)|$. But since any cycle of at least five vertices is $|\{C_3, C_4\}$-free, and has equal order and size, then $H$ is not extremal. By contradiction, $x$ has at least two neighbors of degree at least 2.

We have established that for every $x \in V(G)$, $d(x) \geq 2$, there exists $y, z \in N(x)$ such that $d(y) \geq 2$ and $d(z) \geq 2$. Thus, $N(y) - x$ and $N(z) - x$ are both non-empty and, since $g(G) \geq 5$, disjoint. If $x$ is not in a cycle in $G$, then $y' \in N(y) - x$ is distance 4 from $z' \in N(z) - x$ in $G$. This contradicts our assumption that $G$ is extremal. Thus, for all $x$ in extremal graphs $G$ with $v \geq 5$, $x$ is in a cycle. ∎

**Lemma 2.6.** *For all extremal graphs $G$, with $v \geq 5$, if $x \in V(G)$ and $d(x) \geq 2$, then $x$ is in a cycle of length less than 8.*

Proof: For $5 \leq v \leq 7$, this follows directly from Lemma 2.5. Now let $G$ be an extremal graph on at least 8 vertices, and let $C = (x_1, x_2, \ldots, x_k, x_1)$ be a smallest cycle containing $x = x_1$. If we assume that $k > 7$, then $x_1$ is distance 4 from $x_5$ in $C$. Further, $x_1$ is distance 4 from $x_5$ in $G$; otherwise we contradict our assumption that $C$ is a smallest cycle containing $x_1$. By Proposition 2.1, the existence of a distance 4 pair of vertices contradicts our assumption that $G$ is extremal; the smallest cycle containing $x_1$ must have length less than 8. ∎

**Theorem 2.7.** *For all extremal graphs $G$, where $v \geq 5$, and for all $x \in V(G)$:*

1. *If $d(x) = 2$, then $G$ has a 5-cycle or 6-cycle that contains $x$;*
2. *If $d(x) \geq 3$, then $G$ has a 5-cycle that contains $x$.*

Proof: For $v < 8$, this is true by inspection of $\mathcal{G}_5, \mathcal{G}_6$, and $\mathcal{G}_7$. Now we assume that $v \geq 8$.

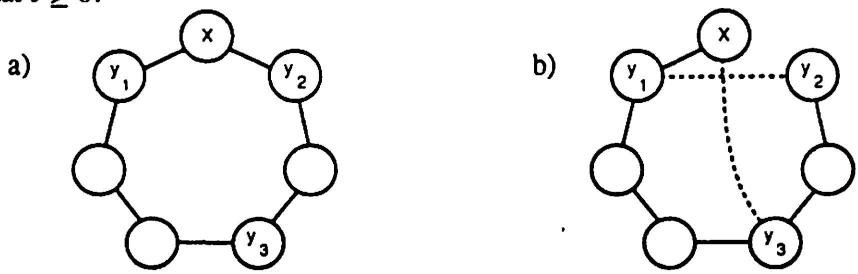



**Figure 2:** Subgraph containing $x$ where $d(x) = 2$

1. If $d(x) = 2$, then $x$ is in a 5-cycle or 6-cycle. By Lemma 2.6, $x$ is in a cycle of length 5, 6, or 7. We assume that $x$ is not in a 5-cycle or 6-cycle. This implies that $G$ contains the subgraph pictured in Figure 2a which depicts a smallest cycle containing $x$. Further, $G$ cannot contain additional edges incident only on vertices in the subgraph, for such edges would complete a triangle, or a quatrilateral. We define $G'$ such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{(y_1, y_2), (x, y_3)\} / \{(x, y_2)\}$. Thus, $G'$ is the result of replacing in $G$ the subgraph in Figure 2a with the subgraph in Figure 2b. Note that $G'$ does not contain $C_3$ or $C_4$ since the presence of such subgraphs would imply that $x$ was in a 5-cycle or a 6-cycle in $G$. Since $|V(G')| = |V(G)| + 1$, then $G$ is not extremal, thus contradicting our assumption. Therefore, for all $x$ in extremal graph $G$ on at least 5 vertices, where $d(x) = 2$, $G$ contains a 5-cycle or 6-cycle that includes $x$.
2. If $d(x) \geq 3$, then $x$ is in a 5-cycle. By Lemma 2.6, $x$ is in a cycle of length 5, 6, or 7. Let us assume that $x$ is not in a 5-cycle. This implies that $G$ contains the subgraph in Figure 3a which depicts some smallest cycle containing $x$; the cycle has length 6 or 7. Further, $G$ cannot contain additional edges incident only on vertices in the subgraph, for any such edge

would complete either a 5-cycle involving $x$, a triangle, or a quadrilateral. We define $G'$ such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{(y_1, y_2), (y_1, y_3), (x, y_4)\}/\{(x, y_1), (x, y_3)\}$. Thus, we form $G'$ by replacing the subgraph in Figure 3a with the subgraph in Figure 3b; the dashed lines indicate new edges. Note that $G'$ does not contain $C_3$ or $C_4$, since the presence of such a cycle would imply that $x$ was in a 5-cycle in $G$. Since $|V(G')| = |V(G)| + 1$, then $G$ is not extremal, thus contradicting our assumption. Therefore, for all $x$ in extremal graph $G$ on at least 5 vertices, where $d(x) \geq 3$, $G$ contains a 5-cycle that includes $x$. ∎

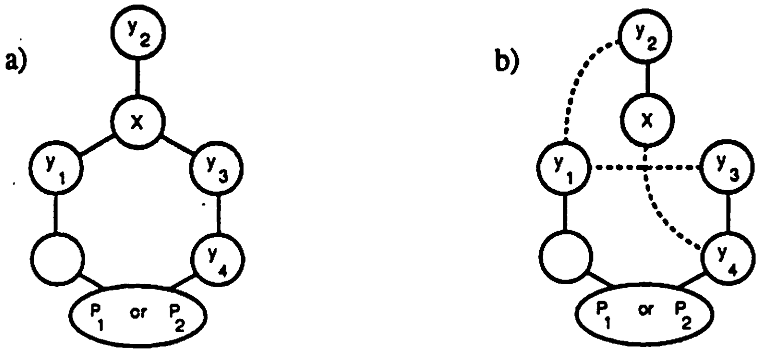



**Figure 3:** Subgraph containing $x$ where $d(x) \geq 3$

In the extremal graphs we have constructed, almost every vertex of degree 2 is in a 5-cycle; an exception is the degree 2 vertex in $G_{11b}$. It is interesting to note that in all of the $\{C_3, C_4\}$-free graphs generated by [4] for their constructive lower bounds on $f(v)$, every vertex of degree at least 3 is in a 5-cycle.

It is now quite straightforward to show that the girth of an extremal graph is 5.

**Corollary 2.8.** *For all extremal graphs G with* $v \geq 7$, $g(G) = 5$.

Proof: For $v \geq 7$, it is trivial to construct a $\{C_3, C_4\}$-free graph with $v$ vertices and $v+1$ edges, thus showing that $f(v) > v$. In turn, this implies that all extremal graphs with at least 7 vertices have average degree greater than 2, and thus have at least one vertex $x$ of degree at least 3. By Theorem 2.7, $x$ is in a 5-cycle. ∎

## 3. Enumeration of extremal graphs

All of the extremal graphs, for $v \leq 10$, were enumerated in [4]. They are shown in Figure 4. We enumerate all of the extremal graphs for $11 \leq v \leq 21$.

**Theorem 3.1.** *For* $11 \leq v \leq 21$, $F(v)$ *has the values in the following table:*

| $v$ | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $F(v)$ | 3 | 7 | 1 | 4 | 1 | 22 | 14 | 15 | 1 | 1 | 3 |

Proof: Generally we construct the elements of $\mathcal{G}_v$ using two techniques. The first is to prove that if $G \in \mathcal{G}_v$ has a particular degree sequence, then it must contain $S \cong S_{m,n}$ for some $m$ and $n$. Then we determine the edges amongst the leaves of $S$ and, if $|V(S)| < |V(G)|$, the vertices in $G$ that are external to $S$.

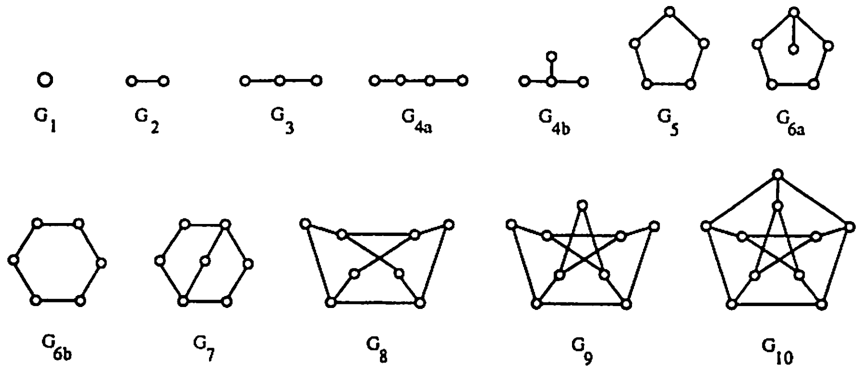



**Figure 4:** All extremal graphs for $v \leq 10$

The second technique involves building some elements of $\mathcal{G}_v$ from elements of $\mathcal{G}_{v-1}$. To show how this is done, we first define the notion of a *cold* set. For extremal graph $G$ of order $v, T \subseteq V(G)$ is *cold* iff $|T| = f(v+1) - f(v)$, and for all distinct $x, y \in T, (x, y) \notin E(G)$ and $N(x) \cap N(y) = \emptyset$. By extension, an extremal graph $G$ is cold if it contains a cold set; if $G$ is not cold, then it is *hot*. Note that if $G \in \mathcal{G}_v$ with cold set $T$, then all $G'$, such that $V(G') = V(G) + u_{v+1}$ and $E(G') = E(G) \cup \{(t, u_{v+1} \mid t \in T\}$, are in $\mathcal{G}_{v+1}$. Thus, by finding all such graphs $G'$, we cover the set of all graphs in $\mathcal{G}_{v+1}$ with minimum degree $f(v+1) - f(v)$. This technique is similar to one used by Clapham, Flockhart, and Sheehan in [2].

We now provide a case by case proof of Theorem 3.1 based on the values of $v$.

F(11) = 3

Since $f(11) = 16$, then for all $\mathcal{G} \in G_{11}$, $1 \leq \delta \leq 2$ and $\Delta \geq 3$ by Propositions 2.2 and 2.3. Let us assume that $\Delta \geq 5$. If $x \in V(G)$ and $d(x) = 1$, then $G - x \in \mathcal{G}_{10}$. But that is not possible since $\Delta(G - x) \geq 4$, and the unique graph in $\mathcal{G}_{10}$ is 3-regular. Thus, if $\Delta \geq 5$, then $G$ contains no pendent vertices; therefore, $G$ contains $S \cong S_{5,1}$. Since $|V(S)| = 11$ and $|E(S)| = 10$, then the induced subgraph in $G$ on the 5 leaves of $S$ contains 6 edges. But since $f(5) < 6$, our assumption that $\Delta \geq 5$ is false.

Since any graph with 11 vertices and $(\delta, \Delta) = (1, 3)$ has at most 15 edges, and $15 < f(11)$, we conclude that $(\delta, \Delta) \in \{(1,4), (2,3), (2,4)\}$. We consider each of these degree ranges in turn.

Case: $(\delta, \Delta) = (1, 4)$

If $x \in V(G)$ such that $d(x) = 1$, then $G - x$ is a cold graph in $\mathcal{G}_{10}$. Since the Petersen graph is the unique graph in $\mathcal{G}_{10}$, and the vertices in the Petersen graph are indistinguishable, then $G_{11_a}$ in Figure 5 is the only member of $\mathcal{G}_{11}$ for which $\delta = 1$.

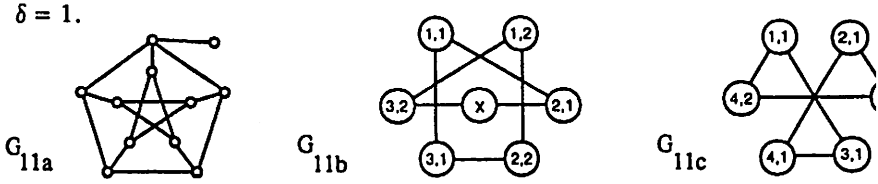



**Figure 5:** The extremal graphs of order 11

Case: $(\delta, \Delta) = (2,3)$

We next determine tbe number of vertices of each degree. In general we do this by solving the following simultaneous equations for $d_i$, $\delta \leq i \leq \Delta$ where $d_i = |\{x : x \in V(G), d(x) = i\}|$.

$$\sum_{i=\delta}^{\Delta} i d_i = 2e \qquad \sum_{i=\delta}^{\Delta} d_i = v \tag{3.1}$$

Since $v = 11$, $e = 16$, $\delta = 2$, and $\Delta = 3$, then $d_2 = 1$ and $d_3 = 10$. Let $x$ be the degree 2 vertex. Since $|N(x) \cup N(N(x))| = 7$, $G$ contains a degree 3 vertex, $r$ that is distance 3 from $x$. Thus, $r$ is the root of $S \cong S_{3,2}$ in $G$, and since $|V(S)| = 10$, the degree 2 vertex, $x$ is the sole vertex external to star $S$. Further $N(x)$ contains only leaves of the star. As we described earlier, we label the leaves of $S$ $b_{i,j}$, $1 \leq i, j \leq 3$ where $b_{i,j}$ is leaf $j$ in branch $i$.

Since $d(x) = 2$, the leaves of one branch, say $b_1$, are not adjacent to $x$. Therefore, $b_{1,1}$ and $b_{1,2}$ are each adjacent to 2 other leaves. Since they cannot have a common neighbor, and no leaf is adjacent to more than one leaf in another branch we may assume that $b_{2,1}, b_{3,1} \in N(b_{1,1})$ and $b_{2,2}, b_{3,2} \in N(b_{1,2})$. Since $x$ is adjacent to one leaf in each of branches $b_2$ and $b_3$, and the neighbors of $x$ cannot be distance 2 apart, $N(x) = \{b_{2,1}, b_{3,2}\}$. The only possible remaining edge is $(b_{2,2}, b_{3,1})$. The resulting graph, $G_{11_b}$, is shown in Figure 5 where $r$ and $N(r)$ have been left out for clarity.

Case: $(\delta, \Delta) = (2,4)$

If $G$ contains edge $(x, y)$ such that $d(x) = d(y) = 4$, then if $G' = \langle N(x) \cup N(y) \rangle$, then $|V(G')| = 8$; since $N(x) \cap N(y) = \emptyset$, then $|E(G')| = 7$. Since each of the 3 vertices in $V(G)/V(G')$ can have at most 2 neighbors in $G'$, and there are at most $f(3) = 2$ edges in $\langle V(G)/V(G') \rangle$, then $|E(G)| \leq 15 \leq f(11)$. Therefore, $G$ does not contain adjacent vertices of degree 4. Since no 2 vertices in an extremal graph have more than 1 common neighbor, any extremal graph with at least 3 degree 4 vertices that are mutually non-adjacent must have at least 1[illegible] vertices. Therefore, $G$ has at most 2 degree 4 vertices. By this fact, together with Equation 3.1, $(d_2, d_3, d_4) \in \{(2,8,1),(3,6,2)\}$.

Since $G$ does not contain adjacent vertices of degree 4, and a $(4, n \geq 1)$-star with more than two 3-branches has more than 11 vertices, $G$ must contain a degree 4 vertex with two neighbors having degree 2, and two neighbors having degree 3. Thus, $G$ contains star $S$ with two 3-branches and two 2-branches; it is easy to show that if $G$ contains a star with three 2-branches, one 3-branch, and one external vertex adjacent only to leaves in the star, then it is not possible to have 16 edges without creating a triangle or quadrilateral. The leaves can be labeled $b_{1,1}$, $b_{2,1}, b_{3,1}, b_{3,2}, b_{4,1}, b_{4,2}$.

Since there is at most one 2-leaf and one 4-leaf, we may assume that $d(b_{3,1}) = 3$ and $d(b_{3,2}) \geq 3$. Since assuming $d(b_{3,2}) = 4$ leads to the presence of $C_3$ or $C_4$ in $G$, $d(b_{3,2}) = 3$. Since leaves in the same branch have a unique common neighbor (their parent), then $N(b_{3,1}) \supseteq \{b_{4,1}, b_{1,1}\}$, and $N(b_{3,2}) \supseteq \{b_{4,2}, b_{2,1}\}$. Since no remaining leaf can have degree 4, then by the permissible degree sequences $d(b_{1,1}) = d(b_{2,1}) = d(b_{4,1}) = d(b_{4,2}) = 3$. Therefore, $b_{4,1} \in N(b_{2,1})$, and $b_{4,2} \in N(b_{1,1})$. This completes $G_{11c}$ which is shown in Figure 5 without the root of the star and its neighbors.

**F(12) = 7**

Since $f(12) = 18$, then for all $G \in \mathcal{G}_{12}$, $2 \leq \delta \leq 3$ by Propositions 2.2 and 2.3. If there exists $x \in V(G)$, $d(x) = 2$, then $N(x)$ is a cold set in a cold graph in $\mathcal{G}_{11}$. Considering all $G$ built from the cold graphs in $\mathcal{G}_{11}$, there are five cases based on the degree sequence of $G$, and on the connectivity amongst the degree 2 and degree 4 vertices in $G$. We will show that there is a unique graph for each case. The notation $(d(p_1), d(p_2), \ldots, d(p_k))$ refers to the degrees of the vertices in a $P_k$, $P = (p_1, p_2, \ldots, p_k)$ that exists in $G$; that is, $G$ contains $P \cong P_k$ where the $k$ vertices in $P$ have the specified degrees.

| case | $(d_2, d_3, d_4)$ | $(d(p_1), d(p_2), \ldots, d(p_k))$ |
|---|---|---|
| a | $(2,8,2)$ | $(4,2,2,4)$ |
| b | $(2,8,2)$ | $(4,2,4,2)$ |
| c | $(2,8,2)$ | $(2,4,3,4,2)$, $p_1$ and $p_5$ non-adjacent |
| d | $(1,10,1)$ | |
| e | $(3,6,3)$ | |

We will construct the unique graph for each of these cases. Additionally, there are two graphs for which $\delta = 3$.

Case a: $(d_2, d_3, d_4) = (2,8,2)$ and $(d(p_1), d(p_2), d(p_3), d(p_4)) = (4,2,2,4)$

Since the two degree 2 vertices are adjacent, then if $x \in V(G)$, $d(x) = 2$ then $\delta(G - x) = 1$, which implies that $G - x \cong G_{11a}$. Thus $N(x)$ is a cold set in $G_{11a}$. All of the cold sets in $G_{11a}$ are indistinguishable; each consists of the

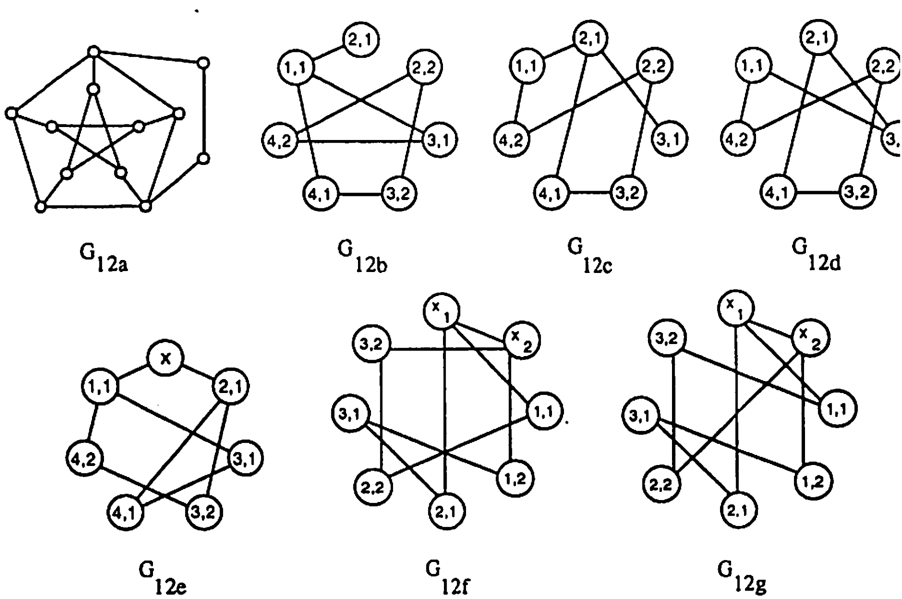



**Figure 6:** The extremal graphs of order 12

pendent vertex and any vertex that is distance 3 from the pendent vertex. Thus, there is a single graph for Case a, which is $G_{12a}$ in Figure 6.

Case b: $(d_2, d_3, d_4) = (2, 8, 2)$ and $(d(p_1), d(p_2), d(p_3), d(p_4)) = (4, 2, 4, 2$

Let us consider the $(4, n)$-star $S$ in $G$ that has as its root $p_1$, the degree 4 vertex that has only one degree 2 neighbor. Thus, in $S$, $d(b_1) = 2$ and $d(b_2) = d(b_3) = d(b_4) = 3$; the star contains all 12 vertices in $G$. By the existence of the path with vertices having degrees $(4, 2, 4, 2)$, $d(b_{1,1}) = 4$, and we may assume that $b_{2,1}$ is the other degree 2 neighbor of $b_{1,1}$. $b_{1,1}$ has 2 other leaves in $S$ as neighbors: $b_{3,1}$ and $b_{4,1}$. $b_{2,2}$ cannot be adjacent to 2 vertices in $N(b_{1,1})$, thus we conclude that $b_{3,2} \in N(b_{2,2})$. Since $b_{4,2}$ has a neighbor that is a leaf in branch 2, and $b_{2,1}$ is full, we conclude that $b_{2,2} \in N(b_{4,2})$. The last two edges must be $(b_{3,1}, b_{4,2})$ and $(b_{3,2}, b_{4,1})$. The graph is $G_{12b}$ in Figure 6; the root of the star and its neighbors have been omitted.

Case c: $(d_2, d_3, d_4) = (2, 8, 2)$ and $(d(p_1), d(p_2), d(p_3), d(p_4), d(p_5)) = (2, 4, 3, 4, 2)$

Let us consider the star $S$ in $G$ that is rooted at $p_2$, one of the degree 4 vertices. Thus, in $S$, $d(b_1) = 2$ and $d(b_2) = d(b_3) = d(b_4) = 3$; the star contains all 12 vertices in $G$. By the existence of $P$ in $G$, $d(b_{1,1}) = 3$, $d(b_{2,1}) = 4$, and since $b_{2,1}$ must have a degree 2 neighbor, we may assume that it is $b_{3,1}$. The remaining leaves in $S$ all have degree 3. Thus, the three leaves in $N(b_{2,1})$ are $b_{1,1}, b_{3,1}$, and $b_{4,1}$. Since no leaf can be adjacent to two siblings, $N(b_{2,2})$ contains $b_{3,2}$ and $b_{4,2}$. The remaining neighbor of $b_{4,1}$ can only be $b_{3,2}$, and the last neighbor of $b_{4,2}$ must be $b_{1,1}$. The connections amongst the leaves in the star are shown as

$G_{12c}$ in Figure 6.

Case d: $(d_2, d_3, d_4) = (1, 10, 1)$

If $x \in V(G)$, $d(x) = 4$, then $N(x)$ includes at least three degree 3 vertices, and, by Proposition 2.4, cannot have four degree 3 neighbors. Therefore, $G$ contains $S_{4,1}$, where branches 2, 3, and 4 are augmented with a second leaf; since $d_2 = 1$, and $d(b_1) = 2$, all of the leaves have degree 3. Since $b_{1,1}$ is adjacent to 2 leaves in the star, we may assume that $b_{1,1}$ has no neighbor in branch 2. Thus, $N(b_{2,1}) \supseteq \{b_{3,1}, b_{4,1}\}$ and $N(b_{2,2}) \supseteq \{b_{3,2}, b_{4,2}\}$. Since the two leaves in $N(b_{1,1})$ cannot already be distance 2 apart, then $N(b_{1,1}) \supseteq \{b_{3,1}, b_{4,2}\}$. The remaining edge must be $(b_{3,2}, b_{4,1})$. The leaves of the star in this graph are shown as $G_{12d}$ in Figure 6.

Case e: $(d_2, d_3, d_4) = (3, 6, 3)$

The only case where this graph occurs is when a twelfth vertex, $x$ is added to $G_{11c}$ and connected to vertices $b_{1,1}$ and $b_{2,1}$ in $G_{11c}$. This graph (excluding the vertices inside the star) is shown as $G_{12e}$, in Figure 6.

Case: $\delta = 3$

We now consider the case where $\delta = 3$. By Proposition 2.4, $\Delta < 4$; therefore $G$ is 3-regular. Thus, $G$ contains $S \cong S_{3,2}$, as well $x_1, x_2 \in V(G)/V(S)$. Since $|E(S)| = 9$, then the induced subgraph on $L = \{\text{leaves of } S\} \cup \{x_1, x_2\}$ has 8 vertices and 9 edges. It is easy to construct the three $\{C_3, C_4\}$-free graphs with $v = 8$ and $e = 9$ which we call $A$, $B$, and $C$; these are shown in Figure 7. By mapping these 3 graphs onto $L$ in every way such that the two degree 3 vertices map onto $x_1$ and $x_2$, and such that $G$ will not contain $C_3$ or $C_4$, we generate the set of graphs in $\mathcal{G}_{12}$ for which $\delta = 3$. Graph $A$ can be mapped onto $L$ in two ways (shown as $G_{12f}$ and $G_{12g}$ in Figure 6). $B$ maps onto $L$ as shown in $H_1$ in Figure 8, and $C$ maps onto $L$ as shown in $H_2$ in the same figure.

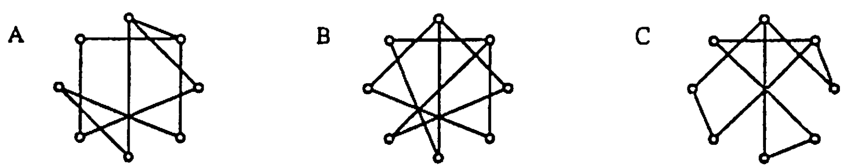


Figure 7: The 3 $\{C_3, C_4\}$-free graphs with 8 vertices and 9 edges

We use the following algorithm to determine that $G_{12f}$ is isomorphic to $H_1$, and $G_{12g}$ is isomorphic to $H_2$. We also use this algorithm throughout the paper to determine the isomorphism (or non-isomorphism) of extremal graphs of a given order. We have included the actual vertex mappings in [5].

**Algorithm ISO**

input: Graphs $G$ and $G'$

1. Partition $V(G)$ into classes, $Q(G) = \{q_0(G), q_1(G), \ldots, q_{\binom{v}{3}}(G)\}$, where

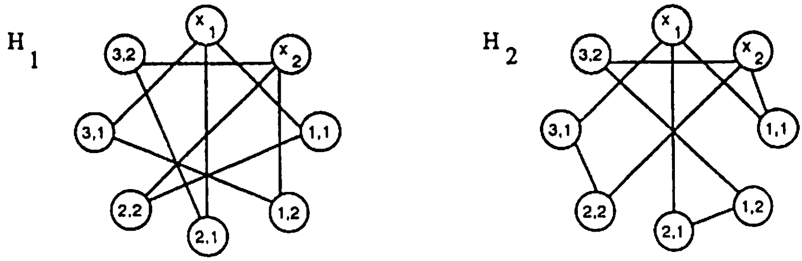



**Figure 8:**
Mappings of $A$, $B$, and $C$ onto the leaves of $S$ and the external vertices 1 and 2

$x \in V(G)$ is in $q_i(G)$ if and only if $|\{C : x \in C \subset V(G), \langle C\rangle \cong C_5\}| = i$. That is, $x$ is in equivalence class $q_i(G)$ if and only if $x$ is in $i$ 5-cycles in $G$. Similarly create $Q(G')$.

2. If there exists an $i$ such that $|q_i(G)| \neq |q_i(G')|$ then the graphs are not isomorphic.
3. If for all $i$, $|q_i(G)| = |q_i(G')|$ then show isomorphism (or non-isomorphism) by attempting to map $V(G)$ onto $V(G')$. Only attempt to map $x \in V(G)$ onto $y \in V(G')$ if $x \in q_i(G)$ and $y \in q_i(G')$.

It is interesting to note that we do not get a finer partition by considering as well the degrees of the vertice. That is, for each graph that we have looked at in this manner, if two vertices are in the same class, then they have the same degree. As well, we have not yet found a pair of non-isomorphic extremal graphs $G$ and $G'$ that were indistinguishable under the invariant based on the classes $Q$.

We can now determine the exact number of non-isomorphic extremal graphs of order 12 with $\delta = 3$. Since $q_3(G_{12f}) = 3$ and $q_3(G_{12g}) = 8$, $G_{12f}$ and $G_{12g}$ are non-isomorphic. The vertex mappings in [5] show that $H_1$ is isomorphic to $G_{12f}$, and $H_2$ is isomorphic to $G_{12g}$. Therefore, there are 2 non-isomorphic graphs in $\mathcal{G}_{12}$ where $\delta = 3$. Thus, $F(12) = 7$, and all the elements of $\mathcal{G}_{12}$ are shown in Figure 6.

**F(13) = 1**

Since $f(13) = 21$, then for all $G \in \mathcal{G}_{13}$, $\delta = 3$ by Propositions 2.2 and 2.3; it follows from Propositions 2.3 and 2.4 that $\Delta = 4$. Therefore, $G$ contains $S \cong S_{4,2}$ which includes all 13 vertices in $G$; by Equation 3.1, $d_3 = 10$ and $d_4 = 3$. Therefore, two leaves in $S$ have degree 4, and the rest of the leaves have degree 3.

We may assume that $d(b_{1,1}) = 4$ and $N(b_{1,1}) \supset \{b_{2,1}, b_{3,1}, b_{4,1}\}$. Since $d(b_{1,2}) \geq 3$, then $N(b_{1,2}) \supset \{b_{2,2}, b_{3,2}\}$. $N(b_{4,1})$ contains either $b_{2,2}$ or $b_{3,2}$; we may assume that $b_{3,2} \in N(b_{4,1})$. The only leaves in $S$ that could be the third degree 4 vertex are $b_{1,2}, b_{2,2}$ or $b_{3,1}$. If $d(b_{2,2}) = 4$ or $d(b_{3,1}) = 4$, then $b_{2,2}$ is adjacent to $b_{3,1}$; but in this case, $N(b_{4,2})$ includes $b_{2,1}$ and $b_{3,1}$, which completes a $C_4$. Thus, $d(b_{1,2}) = 4$, and $b_{4,2} \in N(b_{1,2})$. The only possibilities for the remaining two edges are $(b_{2,1}, b_{4,2})$ and $(b_{2,2}, b_{3,1})$. Thus, there is a unique graph in $\mathcal{G}_{13}$; the connections amongst the leaves of the embedded $(4, 2)$-star are shown as $\mathcal{G}_{13}$ in Figure 9.

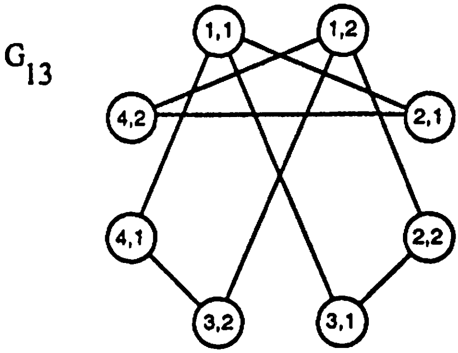



**Figure 9:**
The induced subgraph on the leaves of $S_{4,2}$ embedded in the unique graph in $\mathcal{G}_{13}$

**F(14) = 4**

Since $f(14) = 23$, then for all $G \in \mathcal{G}_{14}$, $2 \leq \delta \leq 3$ by Propositions 2.2 and 2.3. We will first consider the case where $\delta = 2$. Subsequently we will consider several cases where $\delta = 3$.

Since $f(14) = f(13) + 2$, we cover the set of all $G \in \mathcal{G}_{14}$, where $\delta(G) = 2$, using the pairs comprising the cold sets in $G_{13}$. With respect to the symmetry of $G_{13}$ (as shown in Figure 9), the set of cold pairs is $T = \{(b_2, b_{3,2}), (b_2, b_{4,1}), (b_3, b_{2,1}), (b_{2,1}, b_{3,2}), (b_{2,2}, b_{4,1})\}$. Let $H_i$, $1 \leq i \leq 5$ be the five graphs resulting from attaching a vertex $x_1$ to the respective pair of vertices in $T$; thus, $V(H_i) = V(G_{13}) + x_1$ and, for example, $E(H_1) = E(G_{13}) \cup \{(x_1, b_2), (x_1, b_{3,2})\}$. We used algorithm ISO to determine that these five graphs are isomorphic to each other. As usual, the actual mappings are contained in [5]. We show the unique graph in $\mathcal{G}_{14}$ with $\delta = 2$ as $G_{14a}$ in Figure 10; we use the vertex labeling from $H_5$ for the vertices in $G_{14a}$.

We now consider the cases where $\delta = 3$. If $\delta = 3$, then by Propositions 2.3 and 2.4, $\Delta = 4$; by Equation 3.1, $d_4 = 4$. We consider indepentently the cases based on the number of edges in the induced subgraph on the four degree 4 vertices. If $E(G)$ contains $(x, y)$ and $(x, z)$, where $x$, $y$, and $z$ are distinct and $d(x) = d(y) = d(z) = 4$, then $x$ is the root of an augmented $(4, 2)$-star with more than 14 vertices. Therefore, we only need consider the cases where $G$ contains no adjacent degree 4 vertices, one pair of adjacent degree 4 vertices, and two independent pairs of adjacent degree 4 vertices.

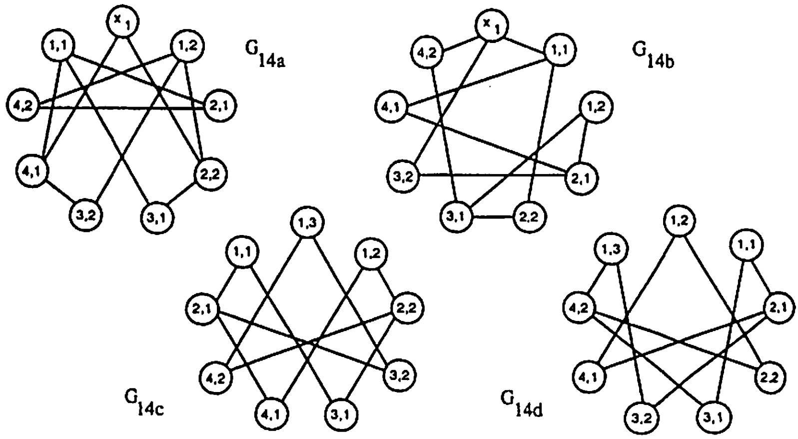



**Figure 10:** The extremal graphs of order 14

Case: $\delta = 3$ and $G$ contains no adjacent degree 4 vertices.

In this case, $G$ contains $S \cong S_{4,2}$, and a vertex $x_1$ that is external to $S$. We first show that $d(x_1) = 3$.

Let us assume that $d(x_1) = 4$. Since $x_1$ cannot have two neighbors in a single branch of $S$, $N(x_1) = \{b_{1,1}, b_{2,1}, b_{3,1}, b_{4,1}\}$ and by our assumption for this case each neighbor of $x_1$ has degree 3. Therefore, we may assume that the two remaining degree 4 vertices are $b_{1,2}$, and $b_{2,2}$. Since $b_{1,2}$ must be adjacent to a leaf in each of branches 2, 3, and 4, then $N(b_{1,2})$ contains $\{b_{2,1}, b_{3,2}, b_{4,2}\}$. Similarly $b_{2,2}$ must be adjacent to a leaf in each of branches 1, 3, and 4. $N(b_{2,2})$ must contain $b_{1,1}$ and $b_{3,2}$; however, if $b_{2,2}$ is adjacent to either leaf in branch 4, then $G$ will contain $C_4$. Therefore, $d(x_1) = 3$.

Next we establish that of the three degree 4 vertices that are leaves in $S$, each is in a distinct branch of $S$. Let us assume that there is a pair of degree 4 siblings $b_{1,1}$ and $b_{1,2}$. Since $|N(b_{1,1}) - b_1| + |N(b_{1,2}) - b_1| = 6$, and one of the 6 leaves not in branch 1 has degree 4, then at least one of the leaves in branch 1 is adjacent to $x_1$. Since both leaves in a branch cannot have a common neighbor other than their parent, then $\{b_{2,1}, b_{3,1}, x_1\} \subset N(b_{1,1})$ and $\{b_{2,2}, b_{3,2}, b_{4,1}\} \subset N(b_{1,2})$. Since $G$ contains no adjacent degree 4 vertices, $b_{4,2}$ must be the remaining degree 4 vertex; however, it is not possible to complete the neighborhood of $b_{4,2}$ without creating a $C_3$ or $C_4$. Therefore, there does not exist a pair of degree 4 siblings in the star embedded in $G$.

Since the induced subgraph $H$ on the leaves of $S$ and $x_1$ has four vertices of degree 3, and 5 vertices of degree 2, then there must be at least one adjacent pair of degree 3 vertices in $H$; this implies that, in $G$, $x_1$ is adjacent to at least one of the degree 4 vertices that is a leaf of $S$. We now consider as separate cases whether $x_1$

is adjacent to one, two, or three degree 4 vertices amongst the leaves of $S$. In all three cases, we assume that the degree 4 vertices that are leaves in $S$ are $b_{1,1}$, $b_{2,1}$ and $b_{3,1}$.

1. *$x_1$ is adjacent to $b_{1,1}$, but no other degree 4 vertex in $G$.*
   We may conclude that $\{b_{1,2}, b_{3,2}, b_{4,1}\} \subset N(b_{2,1})$. Therefore, $N(b_{3,1})$ contains $b_{1,2}$, $b_{2,2}$ and $b_{4,2}$. $b_{1,1}$ must be adjacent to at least one degree 3 sibling of a degree 4 leaf; without loss of generality we may choose $b_{2,2}$. Thus, $N(x_1) = \{b_{1,1}, b_{3,2}, b_{4,2}\}$. The final edge must be $(b_{1,1}, b_{4,1})$. This graph, $G_{14b}$, is shown in Figure 10.
2. *$x_1$ is adjacent to $b_{1,1}$ and $b_{2,1}$ but not $b_{3,1}$.*
   Since $b_{3,1}$ must be adjacent to a degree 3 vertex in each of branches 1, 2, and 4, we may conclude that $\{b_{1,2}, b_{2,2}, b_{4,1}\} \subset N(b_{3,1})$. Since one of the elements of $U = \{b_{3,2}, b_{4,2}\}$ is in $N(b_{1,1})$, and the other element in $U$ is in $N(b_{2,1})$, and vertices $b_{1,1}$ and $b_{2,1}$ are indistinguishable with respect to the vertices in $U$, then we may conclude that $b_{4,2} \in N(b_{1,1})$ and $b_{3,2} \in N(b_{2,1})$. The final element of $N(b_{1,1})$ can only be $b_{2,2}$, $N(x_1)$ must contain $b_{4,1}$, and $b_{1,2}$ is in $N(b_{2,1})$. The final edge is $(b_{3,2}, b_{4,2})$. By Algorithm ISO, this graph is isomorphic to $G_{14b}$.
3. $N(x_1) = \{b_{1,1}, b_{2,1}, b_{3,1}\}$
   Each of the degree 4 leaves must have two neighbors amongst the five degree 3 leaves. By the pigeonhole principle, there must be a degree 3 leaf $s$ that is adjacent to two degree 4 leaves $t$ and $u$. Since vertices $s$, $t$, $u$, and $x_1$ are the corners of a quatrilateral, our assumption that $x_1$ is adjacent only to degree 4 vertices is false.

Case: $\delta = 3$ and $G$ contains one pair of adjacent degree 4 vertices.

In this case $G$ contains $S_{4,2}$ where branch 1 is augmented with a third leaf. Since $G$ does not contain a $P_3$ of degree 4 vertices, the leaves of branch 1 all have degree 3. Since $|N(b_{1,1}) \cup N(b_{1,2}) \cup N(b_{1,3}) - \{b_1\}| = 6$, then all leaves not in branch 1 are adjacent to leaves in branch 1. Thus, $N(b_{1,1}) \supset \{b_{2,1}, b_{3,1}\}$, $N(b_{1,2}) \supset \{b_{2,2}, b_{4,1}\}$ and $N(b_{1,3}) \supset \{b_{3,2}, b_{4,2}\}$. We may assume that $b_{2,1}$ is one of the two remaining degree 4 vertices, which implies that $b_{3,2}, b_{4,1} \in N(b_{2,1})$. Since $G$ contains only one pair of adjacent degree 4 vertices, the final degree 4 vertex is in $\{b_{2,2}, b_{3,1}, b_{4,2}\}$. We consider each of these cases in turn:

1. *$d(b_{2,2}) = 4$:* Thus, $b_{3,1}, b_{4,2} \in N(b_{2,2})$ and the non-adjacent degree 4 vertices are distance 2 apart. We designate this graph $G_{14c}$ and the induced subgraph on the leaves of the star is shown in Figure 10.
2. *$d(b_{3,1}) = 4$:* Thus, $b_{2,2}, b_{4,2} \in N(b_{3,1})$ and the non-adjacent degree 4 vertices are distance 2 apart. The resulting graph is isomorphic to $G_{14c}$.
3. *$d(b_{4,2}) = 4$:* Thus, $b_{2,2}, b_{3,1} \in N(b_{4,2})$ and the non-adjacent degree 4 vertices are distance 3 apart. This graph is depicted as $G_{14d}$ in Figure 10. By the distance between the non-adjacent degree 4 vertices, $G_{14d}$ is not

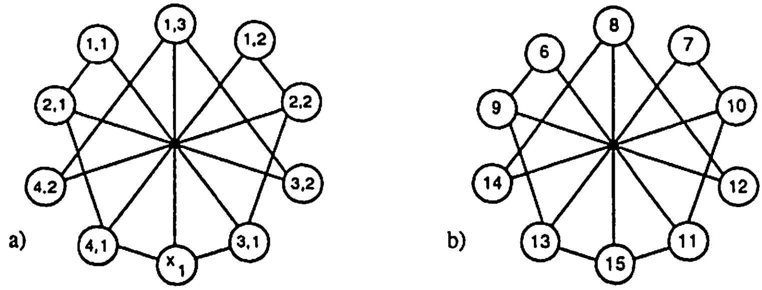



**Figure 11:**
The unique element of $\mathcal{G}_{15}$ and the relabeling of its vertices

isomorphic to $G_{14c}$.

Case: $\delta = 3$ and $G$ contains two independent pairs of adjacent degree 4 vertices.
As in the previous case, $G$ contains $S_{4,2}$, where branch 1 is augmented with a third leaf. Since $G$ does not contain a $P_3$ of degree 4 vertices, the leaves of branch 1 all have degree 3. Since $|N(b_{1,1}) \cup N(b_{1,2}) \cup N(b_{1,3}) - \{b_1\}| = 6$, then all leaves not in branch 1 are adjacent to leaves in branch 1. Thus, $N(b_{1,1}) \supset \{b_{2,1}, b_{3,1}\}$, $N(b_{1,2}) \supset \{b_{2,2}, b_{4,1}\}$ and $N(b_{1,3}) \supset \{b_{3,2}, b_{4,2}\}$. We may assume that $b_{2,1}$ is one of the two remaining degree 4 vertices, which implies that $b_{3,2}, b_{4,1} \in N(b_{2,1})$ and $d(b_{3,2}) = 4$ or $d(b_{4,1}) = 4$. Since it is not possible to add an edge incident on $b_{3,2}$ or $b_{4,1}$ without completing a triangle or quadrilateral, this case does not yield an element of $\mathcal{G}_{14}$.

**F(15) = 1**

Since $f(15) = 26$, then by Propositions 2.2 and 2.3, for all $G \in \mathcal{G}_{15}$, $\delta = 3$. Thus, all elements of $\mathcal{G}_{15}$ contain a cold element of $\mathcal{G}_{14}$. With respect to the symmetry of the graphs, $G_{14a}$ contains cold set $\{b_3, b_{2,1}, x\}$, $G_{14b}$ contains cold set $\{b_4, b_{2,2}, b_{3,2}\}$, and $G_{14c}$ contain cold set $\{b_{1,3}, b_{3,1}, b_{4,1}\}$. $G_{14d}$ is hot. Thus, the following three graphs cover $\mathcal{G}_{15}$:

$H_1$ where $V(H_1) = V(G_{14a}) + x_2$ and
$$E(H_1) = E(G_{14a}) \cup \{(x_2, b_3), (x_2, b_{2,1}), (x_2, x_1)\}$$
$H_2$ where $V(H_2) = V(G_{14b}) + x_2$ and
$$E(H_2) = E(G_{14b}) \cup \{(x_2, b_4), (x_2, b_{2,2}), (x_3, x_2)\}$$
$H_3$ where $V(H_3) = V(G_{14c}) + x_1$ and
$$E(H_3) = E(G_{14c}) \cup \{(x_1, b_{1,3}), (x_1, b_{3,1}), (x_1, b_{4,1})\}.$$

These three graphs are isomorphic to one another; thus, there is a unique extremal graph of order 15. Figure 11a shows $G_{15}$, using the labeling from $H_3$.

**F(16) = 22**

We derive most of the graphs in $\mathcal{G}_{16}$ by adding vertices and edges to elements of $\mathcal{G}_{14}$ and $\mathcal{G}_{15}$. To simplify this process, we relabel the vertices in the graphs in $\mathcal{G}_{14}$ and $\mathcal{G}_{15}$. We relabel $(r, b_1, b_2, \ldots, b_{1,1}, b_{1,2}, \ldots, b_{2,1}, b_{2,2}, \ldots, x_1, x_2, \ldots)$ a $(u_1, u_2, u_3, \ldots)$. When adding a fifteenth (or sixteenth) vertex to a graph, it will be labeled $u_{15}$ (or $u_{16}$). We leave out the $u$ in tables of vertex sets when its absence does not create an ambiguity. As an example, Figure 11b shows the relabeling of the vertices in Figure 11a.

We now derive the elements of $\mathcal{G}_{16}$. Since $f(16) = 28$, then for all $G$ in $\mathcal{G}_{16}$, $2 \leq \delta \leq 3$ by Propositions 2.2 and 2.3.

Case: $\delta = 2$.

Since $f(16) = f(15) + 2$, we cover the elements of $\mathcal{G}_{16}$ with a degree two vertex by adding a vertex $u_{16}$ to the unique extremal graph on 15 vertices, and adjoining it, in turn, to each cold set in $G_{15}$. We can partition the resulting graphs into those for which $\Delta = 4$ and those for which $\Delta = 5$. With respect to the symmetry of $G_{15}$ the cold sets which yield a graph for which $\Delta = 4$ are $\{3, 15\}$, $\{4, 7\}$, $\{4, 14\}$, and $\{6, 14\}$. By Algorithm ISO, the first three resulting graphs are non-isomorphic. We label these graphs $G_{16a}$, $G_{16b}$, and $G_{16c}$ respectively. The fourth graph is isomorphic to $G_{16b}$.

With respect to the symmetry of $G_{15}$, the cold sets which yield a graph for which $\Delta = 5$ are $\{1, 15\}$, $\{3, 8\}$, $\{4, 13\}$, and $\{9, 14\}$. We label these four graphs $G_{16d}$, $H_1$, $G_{16e}$ and $H_2$ respectively. Since $G_{16d}$ contains four cold sets, and $G_{16e}$ contains two cold sets, $G_{16d}$ and $G_{16e}$, are non-isomorphic. $H_1$ is isomorphic to $G_{16d}$ and $H_2$ is isomorphic to $G_{16e}$.

Next we consider the graphs $G$ in $\mathcal{G}_{16}$ for which $\delta = 3$. First we note that by Propositions 2.3 and 2.4, $4 \leq \Delta \leq 5$. We consider separately the cases where $G$ does not contain adjacent vertices of degree 3, and where $G$ does have such an adjacent pair.

Case: $\delta = 3$ and there are no adjacent degree 3 vertices.

If there exists $x \in V(G)$, $d(x) = 5$, then by Equation 3.1 $d_3 \geq 9$. Five of the degree 3 vertices must be in $N(x)$; otherwise the $(5, n)$-star, $S$, rooted at $x$ will have more than 16 vertices. Thus, since $S \cong S_{5,2}$ $|V(S)| = 16$, and the four vertices of degree 3 that are not in $N(x)$ are leaves of $S$. Thus $G$ contains adjacent degree 3 vertices; our assumption is contradicted. Therefore, $\Delta = 4$ and, by Equation 3.1, $d_3 = d_4 = 8$. Next we determine that every degree 4 vertex in $G$ has exactly three neighbors of degree 3.

1. If there exists $x \in V(G)$, $d(x) = 4$ and $N(x)$ contains no degree 3 vertices, then $x$ is the root of $S_{4,3}$ which has more than 16 vertices.
2. If there exists $x \in V(G)$, $d(x) = 4$ and $N(x)$ contains one or two vertices of degree 3, then since the 3-branches of star $S$ rooted at $x$ have leaves all of degree 4, by the pigeonhole principle there must be a 4-branch, $B$, with

three leaves of degree 3. The leaves of $B$ have six independent neighbors in $L = V(G)/(N(x) \cup B + x)$; $L$ is the set of leaves not in $B$ and the vertices in $G$ that are external to $S$. But since there are at most five vertices of degree 4 in $L$, at least one of the degree 3 leaves in $B$ must be adjacent to one of the degree 3 vertices in $L$, thus contraticting our assumption.

3. If there exists $x \in V(G)$, $d(x) = 4$ and $N(x)$ contains four vertices of degree 3, then since the 3-branches of star $S$ rooted at $x$ must each have two leaves of degree 4 (to avoid a leaf-parent pair of degree 3 vertices), then $S$ contains nine vertices of degree 4. But this contradicts our calculation that $G$ contains eight degree 4 vertices.

We conclude that if $\delta = 3$, and $G$ contains no adjacent degree 3 vertices, then each of the 8 degree 4 vertices is adjacent to exactly three vertices of degree 3. Next we construct the unique graph in $\mathcal{G}_{16}$ that meets these constraints.

We first note that the graph, $G$ contains $S \cong S_{4,2}$, where branch 1 is augmented with a third leaf. $G$ also contains two vertices, $x_1$ and $x_2$ that are external to $S$. The leaves in branches 2, 3, and 4 must all have degree 4 (to avoid a pair of adjacent degree 3 vertices). Since this accounts for all degree 4 vertices, then $x_1, x_2$ and the leaves in branch 1 must all have degree 3.

We now describe the edges amongst $x_1, x_2$, and the leaves of $S$. The leaves in branch 1 have a total of six independent degree 4 neighbors; therefore $N(b_{1,1}) \supset \{b_{2,1}, b_{3,1}\}$, $N(b_{1,2}) \supset \{b_{2,2}, b_{4,1}\}$, and $N(b_{1,3}) \supset \{b_{3,2}, b_{4,2}\}$. $N(x_1)$ contains three 4-leaves that are not siblings, and do not have a common neighbor amongst the leaves of branch 1; therefore, $N(x_1) = \{b_{2,1}, b_{3,2}, b_{4,1}\}$. Since each degree 4 vertex must have exactly three degree 3 neighbors, then $N(x_2) = \{b_{2,2}, b_{3,1}, b_{4,2}\}$. Finally, there are three remaining edges amongst the degree 4 leaves of $S$. The only cantidate for $N(b_{2,1})$ is $b_{4,2}$. Similarly, $b_{3,2} \in N(b_{2,2})$ and $b_{3,1} \in N(b_{4,1})$. We label this graph $G_{16_f}$.

Case: $\delta = 3$ and there is a pair of adjacent degree 3 vertices.

In this case, for all $G \in \mathcal{G}_{16}$ with $(x, y) \in E(G)$ such that $d(x) = d(y) = 3$, it is the case that $\langle V(G)/\{x, y\}\rangle \in \mathcal{G}_{14}$; this follows from the observation that $|E(\langle V(G)/\{x, y\}\rangle)| = f(16) - 5 = f(14)$. Thus, we use an approach similar to the use of cold sets. We cover the set of graphs in $\mathcal{G}_{16}$ with $\delta = 3$ and with adjacent degree 3 vertices with the set $A$, where $A$ is the set of all $G$ where, for $H \in \mathcal{G}_{14}$, and $S$ and $T$ independent distance 3 pairs of vertices in $V(H)$ such that $(s, t) \notin E(H)$ where $s \in S, t \in T$, $V(G) = V(H) \cup \{u_{15}, u_{16}\}$ and $E(G) = E(H) \cup \{(u_{15}, u_{16})\} \cup \{(u_{15}, s) | s \in S\} \cup \{(u_{16}, t) | t \in T\}$. We then checked the graphs in $A$ for isomorphism with Algorithm ISO.

The resulting graphs, $G_{16_g}$ through $G_{16_v}$ are listed in the following table where each entry specifies the element of $\mathcal{G}_{14}$ from which the graph is constructed, and the set of two pairs of vertices which correspond to the two neighborhood pairs $n_{15} = N(u_{15}) - u_{16}$ and $n_{16} = N(u_{16}) - u_{15}$. We omit from the table all graphs which have a degree 2 vertex, as those were covered by a previous case.

| graph | subgraph | $n_{15}, n_{16}$ | graph | subgraph | $n_{15}, n_{16}$ |
|---|---|---|---|---|---|
| $G_{16_g}$ | $G_{14a}$ | (3, 11), (13, 14) | $G_{16o}$ | $G_{14c}$ | (3, 8), (4, 13) |
| $G_{16_h}$ | $G_{14b}$ | (2, 13), (9, 11) | $G_{16p}$ | $G_{14a}$ | (2, 14), (4, 8) |
| $G_{16_i}$ | $G_{14c}$ | (3, 8), (11, 13) | $G_{16q}$ | $G_{14b}$ | (2, 11), (3, 13) |
| $G_{16_j}$ | $G_{14d}$ | (10, 12), (11, 13) | $G_{16r}$ | $G_{14a}$ | (2, 14), (8, 11) |
| $G_{16_k}$ | $G_{14b}$ | (2, 11), (5, 9) | $G_{16s}$ | $G_{14a}$ | (2, 14), (3, 11) |
| $G_{16_l}$ | $G_{14b}$ | (3, 13), (4, 12) | $G_{16t}$ | $G_{14b}$ | (3, 13), (4, 6) |
| $G_{16_m}$ | $G_{14c}$ | (6, 14), (7, 12) | $G_{16u}$ | $G_{14a}$ | (1, 14), (8, 11) |
| $G_{16_n}$ | $G_{14b}$ | (2, 13), (4, 12) | $G_{16v}$ | $G_{14b}$ | (1, 14), (10, 12) |

**F(17)=14**

$f(17) = 31 = f(16) + 3$, and for all $G \in \mathcal{G}_{17}$, $\delta = 3$ by Propositions 2.2 and 2.3. Thus, we cover $\mathcal{G}_{17}$ with the set of all $G$ where, for $H \in \mathcal{G}_{16}$ and cold set $S \subset V(H)$, $V(G) = V(H) + u_{17}$ and $E(G) = E(H) \cup \{(u_{17}, s) \mid s \in S\}$. Again, we trim the set of resulting graphs with Algorithm ISO. The table below lists each element of $\mathcal{G}_{17}$, the cold graph $H \in \mathcal{G}_{16}$ from which it was constructed, ant the cold set in $H$ which forms $N(u_{17})$.

| graph | subgraph | $N(u_{17})$ | graph | subgraph | $N(u_{17})$ |
|---|---|---|---|---|---|
| $G_{17a}$ | $G_{16a}$ | (4, 7, 16) | $G_{17h}$ | $G_{16f}$ | (1, 15, 16) |
| $G_{17b}$ | $G_{16a}$ | (4, 14, 16) | $G_{17i}$ | $G_{16h}$ | (4, 12, 15) |
| $G_{17c}$ | $G_{16b}$ | (3, 8, 16) | $G_{17j}$ | $G_{16h}$ | (5, 7, 16) |
| $G_{17d}$ | $G_{16b}$ | (5, 6, 16) | $G_{17k}$ | $G_{16k}$ | (3, 13, 15) |
| $G_{17e}$ | $G_{16b}$ | (6, 14, 16) | $G_{17l}$ | $G_{16o}$ | (5, 6, 15) |
| $G_{17f}$ | $G_{16b}$ | (9, 14, 16) | $G_{17m}$ | $G_{16o}$ | (5, 11, 15) |
| $G_{17g}$ | $G_{16d}$ | (9, 14, 16) | $G_{17n}$ | $G_{16r}$ | (10, 13, 15) |

**F(18)=15**

This case is similar to the previous; since $f(18) = 34 = f(17) + 3$, and for all $G \in \mathcal{G}_{18}$, $\delta = 3$, we construct the elements of $\mathcal{G}_{18}$ from the cold sets in elements of $\mathcal{G}_{17}$. The elements of $\mathcal{G}_{18}$ are listed in the following table.

| graph | subgraph | $N(u_{18})$ | graph | subgraph | $N(u_{18})$ |
|---|---|---|---|---|---|
| $G_{18a}$ | $G_{17a}$ | (5, 6, 16) | $G_{18i}$ | $G_{17c}$ | (5, 11, 17) |
| $G_{18b}$ | $G_{17a}$ | (5, 6, 17) | $G_{18j}$ | $G_{17d}$ | (9, 14, 16) |
| $G_{18c}$ | $G_{17a}$ | (5, 12, 16) | $G_{18k}$ | $G_{17f}$ | (1, 15, 17) |
| $G_{18d}$ | $G_{17a}$ | (6, 14, 17) | $G_{18l}$ | $G_{17g}$ | (10, 12, 16) |
| $G_{18e}$ | $G_{17a}$ | (9, 14, 17) | $G_{18m}$ | $G_{17i}$ | (3, 14, 17) |
| $G_{18f}$ | $G_{17b}$ | (5, 12, 16) | $G_{18n}$ | $G_{17i}$ | (5, 7, 16) |
| $G_{18g}$ | $G_{17c}$ | (5, 6, 16) | $G_{18o}$ | $G_{17k}$ | (4, 12, 17) |
| $G_{18h}$ | $G_{17c}$ | (5, 6, 17) | | | |

**F(19)=1**

The authors of [4] described the unique graphs in $\mathcal{G}_{19}$ and $\mathcal{G}_{20}$. Their uniqueness derives from the observation that all $G \in \mathcal{G}_{19}$ are 4-regular; this follows from Propositions 2.2, 2.3, and 2.4. The $(4,5)$-cage, which was discovered by Robertson [7], has 19 vertices. Thus, every element of $\mathcal{G}_{19}$ is a $(4,5)$-cage. Since the Robertson graph is the unique $(4,5)$-cage, $F(19) = 1$. The Robertson graph is shown as $G_{19}$ in Figure 12.

**F(20) = 1**

For all $G \in \mathcal{G}_{20}$, $3 \le \delta \le 4$ by Propositions 2.2 and 2.3. Since $\Delta \ge 5$ by Proposition 2.3, then by Proposition 2.4, $\delta < 4$; therefore, $\delta = 3$. Since $f(20) = f(19) + 3$, $G_{19}$ is cold, and its unique cold set is the set of three vertices in it outer ring. Thus, $G_{20}$ in Figure 12 is the unique element of $\mathcal{G}_{20}$.

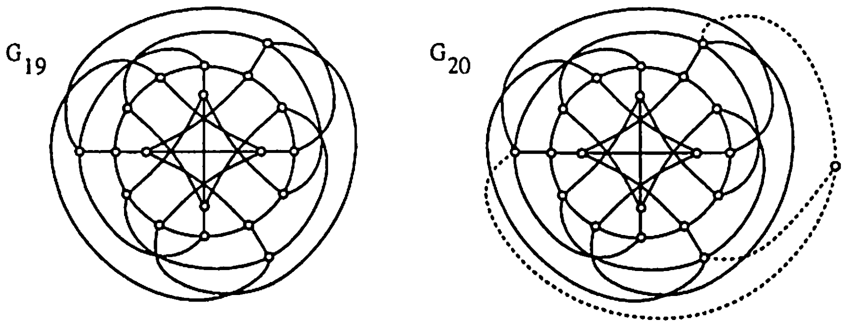



**Figure 12:** The unique extremal graphs of orders 19 and 20.

**F(21) = 3**

Since $f(21) = 44$, then for all $G \in \mathcal{G}_{21}$, $3 \le \delta \le 4$ by Propositions 2.2 and 2.3. We consider the two cases based on the value of $\delta$.

Case: $\delta = 3$

Since for all $x \in V(G)$ where $d(x) = 3$, $|E(G - x)| = f(20)$, then $G - x$ is $G_{20}$, and $N(x)$ contains the vertex in $G_{20}$ that is external to the Robertson graph, and two non-adjacent vertices in the innermost ring of four vertices in the Robertson graph. This graph, $G_{21a}$, is the unique graph in $\mathcal{G}_{21}$ for which $\delta = 3$.

Case: $\delta = 4$

By Propositions 2.3 and 2.4, $\Delta = 5$. Thus, by Equation 3.1, $d_4 = 17$ and $d_5 = 4$. We note that $G$ does not contain an adjacent pair of degree 5 vertices, for then $G$ would contain an augmented $S_{5,3}$ with more than 21 vertices. We also note that, by the pigeonhole principle, for all $G \in \mathcal{G}_{21}$ with $\delta = 4$ that there is at least one degree 4 vertex with at least two degree 5 neighbors. We now consider the cases based on the maximum number of vertices of degree 5 that are adjacent to any vertex of degree 4.

1. There exists $x \in V(G), d(x) = 4$ such that $x$ has four neighbors of degree 5.
   Let $x$ be the root of $S \cong S_{4,4}$; note that $|V(S)| = 21$. $S$ has four branches, each with four leaves, and each leaf in $S$ is adjacent to three other leaves. Thus, each leaf in branch $i$ is adjacent to one leaf in each of the branches other than $i$. Without loss of generality, we may assume that $N(b_{1,1}) \supset \{b_{2,1}, b_{3,1}, b_{4,1}\}, N(b_{1,2}) \supset \{b_{2,2}, b_{3,2}, b_{4,2}\}, N(b_{1,3}) \supset \{b_{2,3}, b_{3,3}, b_{4,3}\}$, $N(b_{1,4}) \supset \{b_{2,4}, b_{3,4}, b_{4,4}\}$, and $N(b_{2,1}) \supset \{b_{3,2}, b_{4,3}\}$. This leads to the following sequence of deductions: $b_{3,4} \in N(b_{4,3})$, $b_{2,2} \in N(b_{3,4})$, $b_{4,1} \in N(b_{2,2})$, $b_{4,4} \in N(b_{3,2})$, $b_{3,3} \in N(b_{4,1})$, $b_{2,4} \in N(b_{3,3})$, $b_{4,2} \in N(b_{2,4})$, $b_{3,1} \in N(b_{4,2})$, and $N(b_{2,3}) \supset \{b_{3,1}, b_{4,4}\}$. The graph is $G_{21b}$.
2. There exists $x \in V(G), d(x) = 4$ such that $x$ has three neighbors of degree 5, and no vertex has more than three degree five neighbors.
   We let $x$ be vertex $b_1$ in $S \cong S_{5,3}$ rooted at one of the degree 5 vertices in $N(x)$. Thus, $S$ has leaves $b_{i,j}$, $1 \leq i \leq 5$, $1 \leq j \leq 3$ where $d(b_{1,1}) = d(b_{1,2}) = 5$; we may assume that $d(b_{2,3}) = 5$. All other leaves in $S$ have degree 4 in $G$. Since $G$ does not contain adjacent degree 5 vertices, we may assume without loss of generality that $N(b_{1,1}) \supset \{b_{2,1}, b_{3,1}, b_{4,1}, b_{5,1}\}$, $N(b_{1,2}) \supset \{b_{2,2}, b_{3,2}, b_{4,2}, b_{5,2}\}$, and $N(b_{2,3}) \supset \{b_{1,3}, b_{3,3}\}$. Therefore, $N(b_{1,3}) \supset \{b_{4,3}, b_{5,3}\}$. Again without loss of generality we may assume that $N(b_{2,3}) \supset \{b_{4,1}, b_{5,2}\}$. This leads to the following sequence of deductions: $b_{3,2} \in N(b_{4,1})$, $b_{5,3} \in N(b_{3,2})$, $b_{3,1} \in N(b_{5,2})$, $b_{2,1} \in N(b_{5,3})$, $b_{4,3} \in N(b_{3,1})$, $b_{2,2} \in N(b_{4,3})$, $b_{4,2} \in N(b_{2,1})$, $b_{5,1} \in N(b_{2,2})$, and $N(b_{3,3}) \supset \{b_{4,2}, b_{5,1}\}$. This graph is $G_{21c}$.
3. Every degree 4 vertex has at most two degree 5 neighbors.
   We first note that $G$ contains $S \cong S_{5,3}$ where the three leaves of degree 5 are in distinct branches; we assume the degree 5 leaves are $b_{1,1}, b_{2,1}$, and $b_{3,1}$. We also note that since each degree 5 vertex is the center of an $S_{5,3}$ with three degree 5 leaves, then the degree 5 vertices in $G$ are mutually distance 2 apart.

   We may assume that $N(b_{1,1}) \supset \{b_{2,2}, b_{3,2}, b_{4,1}, b_{5,1}\}$ and $N(b_{2,1}) \supset \{b_{1,2}, b_{4,2}\}$. Since the degree 5 leaves are mutually distance 2 apart, $b_{2,1}$ is adjacent to either $b_{3,2}$ or $b_{5,1}$.

   First, we assume that $b_{2,1}$, is adjacent to $b_{3,2}$. Thus, $b_{5,2}$ is adjacent to $b_{2,1}$, and without loss of generality $b_{4,3}$ is adjacent to $b_{1,2}$. Either $b_{1,2}$ or $b_{1,3}$ is adjacent to $b_{3,1}$. If $b_{1,2}$ were adjacent to $b_{3,1}$, then $b_{4,1}, b_{5,3} \in N(b_{3,1})$; however, then it would not be possible to complete the neighborhood of $b_{3,2}$ without completing a triangle or quatrilateral. But if $b_{1,3}$ were adjacent to $b_{3,1}$, then $b_{5,3} \in N(b_{3,2})$, $b_{3,3} \in N(b_{1,2})$, $b_{4,3} \in N(b_{2,2})$, $b_{3,1} \in$

$N(b_{2,2})$, and $b_{5,3} \in N(b_{4,3})$; however, then it would not be possible to complete the neighborhood of $b_{3,1}$. Thus, $b_{2,1}$ is not adjacent to $b_{3,2}$.

Since $b_{2,1}$ is adjacent to $b_{5,1}, b_{3,3} \in N(b_{2,1})$. Since no degree 4 vertex is adjacent to three vertices of degree 5, $b_{5,2} \in N(b_{3,1})$. Therefore, $b_{4,3} \in N(b_{5,1})$. Since $b_{1,2}$ cannot have a neighbor in branch 4, it must have one neighbor in each of branches 3 and 5. Thus, $b_{3,1} \in N(b_{1,2})$, and it follows that $b_{5,3} \in N(b_{1,2})$. Similarly, $b_{2,2}$ cannot have a neighbor in branch 4, and so $b_{2,2}$ must have one neighbor in each of branches 3 and 5; however, no such pair of vertices can be added to $N(b_{2,2})$ without creating a triangle or quatrilateral in $G$. Thus, there is no graph in $\mathcal{G}_{21}$ satisfying this case. ∎

This completes Theorem 3.1. We have found three elements of $\mathcal{G}_{22}$: two using cold graph in $\mathcal{G}_{21}$, and one using the hill-tracking algorithm described in [4]. Similarly, we have found five elements of $\mathcal{G}_{23}$ using cold graphs, and one additional element using hill-tracking. Thus, $F(22) \geq 3$ and $F(23) \geq 6$. It is straightforward to show that there is a unique degree sequence for the elements of $\mathcal{G}_{24}$, and we have only found a single extremal graph with that degree sequence. We conjecture that $F(24) = 1$.

## 4. Values of $f(v)$ for $v$ from 25 to 27

Using the elements in $\mathcal{G}_{21}$ we are able to determine the value of $f(25)$. This allows us to derive $f(26)$, which in turn allows us to derive $f(27)$.

**Theorem 4.1.** $f(25) = 57, f(26) = 61,$ *and* $f(27) = 65$.

Proof: We proceed case by case based on the values of $v$. For each case we find a $\{C_3, C_4\}$-free graph with $v$ vertices and $e = f(v)$ edges, and show by contradiction that there cannot exist a $\{C_3, C_4\}$-free graph with $v$ vertices and $e + 1$ edges.

**f(25) = 57**

The elements of $\mathcal{G}_{25}$ listed in [5] show that $f(25) \geq 57$. We now assume that there exists $\{C_3, C_4\}$-free $G$ where $v = 25$ and $e = 58$. Propositions 2.2 and 2.3 imply that $\delta = 4$. Propositions 2.3 and 2.4 imply that $5 \leq \Delta \leq 6$.

By Proposition 2.2, $G$ cannot contain an adjacent pair of degree 4 vertices. If there exists $x \in V(G)$ such that $d(x) = 6$, then all neighbors of $x$ have degree 4; otherwise $G$ contains a star with more than 25 vertices. Thus, the presence of $x$, $d(x) = 6$ implies that $G$ contains $S \cong S_{6,3}$. Since $v = 25$, $e = 58$, and $(\delta, \Delta) = (4, 6)$, then Equation 3.1 implies that $d_4 \geq 10$; thus, at least four of the leaves of $S$ must also be of degree 4 in $G$. But then $G$ will contain adjacent vertices of degree 4, which contradicts our above statement. Therefore $G$ contains no vertices of degree 6, and $\Delta = 5$. By Equation 3.1, $d_4 = 9$ and $d_5 = 16$.

We now show that every vertex of degree 5 must have either 3 or 4 neighbors of degree 5. If there exists $x \in V(G)$ such that $d(x) = 5$ and all vertices in $N(x)$

have degree 5, then $x$ is the root of $S_{5,4}$. But since $|V(S_{5,4})| = 26$, then every degree 5 vertex in $G$ has at least one degree 4 neighbor. Now let us assume that there is $x \in V(G)$ where $d(x) = 5$ and $y_1, y_2, y_3 \in N(x)$ such that $d(y_1) = d(y_2) = d(y_3) = 4$. If $H = \langle V(G)/\{x, y_1, y_2, y_3\}\rangle$, then $H \in \mathcal{G}_{21}$ and each $y_i$, $1 \leq i \leq 3$ is adjacent to an independent set of three mutually distance 3 vertices in $H$. However, no element of $\mathcal{G}_{21}$ contains 3 independent sets of mutually distance 3 vertices. Therefore, no degree 5 vertex in $G$ has three degree four neighbors. Therefore, every vertex of degree 5 has either 3 or 4 neighbors of degree 5.

Thus, since $5d_5 = 80, 4d_4 = 36$, and because there are no adjacent vertices of degree 4, then there are $(80 - 36)/2 = 22$ edges in the induced subgraph on the set of vertices of degree 5. If $s$ is the number of degree 5 vertices with 3 degree 5 neighbors, and $t$ is the number of degree 5 vertices with 4 degree 5 neighbors, then $s + t = 16$ and $3s + 4t = 44$. The solution to these equations requires that $d_5 = 20$, which contradicts our earlier calculation of $d_6 = 16$. Thus, there is no $\{C_3, C_4\}$-free graph $G$ with $v = 25$ and $e = 58$. Therefore, $f(25) = 57$.

f(26) = 61

By construction, $f(26) \geq 61$ (see [5]). Assume that there exists $\{C_3, C_4\}$-free $G$ with $v = 26$ and $e = 62$. By Propositions 2.2 and 2.3, $\delta \geq 5$ and $\Delta \geq 5$. With Proposition 2.4 this implies that $G$ is 5-regular. But if $G$ is 5-regular, then $e = 65$, which contradicts our assumption that $e = 62$. Therefore, $f(26) = 61$.

f(27) = 65

Graph $G_{27}$ in [5] shows that $f(27) \geq 65$. If we assume there exists $\{C_3, C_4\}$-free $G$ with $v = 27$ and $e = 66$, then by Propositions 2.2 and 2.3 $\delta \geq 5$ and $\Delta \geq 6$. But by Proposition 2.4, this implies that $v > 27$, which is a contradiction. Therefore, $f(27) = 65$. ∎

Using cold sets and the hill-tracking algorithm from [4], we have found four elements of $\mathcal{G}_{25}$, two elements of $\mathcal{G}_{26}$, and one element of $\mathcal{G}_{27}$.

## 5. Concluding remarks

We summarize in the following theorem the principal result in this paper concerning the girth of extremal graphs.

**Theorem 5.1.** *For all extremal graphs $G$ of order $v$:*

1. *If $v \geq 5$ then if $x \in V(G)$ such that $d(x) = 2$, then $x$ is in a 5-cycle or 6-cycle in $G$;*
2. *If $v \geq 5$ then if $x \in V(G)$ such that $d(x) \geq 3$, then $x$ is in a 5-cycle in $G$;*
3. *If $v \geq 7$, then $g(G) = 5$.*

We now combine the results from this paper, concerning the known values of $f(v)$ and $F(v)$, with those from [4].

**Theorem 5.2.** *$f(v)$ and $F(v)$ have the following values (where the underlined values are lower bounds):*

| $v$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(v)$ | 0 | 1 | 2 | 3 | 5 | 6 | 8 | 10 | 12 | 15 | 16 | 18 | 21 | 23 | 26 |
| $F(v)$ | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 1 | 1 | 1 | 3 | 7 | 1 | 4 | 1 |

| $v$ | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 50 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f(v)$ | 28 | 31 | 34 | 38 | 41 | 44 | 47 | 50 | 54 | 57 | 61 | 65 | 175 |
| $F(v)$ | 22 | 14 | 15 | 1 | 1 | 3 | $\underline{3}$ | $\underline{6}$ | $\underline{1}$ | $\underline{4}$ | $\underline{2}$ | $\underline{1}$ | 1 |

Several questions arose while generating the elements of the sets $\mathcal{G}_v$. We pose these questions here to stimulate further research into the nature of extremal graphs.

1. We note that for several values of $v$, $F(v) = 1$; we ask if there are infinite values of $v$ for which this is true. More generally, what are the conditions under which $F(v) = 1$?
2. For $1 < v \leq 27$, there exists an extremal graph $G$ on $v$ vertices such that there is a vertex $x$ with degree $f(v) - f(v-1)$; thus, $G - x$ is an extremal graph on $v - 1$ vertices. This leads us to ask if for all $v > 1$ there exists extremal graph $G$ where $\delta(G) = f(v) - f(v-1)$.
3. Also, we note that for all extremal graphs on $1 < v \leq 21$ vertices (as well as all that we have generated in $\mathcal{G}_v$, $22 \leq v \leq 27$) the minimum degree is $f(v) - f(v-1)$ or $f(v) - f(v-1) + 1$. Is this true for all $v > 1$?
4. We showed that if $G$ is extremal with $v \geq 7$, then the girth of $G$ is 5. This leads us to ask whether for all $\{C_3, C_4, \ldots, C_n\}$-free $G$ with maximal size and $v > cn$, for some constant $c$, it is the case that $g(G) = n + 1$.

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