Generalized Quadrangles Derived from Groups and Having $s = t + 2$

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Abstract. The known generalized quadragles with parameters (s,t) where $|s-t| = 2$ have been characterized in several ways by M. De Soete [D], M. De Soete and J. A. Thas [DT1], [DT2], [DT4], and the present author [P]. Certain of these results are interpreted for a coset geometry construction.

I. Introduction

Let G be a group of order s^3, $s > 1$. Suppose $\mathcal{F}^+ = \{A_*, A_\infty, A_1, \ldots, A_s\}$ is a family of $s+2$ subgroups of G, each of order s. Further, suppose $A_iA_j \cap A_k = \{e\}$ for distinct i, j, k in $I^+ = \{*, \infty, 1, \ldots, s\}$. Then $\mathcal{F}^+$ is called a *4-gonal partition* of G. Recall 10.2.1 of [PT].

I.1

Let $\mathcal{F}^+$ be a 4-gonal partition of G with notation as above.

(i) A generalized quadrangle (GQ) $S^+ = S(\mathcal{G}, \mathcal{F}^+)$ of order $(s-1, s+1)$ is constructed as follows: the points of S^+ are the elements of G; the lines of S are the right cosets of members of $\mathcal{F}^+$; incidence is containment.

(ii) If $A_* \lhd G$, then $\mathcal{F} = \mathcal{F}^+ \setminus \{A_*\}$ is a 4-gonal family for G with $A_i^* = A_*A_i$, $i \in I^\infty = \{\infty, 1, \ldots, s\}$. So there is a GQ $S = S(G, \mathcal{F})$ of order s, and $S(G, \mathcal{F}^+)$ is the GQ $\mathcal{P}(S(G, \mathcal{F}), (\infty))$ obtained by expanding $S(G, \mathcal{F})$ about the regular point (∞). (In this case the construction is well known, so we ask the reader to see [PT] for terminology, notation and construction. Expansion about a regular point is given in 3.1.4 of [PT].)

(iii) If two members of $\mathcal{F}^+$ are normal in G, then G is elementary abelian and $s = 2^e$.

In case (iii) of 1.1, we can also construct a GQ, this time of order $(s+1, s-1)$. However, here we want to give a construction of a GQ of order $(s+1, s-1)$ that includes both the case $s = 2^e$ and the case $s = p^e$, p an odd prime. So let G be elementary abelian of order $s^3 > 1$ with 4-gonal family $\mathcal{F} = \{A_\infty, A_1, \ldots, A_s\}$. This means that for each $i \in I^\infty$, A_i is a subgroup of G having order s, and $A_i \leq A_i^*$ where A_i^* is a subgroup (called the *tangent space of $\mathcal{F}$ at A_i*) of order s^2. Also the usual properties of W. M. Kantor (cf. [PT]) are satisfied:

K1. $A_iA_j \cap A_k = \{e\}$ for distinct $i, j, k \in I^\infty$.
K2. $A_i^* \cap A_j = \{e\}$ for distinct $i, j \in I^\infty$.

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The corresponding GQ $S(G,\mathcal{F})$ is a *translation* GQ (TGQ) (see especially chapters 8 and 9 of [PT]) . When $s = 2^e$, there is a group A_* for which $\mathcal{F}^+ = \mathcal{F} \cup \{A_*\}$ is a 4-gonal partition, and $A_i^* = A_*A_i$. For odd s there is no such A_*, but we can put $\mathcal{F}^- = \{A_1, \ldots, A_s\}$ and describe a GQ $S^- = S(G,\mathcal{F}^-)$ of order $(s+1, s-1)$ in either case. The idea is that in $S(G,\mathcal{F})$, in the usual notation, we can expand about the regular line $[A_\infty]$. The resulting GQ is described as follows.

Lines of $S(G,\mathcal{F}^-)$ are the cosets A_jg: $g \in G$, $1 \leq j \leq s$. Points of $S(G,\mathcal{F}^-)$ are of three types: (i) elements $g \in G$; (ii) cosets A_j^*g: $g \in G$, $1 \leq j \leq s$ (here $A_j^*g = A_*A_jg$ if $s = 2^e$); (iii) cosets $A_\infty A_jg$: $g \in G$, $1 \leq j \leq s$. Incidence is containment.

An *ovoid* of $S^- = S(G,\mathcal{F}^-)$ is a set $\mathcal{O}$ of s^2 points of S^-, no two collinear (i.e., each line of S is incident with a unique point of $\mathcal{O}$). Let $\mathcal{O}_*$ be the set of all points of S^- of type (ii):

$$\mathcal{O}_* = \{A_j^*g \colon g \in G, 1 \leq j \leq s\}.$$

We claim $\mathcal{O}_*$ is an ovoid. Clearly $|\mathcal{O}_*| = s^2$. The line A_jg is incident with the point A_i^*h if and only if $A_jg \subseteq A_i^*h$ if and only if $A_j \subseteq A_i^*hg^{-1}$ if and only if $j = i$ and $hg^{-1} \in A_i^*$. This says that the unique point of type (ii) on A_jg is the point A_j^*g, thus proving that $\mathcal{O}_*$ is an ovoid.

Now let $\mathcal{O}_\infty$ be the set of all points of type (iii).

$$\mathcal{O}_\infty = \{A_\infty A_jg \colon g \in G, 1 \leq i \leq s\}.$$

Replacing A_j^*h in the preceding argument with $A_\infty A_jh$ we see also that $\mathcal{O}_\infty$ is an ovoid.

For each $g \in G$, consider the set A_∞^*g as a set of s^2 points of type (i). Suppose x and y are points of A_∞^*g collinear on some line A_jh. So $x = a_jh = a_\infty^*g$ and $y = b_jh = b_\infty^*g$, with $a_j, b_j \infty A_j$; $a_\infty^*, b_\infty^* \infty A_\infty^*$. Then $a_j^{-1}a_\infty^* = hg^{-1} = b_j^{-1}b_\infty^*$, implying $b_ja_j^{-1} = b_\infty^*(a_\infty^*)^{-1} \in A_j \cap A_\infty^* = \{e\}$. Hence $x = y$, implying that each coset of A_∞^* is an ovoid. Hence we have the following.

I.2

$\mathcal{O}_\infty + \mathcal{O}_* + \{A_\infty^*g \colon g \in G\}$ partitions the pointset of $S^- = S(G,\mathcal{F}^*)$ into ovoids.

The fact that I.2 holds suggests that we should interpret the results of [P] and [D] in the present context to obtain conditions on $\mathcal{F}^-$ (or on $\mathcal{F}$) that characterize the known constructions. In fact, this is the raison d'etre of this essay. So we next recall the known constructions.

Let $\pi = PG(2,q)$ be embedded as a hyperplane of $PG(3,q)$, and let Ω^- be a q-arc of π. For q odd, there is a unique point a of π for which $\Omega = \Omega^- \cup \{a\}$ is an oval (indeed, a conic) of π. For q even, there are points a, b of π for which

$\Omega^+ = \Omega^- \cup \{a, b\}$ is a hyperoval. For $q \geq 4$, Ω^+ is the unique extension of Ω^- to a hyperoval. In all cases we assume that a suitable point a is chosen so that $\Omega = \Omega^- \cup \{a\}$ is an oval. And when q is even, we assume that a suitable point b is chosen so that $\Omega^+ = \Omega \cup \{b\}$ is a hyperoval. (See J. A. Thas [T] for an excellent study of the extension of q-arcs to ovals.)

Now a GQ $S(\Omega^-) = (P^-, B^-, I^-)$ of order $(q+1, q-1)$ is constructed as follows. The pointset P^- is the union $P^- = P \cup \mathcal{O}_\infty \cup \mathcal{O}_*$, where P is the set of points of $PG(3,q)\backslash\pi$, $\mathcal{O}_\infty$ is the set of planes meeting π in a line secant to $\mathcal{O}$ and containing the point a, and $\mathcal{O}_*$ is the set of planes meeting π at a line tangent to Ω at a point of Ω^-. (When q is even, $\mathcal{O}_\infty$ is the set of planes of $PG(3,q)$ containing a but not b, and $\mathcal{O}_*$ is the set of planes of $PG(3,q)$ containing b but not a.) B^- is the set of lines of $PG(3,q)$ not in π but meeting π at a point of Ω^-. The incidence relation I^- is the natural one induced by incidence in $PG(3,q)$.

When q is odd, a famous result of B. Segre guarantees that Ω is a conic. And for q even or odd, if Ω is a conic, then $S(\Omega^-)$ is isomorphic to the GQ $P(Q(4,q), L)$ obtained by expanding $Q(4,q)$ about any (necessarily regular) line L of $Q(4,q)$. If Ω is any oval (q odd or even), the corresponding GQ of order (q,q) is denoted $T_2(\Omega)$. And with the notation used above, $S(\Omega^-) \cong P(T_2(\Omega), L)$. This makes sense, because the point a of Ω plays the role of a regular line in $T_2(\Omega)$. For more details, see [PT] and also [P].

The characterizations of $S(\Omega^-)$ given in [P] and [D] will be interpreted for $S(G, \mathcal{F}^-)$. These characterizations are recalled in Section II and are applied to $S(G, \mathcal{F}^-)$ in the following sections.

II. The Results of De Soete and of Payne

Let $S = (P, B, I)$ be a GQ of order $(q+1, q-1)$ having a *normal* ovoid $\mathcal{O}_\infty$, i.e., $\mathcal{O}_\infty$ is an ovoid such that each pair (x, y) of distinct points of $\mathcal{O}_\infty$ is a regular pair and $\{x,y\}^{\perp\perp} \subseteq \mathcal{O}_\infty$. Then construct an affine plane $\mathcal{A}_\infty$ of order q as follows. Let P_∞ be the set of points of $\mathcal{O}_\infty$; $B_\infty = \{\{x,y\}^{\perp\perp} : x, y \in \mathcal{O}_\infty, x \neq y\}$; I_∞ is the natural incidence relation. Then the structure $\mathcal{A}_\infty = (P_\infty, B_\infty, I_\infty)$ is an affine plane of order q. Denote by $\mathcal{O}$ the union of the perps of elements of a fixed but arbitrary parallel class of lines of $\mathcal{A}_\infty$. Clearly $\mathcal{O}$ is again an ovoid of S.

Let $\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q$ be the ovoids obtained from $\mathcal{O}_\infty$ in this way using the $q+1$ parallel classes of $\mathcal{A}_\infty$. Then $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ is a partition of P into a family $\mathcal{M}$ of ovoids for which $\mathcal{O}_\infty$ is *pivotal*, i.e., in addition to having $\mathcal{O}_\infty$ normal, for each pair (x, y) of noncollinear points of $\mathcal{O}_\infty$, there is some i, $0 \leq i \leq q$, for which $\{x,y\}^\perp \subseteq \mathcal{O}_i$. Because $\mathcal{O}_\infty$ is pivotal for $\mathcal{M}$, a GQ $S_\infty = (P^\infty, B^\infty, I^\infty)$ of order (q,q) may be constructed as follows. $P^\infty = (P\backslash\mathcal{O}_\infty) \cup \{(\mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$; $B^\infty = B \cup \{\{x,y\}^\perp : x, y \in \mathcal{O}_\infty, x \neq y\} \cup \{L_\infty\}$. Then I^∞ is defined in a rather natural way: $L_\infty I^\infty \mathcal{O}_i, 0 \leq i \leq q$. If $M = \{x,y\}^\perp$, $x, y \in \mathcal{O}_\infty$, $x \neq y$, and if $z \in P^\infty$, then $z I^\infty M$ if and only if $z \in M$ or $M \subseteq z$. If $M \in B$ and $x \in P^\infty$, then $x I^\infty M$ if and only if $x I M$. It is easy to check that L_∞

is regular as a line of $\mathcal{S}_\infty$. Moreover, the point $\mathcal{O}_i$ of $\mathcal{S}_\infty$, $0 \le i \le q$, is regular in $\mathcal{S}_\infty$ if and only if $\mathcal{O}_i$ is pivotal for $\mathcal{M}$.

Note that since $\mathcal{S} = (P, B, I)$ is a GQ of order $(q+1, q-1)$ whose pointset P is partitioned into a family $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ of ovoids for which $\mathcal{O}_\infty$ is pivotal, the axioms A1 and A2 of [P] are automatically satisfied.

By combining parts of IV.1 and IV.2 of [P], we have the following:

II.1

Let $\mathcal{M}' = \mathcal{M} \backslash \{\mathcal{O}_\infty\}$

(i) Let $b, d \in \mathcal{O} \in \mathcal{M}'$, $b \ne d$. If $\{b,d\}^\perp \cap \mathcal{O}_\infty \ne \emptyset$, then $\{b,d\}^\perp \subseteq \mathcal{O}_\infty$.

(ii) Each $\mathcal{O}$ in $\mathcal{M}'$ is partitioned into spans of pairs of points whose perps partition $\mathcal{O}_\infty$.

(iii) $x \in \mathcal{O}_i$ and $y \in \mathcal{O}_j$, $0 \le i < j \le q$, $x \not\sim y$, then $|\{x,y\}^\perp \cap \mathcal{O}_\infty| = 1$.

(iv) If $a \in \mathcal{O}_\infty$, $x \in \mathcal{O}_i$, $x \not\sim a$, and $0 \le i, j \le q$, $i \ne j$, then $|\{a,x\}^\perp \cap \mathcal{O}_j| = 1$.

Let $\mathcal{O}$ be any fixed ovoid in $\mathcal{M}'$. For $a_1, a_2 \in \mathcal{O}_\infty$, put $a_1 \equiv a_2$ if and only if $a_1^\perp \cap \mathcal{O} = a_2^\perp \cap \mathcal{O}$. Clearly "$\equiv$" is an equivalence relation on $\mathcal{O}_\infty$. For $a \in \mathcal{O}_\infty$, let $[a]$ denote the equivalence class of a with respect to this relation. And note that $[a_1] = [a_2]$ if and only if $a_1^\perp \cap \mathcal{O} \cap a_2^\perp \ne \emptyset$.

Similarly, for $b \in \mathcal{O}$, put $[b] = \{b_1 \in \mathcal{O} : b^\perp \cap \mathcal{O}_\infty \cap b_1^\perp \ne \emptyset\}$. For $a \in \mathcal{O}_\infty$, let $[a]^\perp = [b]$. Then the q^2 lines joining points of $[a]$ with points of $[b]$ form a set of lines called a *quiver*. Two lines of a quiver are concurrent if and only if they meet at a point of $\mathcal{O}_\infty \cup \mathcal{O}$. Two lines meeting at a point of $\mathcal{O}_\infty \cup \mathcal{O}$ are in a unique quiver. Two nonconcurrent lines L and M are in the same quiver if and only if the point of $\mathcal{O}_\infty$ on L is collinear with the point of $\mathcal{O}$ on M, and the point of $\mathcal{O}$ on L is collinear with the point of $\mathcal{O}_\infty$ on M. Each point of $\mathcal{O}_\infty$ (respectively, $\mathcal{O}$) determines a unique quiver. Hence there are q quivers, each containing q^2 lines, and each line not in a given quiver is concurrent with exactly q pairwise nonconcurrent lines of that quiver.

Let L and M be nonconcurrent lines. Let a_1, a_2 be the points of L in $\mathcal{O}_\infty$, $\mathcal{O}$, respectively, and let b_1, b_2 be the points of M in $\mathcal{O}$, $\mathcal{O}_\infty$, respectively. Suppose $a_1 \sim b_1$, and let a_3 be the point of L collinear with b_2. Since $b_1 \in \{a_1, b_2\}^\perp$, $[a_1] = [b_2]$. And since $a_1 \in \{a_2, b_1\}^\perp$, $[a_2] = [b_1]$. Hence $a_2 \sim b_2$, i.e., $a_2 = a_3$. But this says that L and M are in the same quiver. This proves:

II.2

If L and M are nonconcurrent lines for which the point a_1 of L in $\mathcal{O}_\infty$ is collinear with the point b_1 of M in $\mathcal{O}$, then L and M are in the same quiver.

Keep in mind that the concept of quiver always depends on the ovoid $\mathcal{O}_\infty$ and all the quivers discussed above depend on the ovoid $\mathcal{O}$. But $\mathcal{O}$ could be any member of $\mathcal{M}' = \mathcal{M} \backslash \{\mathcal{O}_\infty\}$. So the quivers discussed so far are called $\mathcal{O}$-quivers.

Fix $i \in \{0, 1, \ldots, q\}$. In [P] there appears the following "axiom":

A3(I). Let L_1, M_1 be nonconcurrent lines of S meeting lines L_2, M_2 at four distinct points belonging to members of $\mathcal{M}_i = \mathcal{M}'\backslash\{\mathcal{O}_i\}$. Let a_j be the point of $\mathcal{O}_\infty$ incident with L_j, $j = 1,2$, and let b_j be the point on M_j collinear with a_j, $j = 1,2$. Then $b_1 \in \mathcal{O}_i$ if and only if $b_2 \in \mathcal{O}_i$.

It was observed in Section V of [P] that A3(i) is equivalent to having $\mathcal{O}_i$ be a coregular point of S_∞. And this was also observed to be equivalent to the following: If L, M are any nonconcurrent lines in some $\mathcal{O}_i$-quiver Q, then the lines of $\{L, M\}^\perp$ not in Q (i.e., not incident with any of the points of $\mathcal{O}_\infty \cup \mathcal{O}_i$ on L or M) are all in a common $\mathcal{O}_i$-quiver Q'. By II.2 this is equivalent to the following: If K and N are any distinct lines of $\{L, M\}^\perp$ not in $\mathcal{O}$, the point of $\mathcal{O}_\infty$ on K is collinear with the point of $\mathcal{O}_i$ on N. Hence we may collect from [D] and [P] the following characterizations of $\mathcal{P}(Q(4,q), L)$.

II.3

Let $S = (P, B, I)$ be a GQ of order $(q+1, q-1)$, $q \geq 4$. Then S is isomorphic to $\mathcal{P}(Q(4,q), L)$ if and only if there is a partition $\mathcal{M} = \{(\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ of the pointset P into ovoids in such a way that $\mathcal{O}_\infty$ is pivotal for $\mathcal{M}$ and any one of the following equivalent conditions holds:

(i) let L_1 and M_1 be any nonconcurrent lines meeting nonconcurrent lines L_2 and M_2 in four points of $\mathcal{O}_0 \cup \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_q$. Let a_i be the point of $\mathcal{O}_\infty$ on L_i, $i = 1,2$. If $a_i \sim b_i I M_i$, $i = 1,2$, then b_1 and b_2 must belong to the same member of $\mathcal{M}'$.

(ii) With the same notation as in (i), it must be that $\{b_1, b_2\}^\perp \cap \mathcal{O}_\infty = \emptyset$.

(iii) For any pair (L, M) of nonconcurrent lines, let a be the point of $\mathcal{O}_\infty$ on L. Then $a \sim bIM$ determines the point b in some $\mathcal{O}_i$, $0 \leq i \leq q$. So L and M belong to some $\mathcal{O}$-quiver Q. Then for each pair (K, N) of distinct lines of $\{L, M\}^\perp$ not in Q, the point of $\mathcal{O}_\infty$ on K must be collinear with the point of $\mathcal{O}_i$ on N.

From now on we suppose that S satisfies A3(0), and all quivers to be mentioned are $\mathcal{O}_0$-quivers. Then we quote the result V.1 of [P].

II.4.

Let L and M be nonconcurrent lines of some quiver Q. Then the following hold:

(i) The lines of $\{L, M\}^\perp$ not in Q (i.e., not incident with the points of $\mathcal{O}_\infty \cup \mathcal{O}_0$ on L and M) are all in a common quiver Q'.

(ii) L and M belong to a $q \times q$ grid Γ having q lines (of $\{L, M\}^\perp$ in Q' and q lines in Q. The points of Γ are precisely those points on lines of $\{L, M\}^\perp$ not in $\mathcal{O}_\infty \cup \mathcal{O}_0$.

(iii) If (L, M, K) is a triad of lines in Q, then K belongs to the $q \times q$ grid containing L and M if and only if (L, M, K) is centric.

Axiom A4 of [P] was included to force q to be even. Here we omit A4 since we want to allow both q odd and q even. Nevertheless, the two cases are treated separately.

For $q = 2^e$, we give axioms A5 and A6 of [P].

A5. Let (L_1, L_2, L_3) be a centric triad of lines for which the points of $\mathcal{O}_\infty$ on L_1, L_2, L_3 are each collinear with the points of $\mathcal{O}_0$ on L_1, L_2, L_3. If for some $(a, b) \in \mathcal{O}_\infty \times \mathcal{O}_0$, $a \not\sim b$, both L_1 and L_2 are incident with points of $\{a, b\}^\perp$, then L_3 is also incident with a point of $\{a, b\}^\perp$.

A6. Let X_1, X_2, X_3 be distinct points of some $\mathcal{O}_j$, $1 \leq j \leq q$. Let (L_1, L_2, L_3) and (M_1, M_2, M_3) be two triads of lines such that L_i meets M_i at X_i, $i = 1, 2, 3$. Suppose the three points of $\mathcal{O}_\infty$ on L_1, L_2, L_3 (respectively, M_1, M_2, M_3) are each collinear with the three points of $\mathcal{O}_0$ on L_1, L_2, L_3 (respectively, M_1, M_2, M_3). Then (L_1, L_2, L_3) is centric if and only if (M_1, M_2, M_3) is centric.

In [P] there was one additional axiom A7, which we no longer need. (The proof in [P] depended in a very direct way on the main proof in [DT2]. This meant that A7 was required in order to satisfy a specific last property in [DT2], which property was shown in [DT4] to be superfluous.) Finally, the main result of [P] may be stated as follows:

II.5.

Let $\mathcal{S}$ be a GQ of order $(q+1, q-1)$, q even, whose pointset P is partitioned into a family $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ of ovoids for which $\mathcal{O}_\infty$ is pivotal. Suppose $\mathcal{S}$ satisfies A3(0), A5 and A6. Then there must be some q-arc Ω^- of $\pi = PG(2, q)$ for which $\mathcal{S} \cong \mathcal{S}(\Omega^-)$. Since q is even, there is a hyperoval $\Omega^+ = \Omega^- \cup \{a, b\}$ containing Ω^-. Let $\Omega = \Omega^- \cup \{a\}$. Then the point a plays the role of a regular line in $T_2(\Omega)$, and $\mathcal{S}(\Omega^-) \cong \mathcal{P}(T_2(\Omega), a)$.

Now let $q = p^e$, p an odd prime. The axiom $A_{\mathcal{O}_0}(1)$ of [D] is our axiom A3(0). But M. De Soete [D] restated A5 and A6 in a way that still depends on the pivotal ovoid $\mathcal{O}_\infty$ and on the choice of $\mathcal{O}_0 \in \mathcal{M}' = \mathcal{M} \backslash \{\mathcal{O}_\infty\}$ but for which q is assumed to be odd.

$A_{\mathcal{O}_0}(2)$. Let (x, y, z) be a triad of points of $\mathcal{S}$ such that $|\{x, y, z\}^\perp \cap \mathcal{O}_\infty| = 1$. If L_1, L_2, L_3 (respectively, M_1, M_2, M_3) are nonconcurrent lines of some quiver Q (respectively, Q') such that $L_1 I x I M_1, L_2 I y I M_2, L_3 I z I M_3$, then (L_1, L_2, L_3) is centric if and only if (M_1, M_2, M_3) is centric.

$A_{\mathcal{O}_0}(3)$. Let (L_1, L_2, L_3) be a centric triad of lines in some quiver Q. If for some $a, b \in \mathcal{O}_0$, $a \neq b$, both L_1 and L_2 are incident with points of $\{a, b\}^\perp$, then L_3 is also incident with a point of $\{a, b\}^\perp$.

The result of [D] may now be formulated as follows.

II.6

Let $\mathcal{S}$ be a GQ of order $(q+1, q-1)$, q odd, $q \geq 3$, whose pointset P is partitioned into a family $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_0, \mathcal{O}_1, \ldots, \mathcal{O}_q\}$ of ovoids for which $\mathcal{O}_\infty$ is pivotal. Suppose S satisfies A3(0), $A_{\mathcal{O}_0}(2)$, $A_{\mathcal{O}_0}(3)$ Then there must be some q-arc Ω^- of $\pi = PG(2,q)$ for which $S \cong S(\Omega^-)$. Since q is odd, Ω^- is contained in a unique conic Ω and $S(\Omega^-) \cong \mathcal{P}(T_2(\Omega), L)$ for any line L. (Of course, $T_2(\Omega) \cong Q(4,q)$.)

It is helpful to note that for q even (respectively, odd), the axioms A5 and A6 (respectively, $A_{\mathcal{O}_0}(2)$ and $A_{\mathcal{O}_0}(3)$) are designed to guarantee exactly the following: If y, z are any noncollinear points of $\mathcal{P}\backslash(\mathcal{O}_\infty \cup \mathcal{O}_0)$ then there is a unique set yz of q points contained in any grid Γ which contains both y and z.

III. The GQ $S(G,\mathcal{F}^-)$

Let G be an elementary abelian group of order q^3, $q > 1$, with 4-gonal family $\mathcal{F} = \{A_\infty, A_1, \ldots, A_q\}$. Put $\mathcal{F}^- = \mathcal{F}\backslash\{A_\infty\}$ and recall the construction of the GQ $S = S(G,\mathcal{F}^-)$ of order $(q+1, q-1)$. Let $\mathcal{M} = \{\mathcal{O}_\infty, \mathcal{O}_*\} \cup \{A_\infty^* g : g \in G\}$ be the ovoids (cf. I.2) that partition the points of S. We want eventually to interpret for $\mathcal{M}$ the hypotheses of II.3, II.5 and II.6.

III.1

Let $w_1 = A_\infty A_j g_1$ and $w_2 = A_\infty A_j g_2$, $1 \le j \le q$, be distinct points of type (iii). Then $\{w_1, w_2\}^\perp = \{A_j^* h : h \in G\}$, and $\{w_1, w_2\}^{\perp\perp} = \{A_\infty A_j g : g \in G\}$.

Proof: We know $A_j^* \cap A_\infty A_j = A_j$. If $x, y \in A_j^* g \cap A_\infty A_j$, $g \in G$, then $xy^{-1} \in A_j^* \cap A_\infty A_j = A_j$. Hence if $x \in A_j^* g \cap A_\infty A_j$, then $A_j x = A_j^* g \cap A_\infty A_j$. So $|A_j^* g \cap A_\infty A_j| = 0$ or q. But there are only q distinct cosets of A_j^* and $|A_\infty A_j| = q^2$, so each coset of A_j^* meets $A_\infty A_j$ in exactly q elements. To finish the proof it will suffice to show that each point of the form $A_j^* h$, $h \in G$, is collinear with each point of the form $A_\infty A_j g$, $g \in G$. So choose $x \in A_j^* h g^{-1} \cap A_\infty A_j$, and put $b = xg$. Then $A_j b = A_j x g \subseteq A_j^* h \cap A_\infty A_j g$. Hence $A_j b$ is the line joining $A_j^* h$ and $A_\infty A_j g$. ∎

III.2

Consider points $w_1 = A_\infty A_i g_1$, $w_2 = A_\infty A_j g_2$, $1 \le i, j \le q$, $g_1, g_2 \in G$. Write $g_2 g_1^{-1} = a_\infty a_i a_j^{-1}$ (uniquely!), where $a_k \in A_k$ for any appropriate index k. Put $b = a_\infty a_i g_1 = a_j g_2$. Then $\{w_1, w_2\}^\perp = A_\infty b$ and $\{w_1, w_2\}^{\perp\perp} = \{A_\infty A_k b : 1 \le k \le q\}$.

Proof: Let $c_\infty b$ be a typical element of $A_\infty b$. Then $c_\infty b = c_\infty a_\infty a_i g_1 \in A_i c_\infty a_\infty g_1 \subseteq A_\infty A_i g_1$. So $A_i c_\infty a_\infty g_1$ is the line joining $c_\infty b$ and $A_\infty A_i g_1$. Similarly, $c_\infty b = c_\infty a_j g_2 \in A_j c_\infty g_2 \subseteq A_\infty A_j g_2$, so $A_j c_\infty g_2$ is the line joining $c_\infty b$ and $A_\infty A_j g_2$. Hence $\{w_1, w_2\}^\perp = A_\infty b$. For $c_\infty b \in A_\infty b$, $c_\infty b \in A_k c_\infty b \subseteq A_\infty A_k b$, so $\{w_1, w_2\}^{\perp\perp} = (A_\infty b)^\perp = \{A_\infty A_k b : 1 \le k \le b\}$. ∎

Combining III.1 and III.2 we have the following immediate corollary.

III.3 For distinct points $w_1, w_2 \in \mathcal{O}_\infty$, the pair (w_1, w_2) is regular with $\{w_1, w_2\}^{\perp\perp} \subseteq \mathcal{O}_\infty$. Moreover, there is a single ovoid $\mathcal{O} \in \mathcal{M}' = \mathcal{M}\backslash\{\mathcal{O}_\infty\}$ for which $\{w_1, w_2\}^\perp \subseteq \mathcal{O}$. This says $\mathcal{O}_\infty$ is a normal ovoid which is pivotal for $\mathcal{M}$.

There is a companion result giving $\{u, u'\}^\perp$ for distinct $u, u' \in \mathcal{O}_*$.

III.4.

(i) If $u = A_j^* g_1$, $u' = A_j^* g_2$, $g_1 g_2^{-1} \in A_j^*$, then $\{u, u'\}^\perp = \{A_\infty A_j h: h \in G\}$. And $\{u, u'\}^{\perp\perp} = \{A_j^* g: g \in G\}$.

(ii) If $u = A_i^* g_1$, $u' = A_j^* g_2$, $i \neq j$, write $g_2 g_1^{-1} = a_i^* a_j^{-1}$ (uniquely.) with $a_i^* \in A_i^*$, $a_j \in A_j$. Then $b = a_i^* g_1 = a_j g_2 \in A_i^* g_1 \cap A_j^* g_2$. Then $\{u, u'\}^\perp = A_i^* g_1 \cap A_j^* g_2 = b(A_i^* \cap A_j^*)$. But $\{u, u'\}^{\perp\perp}$ might have fewer than q elements.

Since $\mathcal{O}_\infty$ is pivotal for $\mathcal{M}$, clearly II.1 holds. We may adapt the discussion following II.1 to the present situation and see that II.2 holds. Now we choose the ovoid $\mathcal{O}_*$ to play the role of $\mathcal{O}_i$ in the discussion following II.2, and to play the role of $\mathcal{O}_0$ in II.4 and what follows II.4. Put $\mathcal{M}'' = \mathcal{M}'\backslash\{\mathcal{O}_*\} = \{A_\infty^* g: g \in G\}$. And all quivers to be mentioned in III.5 and its proof are $\mathcal{O}_*$-quivers.

III.5. Let L and M be nonconcurrent lines of some quiver Q. Then the lines of $\{L, M\}^\perp$ not in Q (i.e., not incident with any of the points of $\mathcal{O}_\infty \cup \mathcal{O}_*$ on L or M) are all in a common quiver Q'. In other words, $\mathcal{M}$ satisfies A3(0) with $\mathcal{O}_*$ playing the role of $\mathcal{O}_0$.

Proof: First we consider what quivers look like. For a typical point $x = A_i^* g \in \mathcal{O}_*$, $[x] = \{A_i^* g: g \in G\}$ by III.1. And $[x]^\perp = \{A_\infty A_i g: g \in G\}$. So for a fixed i, $1 \leq i \leq q$, the cosets of A_i form a quiver Q_i of q^2 lines joining points $A_i^* g$ with points $A_\infty A_i h$, $g, h \in G$.

Suppose $L = A_i g$ and $M = A_i h$ are two lines of the quiver Q_i, and consider what it means for L and M to meet at some point, i.e., at some point of $\mathcal{O}_\infty \cup \mathcal{O}_*$. $A_i g$ and $A_i h$ meet at some point of $\mathcal{O}_*$ if and only if there is some $b \in G$ for which $A_i^* b$ is on them both. But $A_i g \subseteq A_i^* b$ if and only if $A_i \subseteq A_i^* b g^{-1}$ if and only if $bg^{-1} \in A_i^*$. And $A_i h \subseteq A_i^* b$ if and only if $bh^{-1} \in A_i^*$. So for given $g, h \in G$, there is a $b \in G$ such that $A_i g$ and $A_i h$ meet at the point $A_i^* b$ of $\mathcal{O}_*$ if and only if $gh^{-1} \in A_i^*$. Similarly, replacing A_i^* with $A_\infty A_i$, the lines $A_i g$ and $A_i h$ meet at a point of $\mathcal{O}_\infty$ if and only if $gh^{-1} \in A_\infty A_i$. So to have the lines $L = A_i g$ and $M = A_i h$ of the quiver Q_i *nonconcurrent* means that $gh^{-1} \notin A_i^* \cup A_\infty A_i$, i.e., $gh^{-1} = a_i^* a_\infty$ where $a_i^* \in A_i^* \backslash A_i$ and $a_\infty \in A_\infty \backslash \{e\}$.

Recall (from [PT]) that

$$G = A_i^* + (A_i A_\infty \backslash A_i) + (A_i A_1 \backslash A_i) + \cdots + (A_i \widehat{A_i} \backslash A_i) + \cdots + (A_i A_q \backslash A_i)$$

partitions G. So if L, M (as above) are nonconcurrent, there is some j, $i \neq j$, $1 \leq j \leq q$, for which $gh^{-1} \in A_i A_j \backslash A_i$. We claim that the lines of $\{L, M\}^\perp$ belong to the quiver $\mathcal{O}_j$. Say $gh^{-1} = b_i b_j = a_i^* a_\infty$, where $b_i \in A_i$, $e \neq b_j \in A_j$,

$a_i^* \in A_i^* \backslash A_i$, $e \neq a_\infty \in A_\infty$. Put $b = ea_\infty^{-1}g = a_i^*h$. Then $A_ib \subseteq A_i^*h \cap A_\infty A_ig$. And with $c = A_\infty h = (a_i^*)^{-1}g$, $A_ic \subseteq A_\infty A_ih \cap A_i^*g$. For arbitrary $a_i \in A_i$, $a_ib_ih = a_ib_j^{-1}g \subseteq A_ja_ig$. So as a_i varies over the elements of A_i, the line A_ja_ig is the line of $\{L, M\}^\perp = \{A_ig, A_ih\}^\perp$ not in Q_i joining the point a_ig of L with the point $a_ib_ih = a_ib_j^{-1}g$ of M. So, the lines of $\{L, M\}^\perp$ not in Q_i all lie in Q_j. ∎

Note: The $q \times q$ grid determined by L and M in the preceding proof contains the points of $A_iA_jg = A_iA_jh$. One of the rulings consists of certain cosets of A_i, the other consists of certain cosets of A_j.

IV. Result II.3 Interpreted for $\mathcal{S}(G, \mathcal{F}^-)$

We now consider $\mathcal{O}$-quivers for $\mathcal{O} \in \mathcal{M}'' = \mathcal{M}\backslash\{\mathcal{O}_\infty, \mathcal{O}_*\}$. We may assume that $\mathcal{O} = A_\infty^*$ (if $\mathcal{O} = A_\infty^*g$, just translate everything by g^{-1}). For $a \in A_\infty$ consider the panel $(a, A_ka, A_\infty A_ka)$, $1 \leq k \leq q$. Then $[a] = A_\infty$ and $[a]^\perp = \{A_\infty A_k : 1 \leq k \leq q\}$. The corresponding quiver consists of lines of the form $A_ka : a \in A_\infty, 1 \leq k \leq q$.

Let $A_\infty = \{a_1, \ldots, a_q\}$ and let $\{g_1, \ldots, g_q\}$ be a set of distinct representatives of A_∞ in A_∞^*. Then also for each m, $1 \leq m \leq q$, the $g_1, \ldots, g_q$ are distinct representatives for $A_\infty A_m$ in G, and $\{a_ig_j : 1 \leq i, j \leq q\}$ is a set of coset representatives of A_m in G.

Translating the quiver above by an element of A_∞^* (to keep it an A_∞^*-quiver), we see that a typical A_∞^*-quiver consists of a set of lines of the form $\{A_ka_ig_j : 1 \leq i, k \leq q\}$ for a fixed j, $1 \leq j \leq q$.

For the time being we suppose $g_j = e$. Then A_ma_i and A_na_j meet at a point $A_\infty A_tg$ of $\mathcal{O}_\infty$ if and only if $m = n = t$. Suppose $m \neq n$. Then A_ma_i and A_na_j meet at a point $h \in A_\infty^*$ if and only if $a_i = a_j = h$. So suppose $a_i \neq e = a_j$. Then A_ma_i and A_n are nonconcurrent lines of an A_∞^*-quiver. Define c and d by: $\{c\} = A_m^*a_i \cap A_n$; $\{d\} = A_ma_i \cap A_n^*$ Then for $b_m \in A_m$ equal to any one of the $q - 2$ elements of A_m for which $b_ma_i \notin A_n^* \cup A_\infty A_n$, the accompanying diagram indicates all lines of $\{A_ma_i, A_n\}^\perp$. For such a given b_m, there is a unique $v(v \neq m, n, \infty)$ for which $b_ma_i \in A_nA_v$. If $b_ma_i = b_nb_v$, $b_n \in A_n$, $b_v \in A_v$ then $b_n = b_v^{-1}b_ma_i \in A_vb_ma_i$.

The third column of points contains the points of $\mathcal{O}_\infty$ on those lines of $\{A_ma_i, A_n\}^\perp$ not in the quiver containing A_ma_i and A_n. And the fourth column of points contains those of A_∞^* on the same lines. Condition (iii) of II.3 is that each point of the third column be collinear with each point of the fourth. This is equivalent to $A_\infty A_vb_m \cap A_\infty^* = A_\infty A_nd \cap A_\infty^* = A_\infty A_mc \cap A_\infty^*$. Of course, if this holds, then translating by any element of G gives a condition that is equivalent. This leads to the following characterization.

IV.1. Let G be an elementay abelian group of order q^3, $q > 1$ with 4-gonal family $\mathcal{F} = \{A_\infty, A_1, \ldots, A_q\}$ Put $\mathcal{F}^{-1} = y\backslash\{A_\infty\}$. Then the GQ $S = \mathcal{S}(G, \mathcal{F}^{-1})$

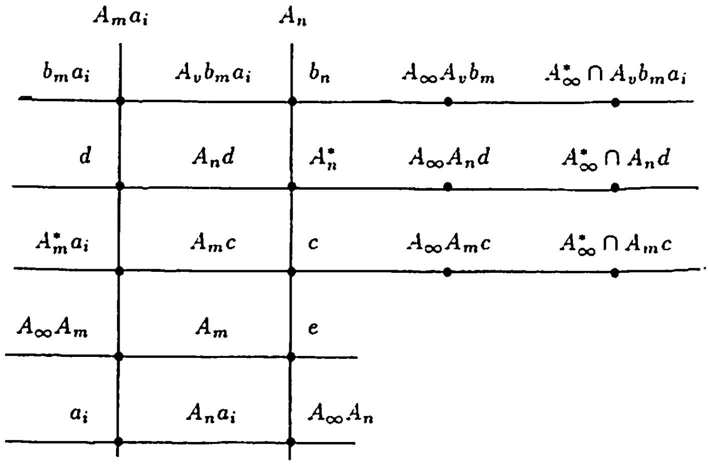

Diagram 1

of order $(q+1, q-1)$ is isomorphic to $\mathcal{P}(Q(4,q), L)$ if and only if the following condition holds:

For each m, n with $1 \leq m, n \leq q$, $m \neq n$, and for each $a \in A_\infty$, $a \neq e$, define the following elements:

$$\{b_m\} = A^*_m \cap A_n a \ (b_m = c_a, \text{ where } a = a_i^{-1} \text{ above});$$
$$\{b_n\} = A_m \cap A^*_n a \ (b_n = d_a, \text{ where } a = a_i^{-1} \text{ above}).$$

For $1 \leq j \leq q$, $m \neq j \neq n$,

$$\{b_j\} = A_m \cap A_n A_j a.$$

Then $K = A_\infty A_j b_j \cap A^*_\infty$ is independent of j, $1 \leq j \leq q$.

V. Result II.5 Interpreted for $S(G, \mathcal{F}^-)$, q Even

Since q is even, there is a group A_* for which $\mathcal{F}^+ = \mathcal{F} \cup \{A_*\}$ is a 4-gonal partition of G and for which $A^*_i = A_i A_*$, $i \in \{\infty, 1, \ldots, q\}$. First we interpret A_5 in this case. Let x, y belong to distinct members of $\mathcal{M}''$, i.e., to distinct cosets of $A^*_\infty = A_\infty A_*$, and suppose $x \not\sim y$. Then there is a unique point $a \in \mathcal{O}_\infty \cap \{x,y\}^\perp$ (respectively, $b \in \mathcal{O}_* \cap \{x,y\}^\perp$). Define the *pseudoline* xy through x and y to be $xy = \{a,b\}^\perp$ (as in [P]). Then (see [P]) A_5 is equivalent to the condition that any grid (i.e., some coset $A_s A_t g$, $1 \leq s < t \leq q$, $g \in G$) containing x and y must contain all q points of the pseudoline xy.

So let x, y be noncollinear points in distinct cosets of $A_\infty A_*$ but in the same coset of $A_s A_t$, $1 \leq s < t \leq q$. Without loss of generality (translate by x^{-1}) we may assume $x = e$, so $y = y_s y_t \in A_s A_t \backslash A_\infty A_*$. There are unique i, j, $1 \leq i$, $j \leq q$, such that $y \in A_\infty A_i = a$ and $y \in A_* A_j = b$. Then $xy = \{a, b\}^\perp = A_\infty A_i \cap A_* A_j$. (Here $i \neq j$ since $x \not\sim y$.) Since $\{x, y\} \subseteq A_s A_t$, A_5 says $A_s A_t \supseteq A_\infty A_i \cap A_* A_j$. So A_5 is equivalent to: $y \in (A_s A_t \cap A_\infty A_i \cap A_* A_j) \backslash A_\infty A_*$ implies $A_s A_t \supseteq A_\infty A_i \cap A_* A_j$.

V.1 A_5 is equivalent to: If $|A_s A_t \cap A_\infty A_i \cap A_* A_j| > 1$, then $A_s A_t \supseteq A_\infty A_i \cap A_* A_j$ (whenever $i, j, s, t, \infty, *$ are distinct).

In [P] A_6 was also restated in terms of grids: Let Γ_i be a $q \times q$ grid with lines from quivers Q_i, Q_i', $i = 1, 2$. Suppose Γ_1 and Γ_2 both have lines incident with points x, y, where x, y are distinct members of the same ovoid in $\mathcal{M}''$ (i.e., x, y belong to the same coset $A_\infty A_* g$ of $A_\infty^* = A_\infty A_*$). Then the set of q points of $A_\infty A_* g$ incident with the lines of Γ_1 is the same as the set of q points of $A_\infty A_* g$ incident with Γ_2. Translating by x^{-1} we may assume $x = e \neq y \in A_\infty A_*$. And there are grids $\Gamma_1 = A_s A_u g_1$, $\Gamma_2 = A_t A_v g_2$ both containing $x = e$ and y. So we may assume $g_1 = g_2 = e$, and $e \neq y \in A_s A_u \cap A_t A_v \cap A_\infty A_*$. Then A_6 says $A_s A_u \cap A_\infty A_* = A_t A_v \cap A_\infty A_*$, i.e., $|A_\infty A_* \cap A_s A_u \cap A_t A_v| = q$. So A_5 and A_6 may be combined into one statement, and the result II.5 may be restated in the present context as:

V.2 There is some q-arc Ω^- of $\pi = PG(2, q)$ for which $S(G, \mathcal{F}^-) \cong S(\Omega^-)$ if and only if for any six distinct indices $i, j, k, \ell, m, n \in \{*, \infty, 1, \ldots, q\}$ including ∞ and $*$, $|A_i A_j \cap A_k A_\ell \cap A_m A_n| = 1$ or q.

VI. Result II.6 Interpreted for $S(G, \mathcal{F}^-)$, q Odd

Letting $\mathcal{O}_*$ play the role of $\mathcal{O}_0$ in Section II, we first interpret $A_{\mathcal{O}_*}(2)$ in $S(G, \mathcal{F}^-)$, q odd. Suppose $A_\infty A_j h$ is a point of $\mathcal{O}_\infty$ collinear with three points x, y, z of the grid $\Gamma = A_s A_t g$. Translating by h^{-1} we assume $h = e$. Then the three points x, y, z are all on the line $A_j g$ if $j = s$ or t. In this case $A_\infty A_j$ is a point on the line $A_j g$, and any grid containing two points of $A_i g$ must contain the points in the set $A_j g$. $A_{\mathcal{O}_*}(2)$ really says that if Γ_1 and Γ_2 are two grids with $x, y, z \in \Gamma_1$ and $x, y \in \Gamma_2$, then $z \in \Gamma_2$, provided $|\{x, y, z\}^\perp \cap \mathcal{O}_\infty| = 1$. When x, y, z lie on a line, we have just seen that this is the case. So suppose no two of x, y, z are collinear. Then $j \notin \{s, t\}$. Suppose $x, y \in A_\infty A_j \cap A_s A_t g$. Then $xy^{-1} \in A_\infty A_j \cap A_s A_t$. If there is a second grid $A_u A_v h$ with $x, y \in A_u A_v h$ then $xy^{-1} \in A_\infty A_j \cap A_u A_v$. So $A_{\mathcal{O}_*}(2)$ says that

$$\{z = cy : c \in A_\infty A_j \cap A_s A_t\} = \{z = dy : d \in A_\infty A_j \cap A_u A_v\}$$

if $|A_\infty A_j \cap A_s A_t \cap A_u A_v| \geq 2$. We have essentially proved the following:

VI.1. $A_{\mathcal{O}_*}(2)$ is equivalent to the following: If $N = |A_\infty A_j \cap A_s A_t \cap A_u A_v| > 1$, then $N = q$, if j, s, t, u, v are distinct members of $\{1, 2, \ldots, q\}$.

Now consider $A_{\mathcal{O}_*}(3)$, which says that if a grid contains two points of $\{u, u'\}^\perp$, for distinct $u, u' \in \mathcal{O}_*$, then it contains q points of $\{u, u'\}^\perp$. By III.4 there are two cases. In both cases if we start with an arbitrary grid A_sA_tg, $1 \le s, t \le q$, $s \neq t$, $g \in G$, we may translate by g^{-1} and assume $g = e$. In the first case if $u, u' \in \mathcal{O}_*$ satisfy $\{u, u'\}^\perp \subseteq \mathcal{O}_\infty$, then we may assume $u = A_s^*g_1$, $u' = A_s^*g_2$. And $\{u, u'\}^\perp = \{A_\infty A_s h : h \in G\}$. But none of these points belongs to any of the grids. So suppose $u = A_i^*g_1$, $u' = A_j^*g_2$, $i \neq j$, $1 \le i, j \le q$. Then $\{u, u'\}^\perp = b(A_i^* \cap A_j^*)$ for any $b \in A_i^*g_1 \cap A_j^*g_2$. And $A_{\mathcal{O}_*}(3)$ is equivalent to: $|b(A_i^* \cap A_j^*) \cap A_sA_t| > 1 \Rightarrow |b(A_i^* \cap A_j^*) \cap A_sA_t| = q$. $A_i^* \cap A_j^*$, is a group of order q. So $A_i^* \cap A_j^* \cap A_sA_t > 1$ (with i, j, s, t distinct) implies $A_i^* \cap A_j^* \subseteq A_sA_t$, in which case any coset of $A_i^* \cap A_j^*$, is contained in A_sA_t or is disjoint from it. It follows that $A_{\mathcal{O}_*}(3)$ is characterized as follows:

VI.2. $A_{\mathcal{O}_*}(3)$ holds in $S(G, \mathcal{F}^-)$, q odd, if and only if for any distinct i, j, s, t in $\{1, \ldots, q\}$, if $|A_i^* \cap A_j^* \cap A_sA_t| > 1$, then $A_i^* \cap A_j^* \subseteq A_sA_t$.

So interpreting II.6 for $S(G, \mathcal{F}^-)$, q odd, we obtain the next result.

VI.3. If q is odd, $S(G, \mathcal{F}^-) \cong \mathcal{P}(Q(4, q), L)$ if and only if the following two conditions hold.

(i) Whenever j, s, t, u, v are distinct members of $\{1, \ldots, q\}$, $|A_\infty A_j \cap A_sA_t \cap A_uA_v| = 1$ or q.

(ii) For any distinct $i, j, s, t \in \{1, \ldots, q\}$, $|A_i^* \cap A_j^* \cap A_sA_t| = 1$ or q.

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