On Steiner Systems $S(5,6,48)$

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Abstract. We prove that there exist precisely 459 pairwise non-isomorphic Steiner systems $S(5,6,48)$ stabilized by the group $PSL_2(47)$.

1. History

In 1976 Denniston published a paper [1] in which he constructed, by hand, Steiner systems with parameters $t = 5$, $k = 6$, $v = 24, 48, 84$ and $t = 5$, $k = 7$, $v = 28$. These systems together with their derived 4-designs were the first known Steiner systems with $t > 3$ other than the well-known classical systems $S(4,5,11)$, $S(5,6,12)$, $S(4,7,23)$ and $S(5,8,24)$ associated with the exceptional Mathieu groups [6]. Denniston's method consisted of assembling orbits of k-element subsets of the set $GF(p^\alpha)\cup\{\infty\}$ under the action of the group $PSL_2(p^\alpha)$, p prime. The designs $S(5,6,p+1)$, $p \equiv -1 \pmod 4$ and prime, form a sequence. A system with the value $v = 72$, omitted by Denniston, was given two years later by Mills [5] using computer calculations. Knowledge of Steiner 5-designs then remained static until recently when systems $S(5,6,108)$ and $S(5,6,132)$ corresponding to the next two values in the sequence were constructed by the present authors [3], [4] also using a computer.

If existence results for Steiner systems $S(t,k,v)$ with $t > 3$ are rare, then enumeration results are even rarer. Each of the Mathieu designs is the unique design, up to isomorphism, with that parameter set. The only other result is that the number of pairwise non-isomorphic designs $S(5,6,24)$ stabilied by the group $PSL_2(23)$ is 3 [2]. In this paper we consider the next value of v in the sequence above mentioned and we prove the following result.

Theorem. *There exist precisely* 459 *pairwise non-isomorphic Steiner systems* $S(5,6,48)$ *stabilized by the group* $PSL_2(47)$.

2. Methodology

Let $V = GF(47) \cup \{\infty\}$ and denote by $\binom{v}{5}$ and $\binom{v}{6}$ the collections of all 5-element and 6-element subsets of V respectively. Represent the group $G = \mathrm{PSL}_2(47)$ as the set of all mappings of the form $z \mapsto (az+b)/(cz+d)$, where $a, b, c, d \in GF(47)$ and $ad - bc$ is a non-zero square, the group operation being functional composition. Firstly compute orbits of $\binom{v}{5}$ under G. Elementary calculations determine that the number of sets in each orbit (the orbit length) is equal to the order of G, i.e. each orbit is full, and there are precisely 33 such orbits. These are listed in Appendix 1 numbered as 0 to 32 with a pair of elements of V given alongside, which, with the three elements $\infty, 0$ and 1, form a generating set for the orbit.

Next compute orbits of $\binom{v}{6}$ under G. In this case the orbit lengths can equal $|G|/n$ where $n = 1, 2, 3$ or 6. We call such orbits full, half, third or sixth and they contain 6, 3, 2 or 1 orbits of $\binom{v}{5}$ respectively. An orbit of $\binom{v}{6}$ in which an orbit of $\binom{v}{5}$ is repeated is said to be unsuitable since it can never appear as part of a Steiner system $S(5, 6, 48)$. Appendix 2 lists all suitable 6-element set orbits, together with a triple of elements of V which, with $\infty, 0$ and 1, form a generating set for the orbit. This appendix also gives the numbers of the 5-element set orbits contained in each 6-element set orbit. Altogether the suitable orbits comprise 144 full orbits (#0 to #143), 74 half orbits (#144 to #217), 3 third orbits (#218 to #220) and 9 sixth orbits (#221 to #229). Appendix 3 lists pairs of orbits of both $\binom{v}{5}$ and $\binom{v}{6}$ which map to one another under the involution $z \mapsto -z$. Orbits which do not appear are those which are stabilized by the mapping (and, in the case of 6-element set orbits, those which are unsuitable).

The next stage is to compute all collections of 6-element subset orbits which together contain all 33 5-element subset orbits precisely once. It is particularly difficult for the reader to verify the completeness of this part of the process without personally writing computer programs, although individual solutions can be checked using Appendix 2. Therefore to ensure the correctness of our results all the calculations were done by the first two authors and the third author independently using different programs running on different machines. We find that there are precisely 918 distinct solutions but these are isomorphic in pairs under the mapping $z \mapsto -z$. Hence there exist at most 459 pairwise non-isomorphic Steiner systems $S(5, 6, 48)$ stabilized by the group $\mathrm{PSL}_2(47)$. Finally the words "at most" can be removed from this statement by examining the derived Steiner triple systems. For each $S(5, 6, 48)$, the derived Steiner triple systems are all isomorphic; a consequence of the triple homogeneity of the automorphism group. We find that for each of the 459 $S(5, 6, 48)$s the derived triple systems all have an automorphism group of order 3 but nevertheless are non-isomorphic. Hence the result is proved. Appendix 4 lists collections of 6-element set orbits which comprise 459 pairwise non-isomorphic $S(5, 6, 48)$s stabilized by the group $\mathrm{PSL}_2(47)$. The systems are numbered 0 to 458 and are ordered firstly in ascending order by the number of orbits which comprise the system and then numerically by orbit number.

3. Analysis

A table showing the number of the 918 distinct solutions in which each of the 230 suitable 6-set orbits appears is given in Appendix 5. Each entry in the body of the table is the number of solutions in which orbit number $10x + y$ appears, where x is the row number and y the column number. Thus, for example, orbit #104 appears in 18 solutions. Some features are of note. Firstly, some full orbits such as #0 and #22 appear in no solutions in spite of being suitable orbits. It may be profitable to investigate this further and try to discover how these suitable but unusable orbits might be identified theoretically so that they could have been eliminated before any computer search. Other suitable full orbits occur in very few systems e.g. just one or two. These too could repay further study since elimination of these would restrict the computer search space whilst still leaving the vast majority of the solutions. This could be extremely important in trying to construct systems $S(5,6,v)$ with larger v. As an example, one observation which the authors have been able to make along these lines is that of the 144 suitable 6-set orbits only 6 are stabilized by the mapping $z \mapsto -z$, namely #34, 50, 65, 80, 139 and 142. Of these only #65 appears in any of the solutions and this in only four of them. It seems therefore that for larger v such orbits could be pre-eliminated before any computer search with minimal loss though also with minimal gain since there will be relatively few such orbits. On the other hand, perhaps systems in which an orbit appears uniquely can be identified as being "special" in some way and some theoretical method of constructing them developed.

At the other extreme, some orbits occur frequently and it would be interesting to identify these too. In particular, we find that of the 918 distinct solutions, 260 comprise short (i.e. not full) orbits only. The table of frequencies of orbits in these systems is also given in Appendix 5. This statistic suggests that one might construct systems using only the short orbits, an approach which has already been used successfully for $S(5,6,84)$, [1] and $S(5,6,108)$, [3]. We might add that of the remaining $918 - 260 = 658$ solutions, 394 contain one full orbit, 224 contain two full orbits and 40 contain three full orbits. None contain four or five full orbits in spite of this being theoretically possible. Finally, in the next section we are able to give a description of a more theoretical way of constructing two of the systems numbered 457 and 458 in Appendix 4.

4. Two special systems

We follow and extend the theory first described in [3]. Consider the 5-set orbits. Define an orbit to be of type A if it is also stabilized by the mapping $z \mapsto -z$. Easily from Appendices 1 and 3, there are 11 type A orbits: #0, 3, 10, 14, 15, 21, 24, 29, 30, 31 and 32. Next define an orbit to be of type B if it is not stabilized by $z \mapsto -z$ but contains a block of the form $\{\infty, -1, 0, 1, a\}$ (or $\{\infty, 0, 1, 2, a+1\}$ by adding 1). Such orbits appear in pairs, each the negative of the other under the mapping $z \mapsto -z$ and there are 4 such pairs: #1 and #7, #2 and #9, #4 and #6, #5 and #8. All other 5-set orbits, which likewise occur in pairs, are of type X.

From Appendix 2 observe that each pair of type B orbits occur as suborbits of two half 6-set orbits; the third 5-set orbit covered in each case being of type A. For the convenience of the reader we list these below.

Half 6-set orbit	5-set orbits covered	Half 6-set orbit	5-set orbits covered
#150	#1, 7 and 14	#162	#4, 6 and 3
#151	#1, 7 and 31	#164	#4, 6 and 24
#159	#2, 9 and 15	#169	#5, 8 and 30
#160	#2, 9 and 21	#170	#5, 8 and 32

These orbits are referred to as of type ABB' and the equivalent orbits play a key role in the construction of $S(5, 6, 108)$ in [3]. By including an appropriate four of the above orbits in an $S(5, 6, 48)$ all orbits of type B will be covered and also four orbits of type A.

Now consider the four orbits of type A which appear in type ABB' orbits but remain uncovered. These may be able to be covered using the following half 6-set orbits which cover two type A orbits and one type X orbit, (type AAX).

Half 6-set orbit	5-set orbits covered	Half 6-set orbit	5-set orbits covered
#196	#14, 15 and 26	#202	#15, 30 and 18
#197	#14, 15 and 27	#205	#15, 30 and 23
#187	#31, 21 and 11	#208	#21, 32 and 17
#191	#31, 21 and 12	#212	#21, 32 and 19
#193	#15, 30 and 13	#214	#24, 32 and 25
#201	#15, 30 and 16	#215	#24, 32 and 28

The three remaining type A orbits #0, 10 and 29 not yet considered are covered by two further type AAX orbits #182 and #183 which cover #10, 29 and 11 and #10, 29 and 12 respectively and the sixth 6-set orbit #221 covering #0.

We leave as an exercise for the reader, the determination of the number of possibilities of choosing orbits from the above so that all type A and type B 5-set orbits are covered. It is completely straightforward and there are in fact just eight such solutions. It only remains to cover the missing orbits of type X and the ingredients that are used for this are simply the third and sixth 6-set orbits. It is immediately clear that four of the partial systems can not be completed using these orbits but the other four (corresponding to our solutions numbered 457 and 458 and their negatives), can using two of the three third orbits and seven of the remaining eight sixth orbits.

To summarize the above, what we are observing is that having calculated all the orbits, we can restrict our attention to only a very small and easily identifiable subset of them, namely the half 6-set orbits of types ABB' and AAX as well as the third and sixth 6-set orbits. From these a simple hand search will produce systems $S(5,6,48)$. It would be interesting to try to adopt and adapt this approach to attempt to construct Steiner systems $S(5,6,v)$ with larger v and the authors intend to pursue investigations along these lines further.

Appendix 1

5-set orbit representatives under $PSL_2(47)$:

0	2 3	1	2 5	2	2 6
3	2 7	4	2 8	5	2 10
6	2 12	7	2 13	8	2 14
9	2 16	10	3 4	11	3 7
12	3 8	13	3 11	14	3 12
15	3 13	16	3 14	17	3 15
18	3 17	19	3 19	20	3 20
21	3 22	22	3 26	23	3 39
24	4 9	25	4 13	26	4 19
27	4 20	28	4 21	29	4 27
30	5 8	31	6 10	32	7 11

Appendix 2

Full 6-set orbit representatives under $PSL_2(47)$:

0	2 3 37	0 1 4 15 16 25	1	2 3 36	0 1 5 10 18 22
2	2 3 12	0 1 6 14 22 25	3	2 3 15	0 1 8 14 17 28
4	2 3 16	0 1 9 10 11 14	5	2 3 40	0 2 3 6 8 15
6	2 3 7	0 2 3 11 13 27	7	2 3 30	0 2 5 10 21 22
8	2 3 29	0 2 6 13 14 19	9	2 3 34	0 2 7 10 12 14
10	2 3 23	0 2 8 11 17 26	11	2 3 10	0 3 4 5 9 15
12	2 3 9	0 3 4 10 11 24	13	2 3 41	0 3 6 10 12 24
14	2 3 43	0 3 9 12 23 26	15	2 3 31	0 4 5 17 22 29
16	2 3 38	0 4 7 14 20 28	17	2 3 22	0 4 8 11 21 25
18	2 3 21	0 4 9 14 17 23	19	2 3 28	0 5 6 12 21 28
20	2 3 35	0 5 7 14 19 25	21	2 3 27	0 5 9 12 19 27
22	2 3 13	0 6 7 15 18 28	23	2 3 19	0 6 8 19 20 29
24	2 3 14	0 7 8 10 16 20	25	2 3 20	0 8 9 10 20 21
26	2 5 42	1 2 3 5 16 28	27	2 5 23	1 2 4 5 13 28

28	2 5 29	1 2 11 13 29 32	29	2 5 6	1 2 14 15 21 22
30	2 5 9	1 3 5 6 31 32	31	2 5 40	1 3 10 15 23 26
32	2 5 7	1 3 17 20 31 32	33	2 5 37	1 4 5 9 13 25
34	2 5 12	1 4 6 7 26 27	35	2 5 21	1 4 13 18 21 27
36	2 5 31	1 4 14 17 24 30	37	2 5 35	1 5 13 18 31 32
38	2 5 30	1 5 14 22 24 30	39	2 5 18	1 6 10 12 13 17
40	2 5 41	1 6 17 18 25 26	41	2 5 28	1 6 24 27 29 31
42	2 5 19	1 8 12 15 18 21	43	2 5 39	1 8 18 20 26 28
44	2 5 22	1 8 27 28 30 31	45	2 5 43	1 9 16 21 30 31
46	2 5 16	1 9 17 22 23 25	47	2 5 20	1 9 21 22 26 27
48	2 6 9	2 3 11 12 19 30	49	2 6 7	2 3 15 19 20 29
50	2 6 12	2 4 5 6 8 9	51	2 6 8	2 4 10 17 29 32
52	2 6 21	2 4 12 20 27 29	53	2 6 37	2 4 18 28 31 32
54	2 6 31	2 4 23 26 30 31	55	2 6 35	2 5 12 15 30 31
56	2 6 30	2 5 13 17 19 30	57	2 6 39	2 6 7 8 23 28
58	2 6 18	2 6 10 12 20 28	59	2 6 28	2 6 13 16 20 27
60	2 6 38	2 7 13 19 20 28	61	2 6 44	2 7 18 21 30 31
62	2 6 13	2 7 20 21 26 27	63	2 6 14	2 8 11 14 16 20
64	2 6 19	2 8 13 21 30 32	65	2 6 16	2 9 11 12 17 19
66	2 7 13	3 4 7 8 31 32	67	2 7 21	3 4 11 15 19 27
68	2 7 31	3 4 26 28 30 32	69	2 7 35	3 5 13 17 18 22
70	2 7 30	3 5 24 27 29 32	71	2 7 18	3 6 12 15 17 26
72	2 7 41	3 6 25 27 30 32	73	2 7 44	3 7 8 9 18 25
74	2 7 34	3 7 10 13 15 27	75	2 7 38	3 7 19 22 31 32
76	2 7 39	3 8 16 19 20 23	77	2 7 22	3 8 24 26 29 32
78	2 7 20	3 9 11 12 17 30	79	2 7 16	3 9 15 17 22 29
80	2 8 30	4 5 6 8 25 28	81	2 8 10	4 5 14 16 26 29
82	2 8 27	4 5 24 25 29 30	83	2 8 34	4 7 10 11 19 23
84	2 8 44	4 7 16 19 27 28	85	2 8 13	4 7 24 26 29 31
86	2 8 19	4 8 12 13 15 27	87	2 8 26	4 9 10 11 22 25
88	2 8 20	4 9 18 22 23 26	89	2 10 12	5 6 11 15 23 26
90	2 10 38	5 7 11 15 16 21	91	2 10 44	5 7 16 22 25 27
92	2 10 13	5 7 25 26 30 31	93	2 10 26	5 9 12 14 18 22
94	2 10 16	5 9 21 23 30 32	95	2 12 39	6 7 8 9 23 25
96	2 12 13	6 7 14 19 24 30	97	2 12 34	6 7 16 21 23 26
98	2 12 19	6 8 14 18 27 29	99	2 12 14	6 8 24 28 29 30
100	2 12 16	6 9 10 19 29 32	101	2 12 26	6 9 11 22 26 29
102	2 12 20	6 9 13 27 30 31	103	2 12 43	6 9 16 25 31 32
104	2 13 14	7 8 14 20 24 30	105	2 13 19	7 8 16 23 31 32
106	2 13 43	7 9 12 23 29 32	107	2 13 20	7 9 14 15 20 21
108	2 14 43	8 9 11 15 30 31	109	2 14 20	8 9 17 19 23 30
110	3 4 32	10 11 17 22 27 32	111	3 4 18	10 11 18 19 21 26
112	3 4 33	10 12 16 17 21 27	113	3 4 19	10 12 19 20 26 32
114	3 4 29	10 13 14 21 25 28	115	3 4 21	10 13 17 21 24 28

116	3 4 22	10 14 21 23 25 28	117	3 4 30	10 19 21 23 24 25
118	3 7 22	11 12 16 21 22 32	119	3 7 41	11 12 18 20 21 32
120	3 7 12	11 13 14 15 18 29	121	3 7 35	11 13 14 16 18 30
122	3 7 29	11 13 16 24 27 28	123	3 7 43	11 15 17 19 23 31
124	3 7 31	11 17 22 28 29 32	125	3 7 38	11 18 20 24 31 32
126	3 8 19	12 13 15 17 19 31	127	3 8 43	12 14 15 16 23 29
128	3 8 37	12 14 16 18 23 30	129	3 8 14	12 16 22 24 31 32
130	3 8 42	12 18 23 24 25 26	131	3 8 27	12 19 20 25 29 32
132	3 11 26	13 22 27 28 31 32	133	3 12 14	14 16 24 25 26 30
134	3 12 17	14 18 24 27 28 30	135	3 13 15	15 17 25 27 28 29
136	3 13 27	15 19 25 26 28 29	137	3 14 15	16 17 24 28 29 31
138	3 14 20	16 20 26 27 30 31	139	3 15 20	17 19 20 22 25 28
140	3 17 19	18 19 24 25 29 31	141	3 17 26	18 22 26 27 30 31
142	3 20 26	20 22 25 26 27 28	143	3 20 43	20 23 25 26 31 32

Half 6-set orbit representatives under $PSL_2(47)$:

144	2 3 24	0 2 20	145	2 3 26	0 9 22
146	2 5 33	1 2 22	147	2 5 8	1 4 30
148	2 5 27	1 5 13	149	2 5 10	1 5 24
150	2 5 13	1 7 14	151	2 5 44	1 7 31
152	2 5 14	1 8 25	153	2 5 26	1 9 16
154	2 5 36	1 28 32	155	2 6 42	2 3 17
156	2 6 10	2 5 31	157	2 6 34	2 7 18
158	2 6 22	2 8 13	159	2 6 20	2 9 15
160	2 6 43	2 9 21	161	2 6 29	2 20 27
162	2 7 40	3 4 6	163	2 7 43	3 9 19
164	2 8 41	4 6 24	165	2 8 14	4 8 31
166	2 8 21	4 12 18	167	2 10 18	5 6 31
168	2 10 34	5 7 28	169	2 10 39	5 8 30
170	2 10 22	5 8 32	171	2 10 20	5 9 23
172	2 10 27	5 13 17	173	2 12 38	6 7 30
174	2 12 18	6 11 16	175	2 13 39	7 8 23
176	2 13 22	7 8 24	177	2 13 26	7 9 20
178	2 13 44	7 25 32	179	2 14 26	8 9 31
180	2 14 19	8 19 23	181	2 16 26	9 22 26
182	3 4 23	10 11 29	183	3 4 28	10 12 29
184	3 4 16	10 14 25	185	3 4 35	10 14 28
186	3 7 39	11 19 23	187	3 7 45	11 21 31
188	3 7 32	11 23 30	189	3 8 21	12 13 17
190	3 8 11	12 13 30	191	3 8 33	12 21 31
192	3 11 35	13 14 28	193	3 11 13	13 15 30
194	3 11 33	13 17 21	195	3 11 15	13 17 32
196	3 12 40	14 15 26	197	3 12 13	14 15 27
198	3 12 20	14 20 27	199	3 12 26	14 22 26

200	3 12 39	14 23 25	201	3 13 14	15 16 30
202	3 13 42	15 18 30	203	3 13 20	15 20 27
204	3 13 31	15 22 26	205	3 13 39	15 23 30
206	3 14 22	16 21 22	207	3 14 37	16 26 31
208	3 15 45	17 21 32	209	3 17 45	18 20 21
210	3 17 42	18 27 31	211	3 19 45	19 21 23
212	3 19 22	19 21 32	213	3 19 43	19 23 32
214	4 9 15	24 25 32	215	4 9 21	24 28 32
216	4 13 27	25 26 29	217	4 20 41	27 28 29

Third 6-set orbit representatives under $PSL_2(47)$:

218	3 7 18	11 12	219	3 11 43	13 23
220	3 14 42	16 18			

Sixth 6-set orbit representatives under $PSL_2(47)$:

221	2 3 25	0	222	3 15 21	17
223	3 19 27	19	224	3 20 38	20
225	3 26 31	22	226	4 13 15	25
227	4 19 40	26	228	4 20 33	27
229	4 21 35	28			

Appendix 3: Isomorphism Mappings

5-orbits:

(1 7) (2 9) (4 6) (5 8) (11 12) (13 23) (16 18)
(17 19) (20 22) (25 28) (26 27)

6-orbits:

(0 22)	(1 24)	(2 16)	(3 20)	(4 9)	(5 11)	(6 14)
(7 25)	(8 18)	(10 21)	(12 13)	(15 23)	(17 19)	(26 73)
(27 95)	(28 106)	(29 107)	(30 66)	(31 74)	(32 75)	(33 57)
(35 97)	(36 96)	(37 105)	(38 104)	(39 83)	(40 84)	(41 85)
(42 90)	(43 91)	(44 92)	(45 61)	(46 60)	(47 62)	(48 78)
(49 79)	(51 100)	(52 101)	(53 103)	(54 102)	(55 108)	(56 109)
(58 87)	(59 88)	(63 93)	(64 94)	(67 71)	(68 72)	(69 76)
(70 77)	(81 98)	(82 99)	(86 89)	(110 113)	(111 112)	(114 116)
(115 117)	(118 119)	(120 127)	(121 128)	(122 130)	(123 126)	(124 131)
(125 129)	(132 143)	(133 134)	(135 136)	(137 140)	(138 141)	(144 145)
(146 177)	(147 173)	(148 175)	(149 176)	(152 168)	(153 157)	(154 178)
(155 163)	(156 179)	(158 171)	(161 181)	(165 167)	(166 174)	(172 180)
(182 183)	(184 185)	(186 189)	(187 191)	(188 190)	(192 200)	(193 205)
(194 211)	(195 213)	(196 197)	(198 199)	(201 202)	(203 204)	(206 209)
(207 210)	(208 212)	(214 215)	(216 217)	(222 223)	(224 225)	(226 229)
(227 228)						

Appendix 4

Block orbit representatives of $S(5,6,48)$ invariant under $PSL_2(47)$:

0	5	111	129	147	172	177	200	217			
1	6	36	103	168	180	183	204	209			
2	15	64	130	151	163	174	185	203			
3	56	77	116	145	151	166	174	203			
4	4	76	156	166	173	194	204	214	217		
5	4	129	161	162	168	180	194	202	216		
6	10	129	148	162	177	184	202	211	217		
7	1	104	159	162	194	207	213	217	218	226	
8	1	104	159	162	195	207	211	217	218	226	
9	1	106	133	158	162	187	203	222	223	229	
10	1	137	160	162	178	180	193	198	218	227	
11	1	155	165	174	177	190	196	211	214	217	
12	1	155	165	174	177	190	197	211	215	216	
13	2	74	137	160	166	170	188	223	224	227	
14	3	62	163	166	167	182	201	214	219	225	
15	4	56	162	175	191	203	215	216	220	225	
16	4	56	162	175	191	204	214	217	220	224	
17	4	60	77	166	167	205	206	222	226	228	
18	7	77	123	153	166	173	192	224	226	228	
19	10	106	114	149	162	201	210	223	224	225	
20	26	83	98	145	191	193	214	222	224	227	
21	26	83	98	181	191	193	214	221	222	224	
22	27	77	145	173	184	186	191	203	220	222	
23	29	76	102	166	168	182	214	221	222	227	
24	30	86	133	145	157	182	211	222	224	229	
25	30	87	144	176	190	196	211	217	220	222	
26	30	88	144	176	184	194	201	217	218	223	
27	30	144	166	176	181	184	186	194	201	217	
28	30	145	161	166	176	185	186	194	201	216	
29	31	52	121	145	167	176	208	223	226	229	
30	31	52	121	145	167	176	212	222	226	229	
31	37	104	112	159	162	186	216	221	225	229	
32	45	60	130	162	170	182	197	221	222	225	
33	49	87	128	154	167	176	194	221	227	228	
34	122	145	147	155	167	178	180	183	196	209	
35	133	145	154	157	162	172	180	182	191	203	
36	133	145	154	157	162	172	180	183	187	203	
37	1	104	159	162	207	208	217	218	219	223	226
38	1	104	159	162	207	212	217	218	219	222	226
39	3	97	156	163	166	182	193	214	224	225	228
40	3	156	162	177	182	190	204	211	214	220	228
41	3	156	163	166	173	182	203	206	214	219	227

42	4	129	158	162	168	202	211	216	222	224	228
43	4	129	158	162	168	205	209	216	222	223	228
44	4	137	157	162	170	190	203	211	225	226	227
45	4	137	157	162	170	190	204	211	224	226	228
46	5	38	166	177	182	194	207	213	226	228	229
47	5	38	166	177	182	195	207	211	226	228	229
48	5	82	151	181	185	194	213	218	220	224	228
49	5	82	151	181	185	195	211	218	220	224	228
50	5	137	148	166	177	184	188	212	225	227	228
51	5	138	150	166	171	182	194	214	223	225	229
52	5	138	150	166	171	182	194	215	223	225	226
53	7	125	153	162	175	190	196	217	222	223	226
54	7	125	153	162	175	190	197	216	222	223	229
55	7	150	162	179	186	190	203	215	216	220	222
56	7	150	162	179	188	189	203	215	216	220	223
57	10	106	149	162	184	193	206	210	223	224	229
58	10	106	149	162	185	193	206	210	223	224	226
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355	149	160	162	178	180	182	189	199	201	210	221	224	229	
356	149	161	162	178	179	183	186	192	202	206	221	222	227	
357	150	156	162	180	181	182	189	201	209	214	221	228	229	
358	150	156	162	180	181	182	189	201	209	215	221	226	228	
359	151	155	164	170	181	183	186	192	201	209	221	226	228	
360	151	155	164	170	181	184	193	211	217	218	220	221	224	
361	151	155	164	170	181	184	201	209	217	218	219	221	223	
362	28	163	166	167	176	184	205	206	221	222	224	227	228	229
363	28	163	166	167	176	185	205	206	221	222	224	226	227	228
364	30	104	159	166	182	206	219	221	222	223	226	227	228	229
365	37	159	162	176	183	188	198	206	221	222	223	226	227	229
366	37	160	162	176	183	186	198	201	221	222	225	226	227	229
367	37	160	162	176	183	186	199	201	221	222	224	226	228	229
368	51	148	163	174	176	191	200	202	221	224	225	227	228	229
369	51	149	163	174	175	191	192	202	221	224	225	226	227	228
370	55	153	162	175	182	192	209	214	221	222	223	225	227	228
371	55	153	162	176	182	192	209	213	221	222	225	226	227	228
372	55	153	162	176	182	195	200	209	221	223	225	227	228	229
373	63	151	162	171	183	194	202	214	221	223	225	227	228	229
374	63	151	162	171	183	194	202	215	221	223	225	226	227	228
375	69	150	160	165	174	183	205	214	221	223	224	227	228	229
376	69	150	160	165	174	183	205	215	221	223	224	226	227	228
377	70	151	160	166	174	180	184	193	221	222	224	225	227	229
378	70	151	160	166	174	180	185	193	221	222	224	225	226	227
379	122	151	160	162	170	183	200	202	221	222	223	224	225	227
380	125	148	160	162	175	183	199	201	221	222	223	226	228	229
381	125	150	160	162	172	180	183	201	221	225	226	227	228	229
382	144	149	162	178	179	182	190	196	211	220	222	225	228	229
383	144	149	162	178	179	182	190	197	211	220	222	225	227	229
384	144	149	162	178	179	183	188	194	196	220	223	225	228	229
385	144	149	162	178	179	183	188	194	197	220	223	225	227	229
386	144	151	163	164	170	183	188	194	196	220	225	226	228	229
387	144	151	163	164	170	183	188	194	197	220	225	226	227	229
388	144	154	162	171	176	182	191	193	199	220	222	223	226	228
389	144	154	162	171	176	183	187	193	199	220	222	223	226	228
390	146	162	168	179	182	190	196	211	214	220	221	222	224	228
391	146	162	168	179	182	190	197	211	214	220	221	222	224	227
392	146	162	168	179	183	188	194	196	214	220	221	223	224	228
393	146	162	168	179	183	188	194	197	214	220	221	223	224	227
394	149	163	166	167	176	182	198	201	208	219	221	226	227	229
395	147	155	168	174	179	183	196	209	214	219	221	223	225	228
396	147	155	168	174	179	183	197	209	214	219	221	223	225	227
397	148	160	162	176	184	202	207	213	217	218	221	222	224	225
398	148	160	162	176	185	201	210	213	216	218	221	222	224	225
399	149	157	162	179	184	193	206	213	217	218	221	222	224	227

400	149	157	162	179	184	194	201	213	217	218	221	224	225	227	
401	149	157	162	179	184	195	201	211	217	218	221	224	225	227	
402	149	157	162	179	184	195	205	206	217	218	221	223	224	227	
403	149	157	162	179	185	193	206	213	216	218	221	222	224	228	
404	149	157	162	179	185	194	201	213	216	218	221	224	225	228	
405	149	157	162	179	185	195	201	211	216	218	221	224	225	228	
406	149	157	162	179	185	195	205	206	216	218	221	223	224	228	
407	149	158	162	178	181	182	191	198	205	220	221	222	223	229	
408	149	158	162	178	181	183	187	198	205	220	221	222	223	229	
409	149	161	162	178	179	182	190	196	211	220	221	222	225	229	
410	149	161	162	178	179	183	188	194	196	220	221	223	225	229	
411	151	155	164	170	181	182	190	197	211	220	221	224	226	229	
412	151	159	162	169	182	189	198	211	214	220	221	225	227	229	
413	151	159	162	169	182	189	198	211	215	220	221	225	226	227	
414	151	159	162	169	182	189	199	211	214	220	221	224	228	229	
415	151	159	162	169	182	189	199	211	215	220	221	224	226	228	
416	151	160	162	169	182	189	200	203	215	220	221	223	225	227	
417	151	160	162	169	182	189	200	204	215	220	221	223	224	228	
418	153	157	162	169	182	191	198	204	214	219	221	222	223	229	
419	153	157	162	169	182	191	198	204	215	219	221	222	223	226	
420	153	157	162	169	182	191	199	203	214	219	221	222	223	229	
421	153	157	162	169	182	191	199	203	215	219	221	222	223	226	
422	30	160	166	176	182	198	201	219	221	222	223	225	226	227	229
423	30	160	166	176	182	199	201	219	221	222	223	224	226	228	229
424	38	159	162	175	182	191	195	220	221	223	224	226	227	228	229
425	38	159	162	175	183	187	195	220	221	223	224	226	227	228	229
426	125	150	159	162	169	183	206	219	221	222	223	226	227	228	229
427	146	162	169	177	182	191	196	214	219	220	221	222	223	228	229
428	146	162	169	177	182	191	196	215	219	220	221	222	223	226	228
429	146	162	169	177	182	191	197	214	219	220	221	222	223	227	229
430	146	162	169	177	182	191	197	215	219	220	221	222	223	226	227
431	146	163	164	169	178	182	191	196	219	220	221	222	224	228	229
432	146	163	164	169	178	182	191	197	219	220	221	222	224	227	229
433	146	163	164	169	178	183	187	196	219	220	221	222	224	228	229
434	146	163	164	169	178	183	187	197	219	220	221	222	224	227	229
435	148	157	162	179	183	188	196	206	214	221	222	223	224	228	229
436	148	157	162	179	183	188	196	206	215	221	222	223	224	226	228
437	148	157	162	179	183	188	197	206	214	221	222	223	224	227	229
438	148	157	162	179	183	188	197	206	215	221	222	223	224	226	227
439	149	157	162	179	182	189	200	201	212	221	224	225	227	228	229
440	149	157	162	179	182	190	196	206	213	221	222	224	226	228	229
441	149	157	162	179	182	190	197	206	213	221	222	224	226	227	229
442	149	157	162	179	183	186	192	201	208	221	224	225	226	227	228
443	149	157	162	179	183	188	195	196	206	221	223	224	226	228	229
444	149	157	162	179	183	188	195	197	206	221	223	224	226	227	229

445	149	157	162	179	184	201	208	217	218	219	221	223	224	225	227		
446	149	157	162	179	184	201	212	217	218	219	221	222	224	225	227		
447	149	157	162	179	185	201	208	216	218	219	221	223	224	225	228		
448	149	157	162	179	185	201	212	216	218	219	221	222	224	225	228		
449	149	157	163	165	174	183	192	205	208	221	224	225	226	227	228		
450	149	157	163	165	174	183	193	200	208	221	224	225	227	228	229		
451	150	155	165	171	174	183	193	209	214	221	223	225	227	228	229		
452	150	155	165	171	174	183	193	209	215	221	223	225	226	227	228		
453	153	157	162	169	182	191	196	214	219	221	222	223	224	225	228	229	
454	153	157	162	169	182	191	196	215	219	221	222	223	224	225	226	228	
455	153	157	162	169	182	191	197	214	219	221	222	223	224	225	227	229	
456	153	157	162	169	182	191	197	215	219	221	222	223	224	225	226	227	
457	150	159	162	169	182	191	214	219	220	221	222	223	224	225	227	228	229
458	150	159	162	169	182	191	215	219	220	221	222	223	224	225	226	227	228

Appendix 5

Orbit frequencies of $S(5,6,48)$ invariant under $PSL_2(47)$:
(i) all 918 solutions

	0	1	2	3	4	5	6	7	8	9
0	0	8	4	4	42	18	6	5	0	42
1	9	18	17	17	6	2	4	0	0	0
2	4	9	0	2	8	5	24	2	10	3
3	15	4	3	14	10	3	4	9	18	6
4	0	3	2	0	1	3	6	7	0	24
5	0	5	4	2	4	31	8	14	2	1
6	6	3	7	10	2	4	15	2	1	9
7	15	2	1	24	4	3	9	15	0	24
8	0	11	5	6	0	3	3	2	1	3
9	2	0	1	10	2	2	4	3	11	5
10	5	4	4	2	18	9	10	3	31	8
11	0	6	6	0	12	2	12	2	1	1
12	3	13	7	2	5	17	2	3	13	17
13	7	5	2	7	7	0	0	8	3	0
14	8	3	0	2	97	97	60	64	78	167
15	56	102	40	87	69	108	96	87	30	60
16	52	82	426	108	122	71	107	71	40	82
17	48	30	81	64	107	78	167	60	69	96
18	81	82	259	259	91	91	69	96	54	69
19	54	96	58	68	116	72	74	74	77	77
20	58	132	132	63	63	68	96	42	79	96
21	42	116	79	72	128	128	105	105	130	164
22	294	494	274	274	309	309	307	366	366	307

(ii) all 260 short orbit solutions

	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	40	40	20	7	23	53
15	18	44	16	54	25	44	44	54	9	12
16	12	25	172	44	52	13	16	13	16	58
17	24	9	20	7	16	23	53	20	25	44
18	20	25	104	104	26	26	26	38	24	26
19	24	38	17	23	32	24	42	42	36	36
20	17	47	47	12	12	23	36	13	18	36
21	13	32	18	24	51	51	26	26	52	62
22	94	180	96	96	91	91	99	106	106	99

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