On the diameter of nth order degree regular trees

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Abstract. For n a positive integer and v a vertex of a graph G, the nth order degree of v in G, denoted by $\deg_n v$, is the number of vertices at distance n from v. The graph G is said to be nth order regular of degree k if, for every vertex v of G, $\deg_n v = k$. For $n \in \{7, 8, \ldots, 11\}$, a characterization of nth order regular trees of degree 2 is obtained. It is shown that, for $n \geq 2$ and $k \in \{3, 4, 5\}$, if G is an nth order regular tree of degree k, then G has diameter $2n - 1$.

1. Introduction

We denote the vertex set and edge set of a graph G by $V(G)$ and $E(G)$, respectively. If $v \in V(G)$, the degree of v in G is written as $\deg v$ and the minimum degree of G is given by $\delta(G) = \min\{\deg v : v \in V(G)\}$, whereas the maximum degree of G is $\Delta(G) = \max\{\deg v : v \in V(G)\}$. For a connected graph G, the distance $d(u, v)$ between two vertices u and v is the length of a shortest $u-v$ path. The eccentricity $e_G(v)$ of a vertex v of G is defined as $\max_{u \in V(G)} d(u, v)$. The diameter diam G of G is $\max_{v \in V(G)} e(v)$. For other graph theory terminology we follow [4].

Observe that the degree of a vertex v in a graph G is the number of vertices at distance 1 from v. This observation suggests a generalization of degree. In [1], for n a positive integer and v a vertex of a graph G, the nth order degree of v in G, denoted by $\deg_n v$, is defined as the number of vertices at distance n from v. Hence $\deg_1 v = \deg v$. Further, in [1], the graph G is defined to be nth order regular of degree k if, for every vertex v of G, $\deg_n v = k$.

Concepts related to the nth order degree of a vertex were introduced by Bloom, Kennedy and Quintas ([2], [3]). In [2] the distance degree sequence of G based at a vertex v is defined as the sequence $\deg_1 v, \deg_2 v, \ldots, \deg_{\text{diam}\, G} v$, and distance degree regular graphs, namely those graphs G in which all the vertices have the same distance degree sequence, were investigated. Randic [8] has examined the role that distance degree sequences of graphs play in chemical applications. Related work was also performed by Hilano and Nomura [7] and by Taylor and Levingston [9].

For $n \in \{7, 8, \ldots, 11\}$, we obtain a characterization of nth order regular trees of degree 2. Further, it is shown that, for $n \geq 2$ and $k \in \{3, 4, 5\}$, if G is an nth order regular tree of degree k, then G has diameter $2n - 1$.

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2. Known results

In this section we list a few known results which will prove useful to us. The following result was proposed as a conjecture in [1] and proven in [5].

Theorem A. *For $n \geq 2$, if G is a connected nth order regular graph of degree 1, then G is either a path of length $2n-1$ or G has diameter n.*

The next result [6] establishes a lower bound on the diameter of nth order degree regular trees.

Theorem B. *For $n \geq 2$ and $k \geq 1$, if G is a tree which is nth order regular of degree k, then diam $G \geq 2n-1$.*

Theorems A and B yield the following characterization of nth order regular trees of degree 1.

Corollary A. *For $n \geq 2$, G is an nth order regular tree of degree 1 if and only if G is a path of length $2n-1$.*

The following result [6] will prove to be useful.

Lemma A. *For $n \geq 2$ and $k \geq 1$, the maximum degree $\Delta(G)$ of a tree G that is nth order regular of degree k is at most $k+1$.*

A necessary condition for a tree to be nth order regular of degree 2 is established in [6].

Theorem C. *For $n \geq 2$, if G is a tree which is nth order regular of degree 2, then the diameter of G is $2n-1$.*

Before proceeding further, we recall the definition of a double star. The double star $S(m,n)$ is obtained from the (disjoint) union of two stars $K(1,n)$ and $K(1,m)$ where $n, m \in \mathbf{Z}^+$, by joining a vertex of maximum degree in $K(1,n)$ to a vertex of maximum degree in $K(1,m)$.

The graph H_k is defined in [6] as follows. For $k, \ell \in \mathbf{Z}^+$ let $H_{\ell,k}$ be the graph obtained from $K(1,k)$ by subdividing each edge ℓ times. Now let F_1 and F_2 be two disjoint copies of $H_{2,k}$ and let v_1 and v_2 be vertices of degree k in F_1 and F_2, respectively. The graph H_k is obtained from $F_1 \cup F_2$ by joining v_1 and v_2 with an edge and then subdividing twice the edge $v_1 v_2$. (The graph H_k is shown in Figure 1.)

In [6], for $k \geq 1$ an integer, the following characterization of nth order regular trees of degree k for $n \in \{2, 3, \ldots, 6\}$ is obtained.

Lemma B. *For $k \geq 1$, a tree G is 2nd order regular of degree k if and only if $G \cong S(k,k)$.*

Lemma C. *For $n \in \{3, 4, 6\}$ and $k \geq 1$, a tree G is nth order regular of degree k if and only if G is a path of length $2n-1$ and $k=1$.*

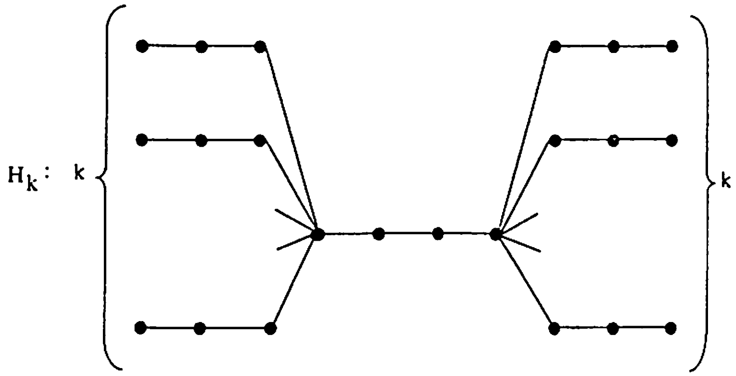


Figure 1 The graph H_k

Lemma D. *For $k \geq 1$, a tree G is 5th order regular of degree k if and only if G is isomorphic to H_k.*

That there exists a tree which is nth order regular of degree k for each $n \geq 7$ and $k \geq 1$ is established in [6].

Theorem D. *Let $n \geq 7$ be an integer. Then for each $k \in \mathbf{Z}^+$, there exists a tree that is nth order regular of degree k.*

3. Characterizations of nth order regular trees of degree 2 for small n.

In this section we investigate nth order regular trees of degree 2 for small n. For $n \geq 2$, let G be a tree which is nth order regular of degree 2. Necessarily (cf. Lemma C) $n \notin \{3,4,6\}$. If $n = 2$, then (cf. Lemma B) $G \cong S(2,2)$, while if $n = 5$, then (cf. Lemma D) $G \cong H_2$. Hence in what follows we restrict our attention to those values of $n \geq 7$. By Theorem D for each $n \geq 7$, there exists a tree that is nth order regular of degree 2.

For $n \in \{7,8,10\}$ and $i \in \{1,2\}$, let G_n and $G_{9,i}$ be the graphs shown in Figure 2. Furthermore, for $i \in \{1,2,\ldots,5\}$, let $G_{11,i}$ be the graph shown in Figure 3. We are now in a position to characterize nth order regular trees of degree 2 for $n \in \{7,8,\ldots,11\}$.

Theorem 1. *For $n \in \{7,8,10\}$, G is an nth order regular tree of degree 2 if and only if $G \cong G_n$. Furthermore, G is a 9th order regular tree of degree 2 if and only if $G \cong G_{9,1}$ or $G \cong G_{9,2}$, while G is an 11th order regular tree of degree 2 if and only if $G \cong G_{11,i}$ where $i \in \{1,2,\ldots,5\}$.*

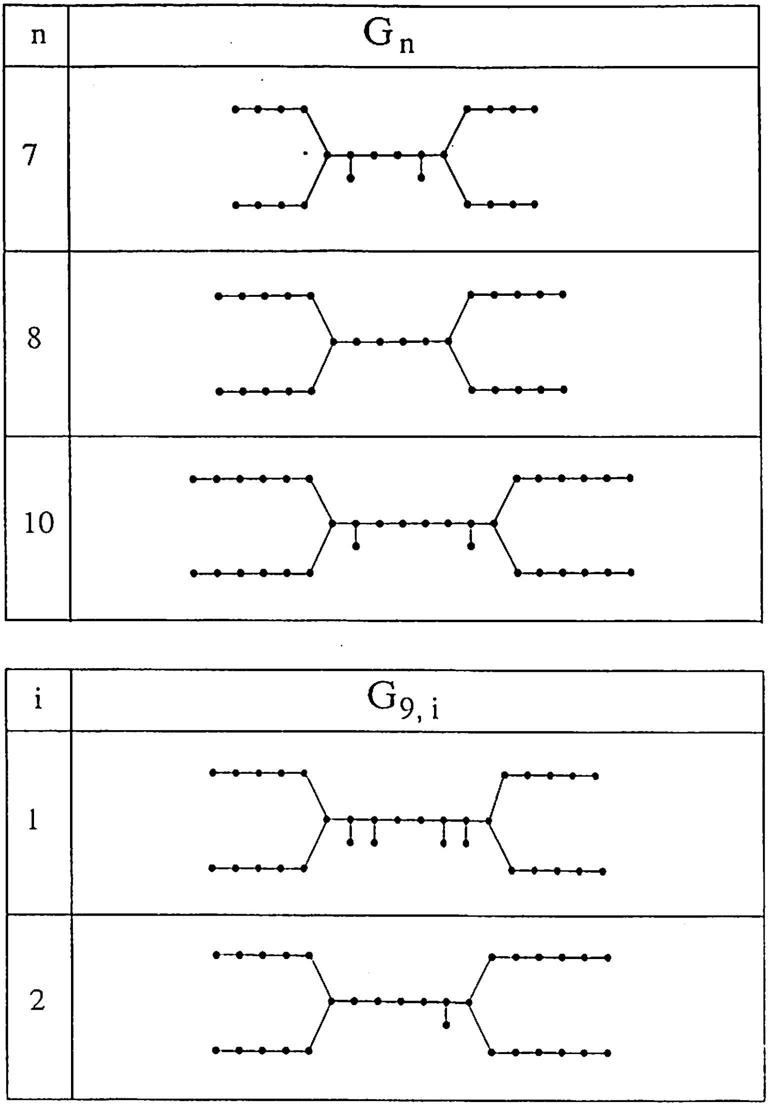


Figure 2: The graphs G_n for $n \in \{7, 8, 10\}$ and $G_{9,i}$ for $i \in \{1, 2\}$.

Proof: Let $n \geq 7$ be an integer and let G be a tree which is nth order regular of degree 2. Then (cf. Theorem C) the diameter of G is $2n - 1$. Let u and v be vertices of G with $d(u, v) = 2n - 1$, and let $P : u = u_0, u_1, u_2, \ldots, u_{2n-1} = v$ be the $u - v$ path in G. Necessarily u and v are end-vertices in G.

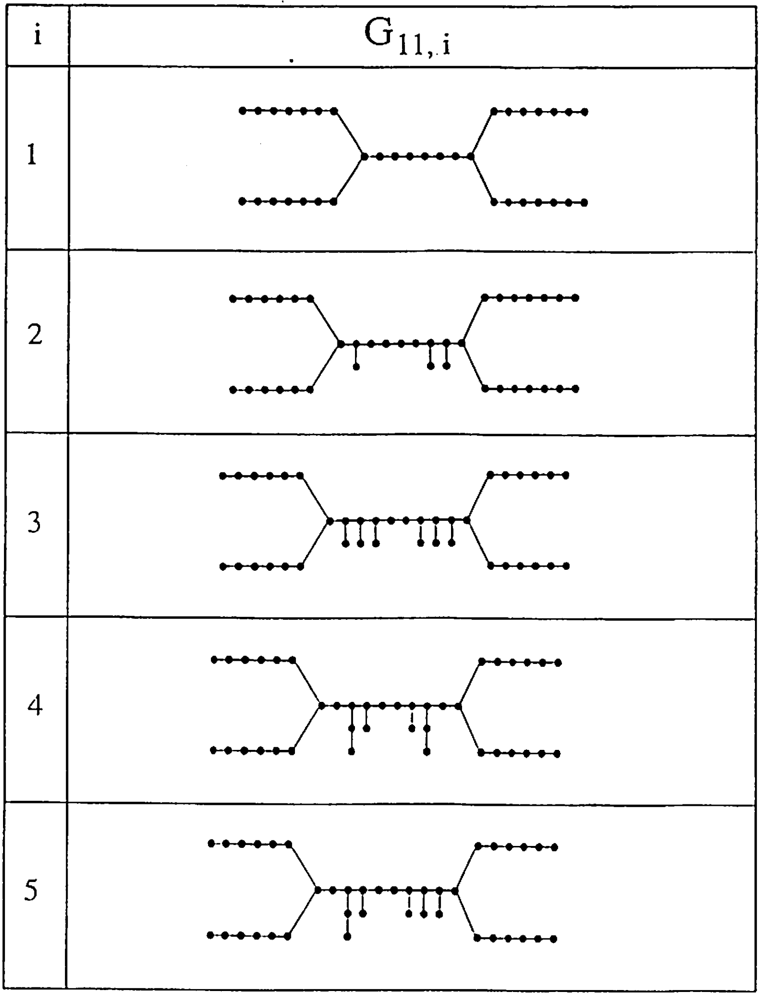


Figure 3: The graph $G_{11,i}$ for $i \in \{1, 2, \ldots, 5\}$.

Since G is nth order regular of degree 2, there exists a vertex w of G, distinct from u_n, that is at distance n from u. Let $Q : u = w_0, w_1, \ldots, w_n = w$ be the $u - w$ path in G. Further let r be the largest integer for which $u_r = w_r$. Lemma A implies that $\deg u_r = 3$. Since P is a longest path in G, it follows that $r \geq n/2$.

We show that $\frac{n}{2} < r < n$. If this is not the case, then $r = \frac{n}{2}$ and w is an end-vertex of G. Consider the vertex u_1. Let u_1^* denote the vertex of G, distinct from u_{n+1}, that is at distance n from u_1. Since P is a longest path in G, the edge $u_{\frac{n}{2}} u_{\frac{n}{2}+1}$ is necessarily an edge of the $u_1 - u_1^*$ path. Furthermore, u_n is a vertex of the $u_1 - u_1^*$ path, for if this were not the case, then $\deg_n u > 2$. Hence the $u_1 - u_1^*$ path is the path $u_1, u_2, \ldots, u_n, u_1^*$. This implies, however, that the three vertices u_1, w_{n-1} and u_{2n-1} are all at distance n from u_1^*, which produces a contradiction. We deduce, therefore, that $\frac{n}{2} < r < n$.

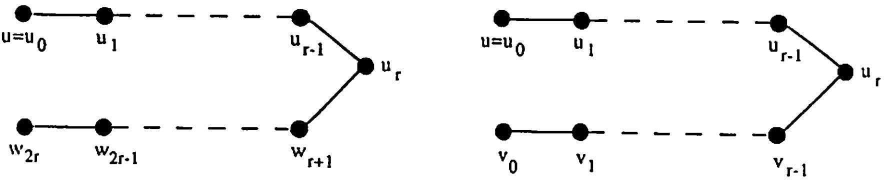
Next we observe that, for each i with $1 \leq i \leq n-r$, the vertex u_{n+r-i} is at distance n from each of the vertices u_{r-i} and w_{r+i} (note that $r-i \geq 2r-n > 0$). So, since G is nth order regular of degree 2, it follows that, for $2 \leq i \leq n-r$, $\deg u_{r-i+1} = 2 = \deg w_{r+i-1}$; or, equivalently, $\deg u_{r-i} = 2 = \deg w_{r+i}$ for $1 \leq i \leq n-r-1$.

We now consider the vertex u_{2r-n}. Since G is nth order regular of degree 2, there exists a vertex z of G, distinct from u_{2r}, that is at distance n from u_{2r-n}. Let R be the $u_{2r-n} - z$ path in G. Since P is a longest path in G, R contains the vertex u_r. We show that R contains the vertex w_{r+1}. If this is not the case, then R contains the vertex u_{r+1}. Since $d(u_r, u_{2r-n}) = n-r = d(u_r, w)$, the vertices u_{2r} and z are both at distance n from w in G. However the vertex u is also at distance n from w and so $\deg_n w > 2$, which produces a contradiction. Hence R contains the vertex w_{r+1}. Thus, since $\deg w_i = 2$ for $r+1 \leq i \leq n-1$, R contains the $w_{2r-n} - w_n$ path as a subpath. Let $R : u_{2r-n} = w_{2r-n}, w_{2r-n+1}, \ldots, w_n, w_{n+1}, \ldots, w_{2r} = z$ be the $u_{2r-n} - z$ path in G.

Since P is a longest path in G, $\deg w_{2r} = 1$. Observe that, for each i with $1 \leq i \leq r$, the vertices u_{r-i} and w_{r+i} are both at distance n from the vertex u_{n+r-i}. Hence since G is nth order regular of degree 2, it follows that $\deg u_{r-i} = 2 = \deg w_{r+i}$ for $1 \leq i \leq r-1$. Hence the component of $G - u_r u_{r+1}$ containing u_r is isomorphic to P_{2r+1}. For convenience, for each i with $0 \leq i \leq r-1$, let us relabel the vertex w_{2r-i} with v_i (see Figure 4).

Next we consider the vertex $v = u_{2n-1}$. Let y be the vertex, distinct from u_{n-1}, that is at distance n from v. Further let m be the largest integer for which the vertex u_{2n-1-m} is contained in the $v - y$ path in G. Then using a similar argument as that used to establish that the component of $G - u_r u_{r+1}$ containing u_r is isomorphic to P_{2r+1}, we may show that the component of $G - u_{2n-2-m} u_{2n-1-m}$ containing u_{2n-1-m} is isomorphic to P_{2m+1}. Let $v = u_{2n-1}, u_{2n-2}, \ldots, u_{2n-1-m}, v_{2n-m}, v_{2n-m+1}, \ldots, v_{2n-1} = y$ be the $v - y$ path in G.

We may assume, without loss of generality, that $m \geq r$. Now let $s = n-r$. Then, since $\frac{n}{2} < r < n$, $1 \leq s < \frac{n}{2}$ (and $n \geq 2s+1$). We show firstly that $s \geq 2$. If this is not the case, then $s = 1$ and $r = n-1$. Now for each i with $0 \leq i \leq n-2$, u_i is at distance n from u_{n+i} and v_{n-2-i}. It follows, therefore, that $\deg_n u_{n+i-1} = 2$ for $1 \leq i \leq n-2$. Furthermore, since $\deg_n u_{n-1} = 2$,it follows that $\deg u_{2n-2} = 3$. However, if $n > 2$, then u_{n-1} is the only vertex at



(a) (b)

Figure 4: The component of $G - u_r u_{r+1}$ containing u_r before (a) and after (b) relabelling.

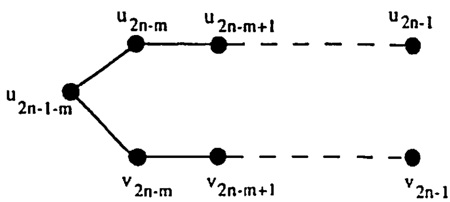


Figure 5: The component of $G - u_{2n-2-m} u_{2n-1-m}$ containing u_{2n-1-m}

distance n from u_{2n-1} in G. Since $n \geq 7$, this produces a contradiction. Hence $s \geq 2$ (and so $r = n - 2$).

Observe that for each i with $0 \leq i \leq r - s$, the vertex u_i is at distance n from u_{n+i} and v_{r-s-i}. It follows, therefore, that $\deg u_{n+i-1} = 2$ for $0 \leq i \leq r - s$. Hence, since $\deg u_{2n-1-m} = 3$, we have $n + r - s \leq 2n - 1 - m$; or, equivalently, $m \leq n + s - r - 1$. Thus $n - s = r \leq m \leq n + s - r - 1$ and so $r \leq 2s - 1$ (with equality if and only if $m = n + s - r - 1$). If $s = 2$, then $r \leq 3$; consequently, $r = 3$ and $n = 5$, which contradicts our choice of $n \geq 7$. Hence $s \geq 3$.

Suppose that $n = 7$. Then $s = 3$ and $r = 4$. Since $\deg u_{n+i-1} = 2$ for $0 \leq i \leq r - s$, we have $\deg u_6 = 2 = \deg u_7$. Furthermore, $r = 4 \leq m \leq 5$. We show that $m = 4$. If this is not the case, then $m = 5$ and $u_{2n-1-m} = u_8$. However the vertices u_3, v_3 and v_{13} are all at distance 7 from the vertex u_{10}, which produces a contradiction. Hence $m = 4$ and $u_{2n-1-m} = u_9$. Since $\deg_7 u_3 = 2$ and

$\deg_7 u_{11} = 2$, it follows that $\deg u_8 = 3$ and $\deg u_5 = 3$, respectively. Moreover, since the vertices u_{10} and v_{10} are both at distance 7 from u_3 and v_3, it follows that the vertex, distinct from u_4 and u_6, that is adjacent to u_5 is an end-vertex and the vertex distinct from u_7 and u_9, that is adjacent to u_8 is an end-vertex. Hence G is 7th order regular of degree 2 if and only if $G \cong G_7$.

Suppose that $n = 8$. Then $s = 3$ and $r = 5$. Since $r = 2s - 1$, $m = n + s - r - 1 = 5$ and so $u_{2n-1-m} = u_{10}$. Since $\deg u_{n+i-1} = 2$ for $0 \leq i \leq r - s$, we have $\deg u_7 = 2 = \deg u_8 = \deg u_9$. Moreover since the vertices u_5 and u_{15} are both at distance 8 from v_{13}, $\deg u_6 = 2$. Hence G is 8th order regular of degree 2 if and only if $G \cong G_8$.

Suppose that $n = 9$. Then $s = 4$ and $r = 5$, and so $\deg u_8 = \deg u_9 = 2$. Since $r \leq m \leq n + s - r - 1$, we have $5 \leq m \leq 7$. We show that $m \leq 6$. If this is not the case, then $m = 7$ and $u_{2n-1-m} = u_{10}$. However, the vertices u_4, v_4 and v_{15} are all at distance 9 from the vertex u_{13}, which produces a contradiction. Hence $m = 5$ or $m = 6$. If $m = 5$ $(m = 6)$, then it is straightforward to see that $G \cong G_{9,1}$ $(G \cong G_{9,2}$, respectively). Hence G is 9th order regular of degree 2 if and only if $G \cong G_{9,1}$ or $G \cong G_{9,2}$.

Suppose that $n = 10$. Then $s = 4$ and $r = 6$, and so $\deg u_9 = \deg u_{10} = \deg u_{11} = 2$. Furthermore, $6 \leq m \leq 7$. If $m = 7$, then $u_{2n-1-m} = u_{12}$. However the vertices u_5, v_5 and v_{19} are all at distance 10 from the vertex u_{15}, which produces a contradiction. Hence $m = 6$ and it is straightforward to see that $G \cong G_{10}$. Thus G is 10th order of degree 2 if and only if $G \cong G_{10}$.

Suppose that $n = 11$ Then either $s = 4$ and $r = 7$ or $s = 5$ and $r = 6$. Suppose firstly that $s = 4$ and $r = 7$. Then since $r = 2s - 1$, $m = n + s - r - 1 = 7$ and so $u_{2n-1-m} = u_{14}$. Moreover $\deg u_{10} = \deg u_{11} = \deg u_{12} = \deg u_{13} = 2$, and since u_7 and u_{21} (u_8 and u_{20}) are both at distance 2 from v_{18} (v_{19}, respectively), $\deg u_8 = 2 = \deg u_9$. Hence in this case $G \cong G_{11,1}$. Suppose then that $s = 5$ and $r = 6$. Then $\deg u_{10} = \deg u_{11} = 2$. Since $r \leq m \leq n + s - r - 1$, we have $6 \leq m \leq 9$. If $m = 8$ or $m = 9$, then $\deg_{11} u_{16} > 2$, which produces a contradiction. Hence $m = 6$ or $m = 7$. If $m = 6$ $(m = 7)$, then it is straightforward to see that G is isomorphic to $G_{11,3}$ or $G_{11,4}$ or $G_{11,5}$ $(G \cong G_{11,2}$, respectively). Hence G is 11th order regular of degree 2 if and only if $G \cong G_{11,i}$ where $i \in \{1, 2, \ldots, 5\}$.

This completes the proof of the theorem. ∎

4. The diameter of nth order regular trees of degree k for small k

The following conjecture is given in [6].

Conjecture 1. *For $n \geq 2$ and $k \geq 1$, if a tree G is nth order regular of degree k, then diam $G = 2n - 1$.*

For $k = 1$ and $k = 2$ Conjecture 1 is true (cf. Corollary A and Theorem C). In this section we prove the conjecture for $k = 3$, $k = 4$ and $k = 5$.

Theorem 2. *For $n \geq 2$, if G is a tree which is nth order regular of degree 3, then the diameter of G is $2n-1$.*

Proof: By Theorem A the diameter of G is at least $2n-1$; so it remains to be shown that the diameter of G is at most $2n-1$. Assume that diam $G = m$ where $m > 2n-1$. Among all the vertices of G with eccentricity equal to m, let u be one for which the number of vertices that the three paths connecting u to the three vertices at distance n from u have in common is a minimum. Let v be a vertex at distance m from u and let $P : u = u_0, u_1, \ldots, u_m = v$ be the $u-v$ path in G. Necessarily u and v are end-vertices in G.

Since G is nth order regular of degree 3, there exist vertices x and y of G, distinct from u_n, that are at distance n from u. Let $u = x_0, x_1, \ldots, x_n = x$ and $u = y_0, y_1, \ldots, y_n = y$ be the $u-x$ path and the $u-y$ path, respectively. Further, let $s(r)$ be the largest integer for which $u_s = x_s (u_r = y_r$, respectively). We may assume, without loss of generality, that $s \leq r (< n)$. Since P is a longest path in G, it follows that $s \geq n/2$ since if $s < n/2$, then the $x_s - v$ section of P together with the $x_s - x_n$ section would give a path of length greater than that of P. We consider two cases.

Case 1: Suppose that $s = n/2$. Then u, x and u_{2n} are all at distance n from u_n in G. Hence, since $\deg_n u_n = 3$, $r > n/2$ (and so $0 < 2r - n < r < n$). We now consider the vertex u_{2r-n}. Let w be a vertex, distinct from u_{2r}, that is at distance n from u_{2r-n}. Further, let Q be the $u_{2r-n} - w$ path in G. Since P is a longest path in G, it follows that Q contains the vertex u_r. The vertex u_{r+1} is also contained in Q, for if this were not the case, then w would be at distance n from u_n in G, which contradicts the fact that $\deg_n u_n = 3$. We observe, therefore, that the path Q does not contain the vertices x and y. Since x, y and u_n are the only vertices at distance n from u in G, the path Q must contain the vertex u_n. We deduce, therefore, that the three vertices at distance n from u_{2r-n} are all contained in the component of $G - u_{n-1} u_n$ that contains the vertex u_n. Hence these three vertices are all at distance n from the vertex y in G. This, together with the fact that u is also at distance n from y, implies that $\deg_n y > 3$, producing a contradiction. Hence Case 1 cannot occur.

Case 2: Suppose that $s > n/2$. Then $0 < 2s - n < s < n$. We now consider the vertex u_{2s-n}. Since P is a longest path in G, it follows that the three paths connecting u_{2s-n} to the three vertices at distance n from u_{2s-n} all contain the vertex u_s. Furthermore at least one of these three paths must contain the vertex x_{s+1}, for if this were not the case, then $\deg_n x > 3$, which produces a contradiction. On the other hand at least two of these paths must contain the vertex u_{s+1}, for if this were not the case, then $\deg_n u_n > 3$, a contradiction. We deduce, therefore, that if $z_{(1)}, z_{(2)}$ and $z_{(3)}$ denote the three vertices at distance n from u_{2s-n} in G, then the $u_{2s-n} - z_{(1)}$ path (say) contains the vertex x_{s+1}, while the $u_{2s-n} - z_{(2)}$ path and the $u_{2s-n} - z_{(3)}$ path both contain the vertex u_{s+1}.

Using a similar argument as in the preceeding paragraph, if $w_{(1)}, w_{(2)}$ and $w_{(3)}$ denote the three vertices at distance n from u_{2r-n} in G, then we may deduce that the $u_{2r-n} - w_{(1)}$ path (say) contains the vertex y_{r+1}, while the $u_{2r-n} - w_{(2)}$ path and the $u_{2r-n} - w_{(3)}$ path both contain the vertex u_{r+1}.

We observe that the vertices $u, u_{2n}, z_{(1)}$ and $z_{(2)}$ are all at distance n from the vertex u_n in G. This produces a contradiction unless $z_{(1)} = w_{(1)}$. This implies, in particular, that $s = r$ and $x_{s+1} = y_{s+1}$. Furthermore, the vertices u_{2s-n}, x and y are therefore all at distance n from u_{2s} in G.

We now consider the vertex $z_{(1)}(= w_{(1)})$. Let z be the vertex, distinct from u_{2s-n} and u_n, that is at distance n from $z_{(1)}$ in G. By our choice of the vertex u, the $z_{(1)} - z$ path must contain the $z_{(1)} - u_s$ path. Now if the $z_{(1)} - z$ path does not contain the vertex $u_{s+1}(u_{s-1})$, then it follows that $\deg_n u_{2s} > 3$ ($\deg_n u > 3$, respectively), which produces a contradiction. Hence Case 2 cannot occur.

Since both Case 1 and Case 2 produce a contradiction, we deduce that our assumption that the diameter of G is greater than $2n-1$ is incorrect. This completes the proof of the theorem. ∎

Theorem 3. *For $n \geq 2$, if G is a tree which is nth order regular of degree 4, then the diameter of G is $2n-1$.*

Proof: By Theorem A the diameter of G is at least $2n-1$; so it remains to be shown that the diameter of G is at most $2n-1$. Assume that diam $G = m$ where $m > 2n-1$. Among all the vertices of G with eccentricity equal to m, let u be one for which the number of vertices that the four paths connecting u to the four vertices at distance m from u have in common is a minimum. Let v be a vertex at distance m from u and let $P : u = u_0, u_1, \ldots, u_m = v$ be the $u - v$ path in G. Necessarily u and v are end-vertices in G.

Since G is nth order regular of degree 4, there exist vertices x, y and w of G, distinct from u_n, that are at distance n from u. Let $u = x_0, x_1, \ldots, x_n = x$, $u = y_0, y_1, \ldots, y_n = y$ and $u = w_0, w_1, \ldots, w_n = w$ be the $u - x, u - y$ and $u - w$ paths, respectively. Further, let s, r and t denote the largest integers for which $u_s = x_s, u_r = y_r$ and $u_t = w_t$, respectively. We may assume, without loss of generality, that $s \leq r \leq t (< n)$. Since P is a longest path in G, it follows that $s \geq \frac{n}{2}$. We consider three cases.

Case 1 Suppose that $s = n/2$. Then u, x and u_{2n} are all at distance n from u_n in G. If $s = t$, then the vertices y and w are also at distance n from u_n, which produces a contradiction. Hence $s < t$ (and so $0 < 2t-n < t < n$). Observe that the vertex x is therefore at distance n from w. We now consider the vertex u_{2t-n}. Since P is a longest path in G, it follows that the four paths connecting u_{2t-n} to the four vertices at distance n from u_{2t-n} all contain the vertex u_t. Furthermore at least two of these four paths must contain the vertex w_{t+1}, for if this were not the case, then $\deg_n w > 4$, which produces a contradiction. However we now arrive at a contradiction since $\deg_n u_n > 4$. Hence Case 1 cannot occur.

Case 2 Suppose that $n/2 < s < t$. Then $0 < 2s - n < s < t < n$. We now consider the vertex u_{2s-n}. Necessarily, the four paths connecting u_{2s-n} to the four vertices at distance n from u_{2s-n} all contain the vertex u_s. Furthermore at least one of these four paths must contain the vertex x_{s+1}, for otherwise, if this were not the case, then $\deg_n x > 4$, which produces a contradiction. Let $x_{(1)}$ denote the vertex at distance n from u_{2s-n} such that the $u_{2s-n} - x_{(1)}$ path contains the vertex x_{s+1}. Observe that the vertices w and u_n are both at distance n from $x_{(1)}$ in G.

We now consider the vertex u_{2t-n}. Since P is a longest path in G, it follows that the four paths connecting u_{2t-n} to the four vertices at distance n from u_{2t-n} all contain the vertex u_t. Furthermore at least two of these four paths must contain the vertex w_{t+1}, for if this were not the case, then $\deg_n w > 4$, which produces a contradiction. However we now arrive at a contradiciton since $\deg_n u_n > 4$. Hence Case 2 cannot occur.

Case 3 Suppose that $n/2 < s = t$. Then $0 < 2s-n < s = r = t < n$. We now consider the vertex u_{2s-n}. Let $x_{(1)}, y_{(1)}$ and $w_{(1)}$ denote the vertices at distance n from u_{2s-n} such that the $u_{2s-n} - x_{(1)}$, $u_{2s-n} - y_{(1)}$ and the $u_{2s-n} - w_{(1)}$ paths, respectively, contain the respective vertices x_{s+1}, y_{s+1} and w_{s+1}. Observe that the vertices $x_{(1)}, y_{(1)}$ and $w_{(1)}$ are all at distance n from u_n in G. Since $\deg_n u_n = 4$, it follows, therefore, that the vertices $x_{(1)}, y_{(1)}$ and $w_{(1)}$ are not all distinct. We may assume, without loss of generality, that $x_{(1)} = y_{(1)}$. In particular, this implies that $x_{s+1} = y_{s+1}$.

We now consider the vertex $x_{(1)}$. By our choice of the vertex u, the four paths connecting $x_{(1)}$ to the four vertices at distance n from $x_{(1)}$ all contain the $x_{(1)} - u_s$ path. Let Q_1 and Q_2 denote the two paths connecting $x_{(1)}$ to the two vertices, distinct from u_{2s-n} and u_n, at distance n from $x_{(1)}$ in G. If neither of the paths Q_1 or Q_2 contain the vertex u_{s+1} (u_{s-1}), then it follows that $\deg_n u_{2s} > 4$ ($\deg_n u > 4$, respectively), which produces a contradiction. Hence we may assume, without loss of generality, that the path Q_1 contains the vertex u_{s-1}, while the path Q_2 contains the vertex u_{s+1}. However the vertex at distance n from $x_{(1)}$ on the path Q_1, together with the four vertices u_{2s-n}, x, y and w, are all at distance n from u_{2s-n} in G, which produces a contradiction. Hence Case 3 cannot occur.

Since Cases 1, 2 and 3 all produce a contradiction, we deduce that our assumption that the diameter of G is greater than $2n - 1$ is incorrect. This completes the proof of the theorem. ∎

One may also show that if G is a tree which is nth order regular of degree 5, then diam G $= 2n - 1$. We omit the proof (a copy of which is available from the author).

Theorem 4. *For $n \geq 2$, if G is a tree which is nth order regular of degree 5, then the diameter of G is $2n - 1$.*

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