Journal of Combinatorial Mathematics and Combinatorial Computing, 122: 301–316
DOI:10.61091/jcmcc122-25
http://www.combinatorialpress.com/jcmcc
Received 11 May 2024, Accepted 16 Septermber 2024, Published 30 September 2024

Article

Decomposition of Complete Graphs into Paths and Stars with Different Number of Edges

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Abstract: Let P_n and K_n respectively denote a path and complete graph on n vertices. By a $\{pH_1, qH_2\}$-decomposition of a graph G, we mean a decomposition of G into p copies of H_1 and q copies of H_2 for any admissible pair of nonnegative integers p and q, where H_1 and H_2 are subgraphs of G. In this paper, we show that for any admissible pair of nonnegative integers p and q, and positive integer $n \geq 4$, there exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p + 4q = \binom{n}{2}$, where S_4 is a star with 4 edges.

Keywords: Graph decomposition, Paths, Stars, Complete graph

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1. Introduction

All graphs considered here are finite. Let K_k denote a complete graph on k vertices. Let P_{k+1}, C_k and $S_k(\cong K_{1,k})$ respectively denote a path, cycle and star each having k edges. Further, we denote a path on $k+1$ vertices $x_1, x_2, \ldots, x_{k+1}$, and edges $x_1x_2, \ldots, x_kx_{k+1}$ by $[x_1 \ldots x_kx_{k+1}]$. If there are $t \geq 1$ stars with same end vertices $x_1, x_2, \ldots, x_k$ and different centers $y_1, y_2, \ldots, y_t$, we denote it by $(y_1, y_2, \ldots, y_t; x_1, x_2, \ldots, x_k)$. Let $\mathbb{Z}_+$ be the set of all positive integers. When $x, y \in \mathbb{Z}$, we define $\lfloor x \rfloor = \max\{y | y \in \mathbb{Z},\ y \leq x\}$ and $\lceil x \rceil = \min\{y | y \in \mathbb{Z},\ y \geq x\}$.

A *decomposition* of a graph G is a partition of G into edge-disjoint subgraphs of G. If the subgraphs in the decomposition are isomorphic to either a graph H_1 or a graph H_2, then it is called a $\{H_1, H_2\}$*-decomposition* of G. We say that G has a $\{pH_1, qH_2\}$*-decomposition* of G if the decomposition contains p copies of H_1 and q copies of H_2 for all possible choices of p and q. Different problems on graph decomposition have been studied for a century. In particular, the problem of decomposing a complete graph into cycles is the center of attraction of many of these studies (e.g., the work of Alspach and Gavlas [1] and its references).

The study of $\{H_1, H_2\}$-decomposition has been introduced by Abueida and Daven [2, 3]. Moreover, Abueida and O'Neil [4] have settled the existence of $\{H_1, H_2\}$-decomposition of λK_m, when $\{H_1, H_2\} = \{K_{1,n-1}, C_n\}$ for $n = 3, 4, 5$. Priyadharsini and Muthusamy [5] gave necessary and sufficient condition for the existence of $\{G_n, H_n\}$-factorization of λK_n, where $G_n, H_n \in \{C_n, P_n, S_{n-1}\}$. Many other results on decomposition of graphs into distinct subgraphs involving paths, cycles or stars have been proved in [6–9]. Recently, Fu, et al. [10] have found the necessary and sufficient conditions for the existence of decomposition of K_n into cycles and stars on four vertices. In this paper, we obtain necessary and sufficient conditions for the existence of a $\{pP_4, qS_4\}$-decomposition of K_n.

Let $M(G)$ denote the set of all pairs (p, q) such that there exists a $\{pP_4, qS_4\}$-decomposition of G and we define the set $I(n)$ in Table 1 which help us to show that $M(K_n) = I(n)$ for all feasible values of n.

n	$I(n)$
$0, 1, 3, 4 \pmod 6$	$\left\{(p, q) \mid p = \frac{n(n-1)}{6} - 4i, q = \frac{n(n-1)}{8} - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{n(n-1)}{24} \right\rfloor \right\}$
$2, 5 \pmod 6$	$\left\{(p, q) \mid p = \frac{n(n-1)-8}{6} - 4i, q = \frac{n(n-1)}{8} - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{n(n-1)-8}{24} \right\rfloor \right\}$

Table 1. The Set $I(n)$

Remark 1. *Let $A + B = \{(x_1 + y_1, x_2 + y_2) \mid (x_1, x_2) \in A,\ (y_1, y_2) \in B\}$ and rA be the sum of r copies of A. If $G = G_1 \oplus G_2$, where $\oplus$ denotes edge disjoint sum of the subgraphs G_1 and G_2, then $M(G) \supseteq M(G_1) + M(G_2)$.*

To prove our main result we state some known results as follows.

Theorem 1. [11] *Let $k, n \in \mathbb{Z}_+$. Then K_n has a P_{k+1}-decomposition if and only if $n \ge k + 1$ and $n(n-1) \equiv 0 \pmod{2k}$.*

Theorem 2. [12, 13] *Let $n, k \in \mathbb{Z}_+$. Then K_n has a S_k-decomposition if and only if $2k \le n$ and $n(n-1) \equiv 0 \pmod{2k}$.*

Theorem 3. [13] *Let $m, n \in \mathbb{Z}_+$ with $m \le n$. Then $K_{m,n}$ has an S_k-decomposition if and only if one of the following holds:*

1. *$m \ge k$ and $mn \equiv 0 \pmod k$;*
2. *$m < k \le n$ and $n \equiv 0 \pmod k$.*

2. Base Constructions

In this section, we provide some useful lemmas which are required in proving our main result. The proof of the Lemmas 1 to 10, are given in the Appendix.

Lemma 1. *There exists a $\{pP_4, qS_4\}$-decomposition of $K_{m,6}$, when $m = 2, 4, 6$.*

Proof. **Case 1.** For $m = 2$.

Let $V(K_{2,6}) = (X_1, X_2)$, where $X_1 = \{x_{1,1}, x_{1,2}\}$ and $X_2 = \{x_{2,i} \mid 1 \le i \le 6\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of $K_{2,6}$ for $p = 4$ and $q = 0$ as

$$[x_{1,1}x_{2,1}x_{1,2}x_{2,2}], [x_{1,1}x_{2,3}x_{1,2}x_{2,4}], [x_{2,2}x_{1,1}x_{2,6}x_{1,2}], [x_{1,2}x_{2,5}x_{1,1}x_{2,4}].$$

Hence, $M(K_{2,6}) = (4, 0)$.

Case 2. For $m = 4$.

Let $V(K_{4,6}) = (X_1, X_2)$, where $X_1 = \{x_{1,i} \mid 1 \le i \le 4\}$ and $X_2 = \{x_{2,i} \mid 1 \le i \le 6\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of $K_{4,6}$ as follows:

1. For $p = 0$ and $q = 6$:
   By Theorem 3, we get the required stars.
2. For $p = 4$ and $q = 3$:
   $[x_{1,1}x_{2,1}x_{1,2}x_{2,2}], [x_{1,2}x_{2,3}x_{1,1}x_{2,2}], [x_{1,3}x_{2,1}x_{1,4}x_{2,2}], [x_{1,4}x_{2,2}x_{1,3}x_{2,3}], (x_{2,4}, x_{2,5}, x_{2,6}; x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4})$.

3. For $p = 8$ and $q = 0$:
   The $4P_4$ along with $[x_{1,1}x_{2,4}x_{1,2}x_{2,5}]$, $[x_{1,2}x_{2,6}x_{1,1}x_{2,5}]$, $[x_{1,3}x_{2,4}x_{1,4}x_{2,5}]$, $[x_{1,4}x_{2,5}x_{1,3}x_{2,6}]$ gives the required paths.

Hence, $M(K_{4,6}) = \{(0,6),(4,3),(8,0)\}$.

Case 3. For $m = 6$.

We can write $K_{6,6} = K_{2,6} \oplus K_{4,6}$. Then $M(K_{6,6}) \supseteq M(K_{2,6}) + M(K_{4,6}) \supseteq (4,0) + \{(0,6),(4,3),(8,0)\} = \{(4,6),(8,3),(12,0)\}$. By Theorem 3, we get $9S_4$. Hence $M(K_{6,6}) = \{(0,9),(4,6),(8,3),(12,0)\}$.

□

Lemma 2. *There exists a $\{pP_4, qS_4\}$-decomposition of K_5.*

Proof. From the definition of $I(n)$, we get $I(5) = (2,1)$. Let $V(K_5) = \{x_i \mid 1 \leq i \leq 5\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_5 for $p = 2$ and $q = 1$ as $[x_2x_4x_3x_5]$, $[x_3x_2x_5x_4]$, $(x_1; x_2, x_3, x_4, x_5)$. Hence, $M(K_5) = I(5) = (2,1)$. □

Lemma 3. *There exists a $\{pP_4, qS_4\}$-decomposition of K_6.*

Proof. From the definition of $I(n)$, we get $I(6) = \{(1,3),(5,0)\}$. Let $V(K_6) = \{x_i \mid 1 \leq i \leq 6\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_6 as follows:

1. $(1,3)$: Let D be an arbitrary $\{pP_4, qS_4\}$-decomposition of K_6. Suppose that $p = 1$ and let $P_4^1 = [x_1x_2x_3x_4]$ be the only P_4 in D. By our assumption $H_1 = K_6 - E(P_4^1)$ has an S_4-decomposition. Let $d(x_i)$ is degree of x_i. In H_1 , $d(x_1) = d(x_4) = 4$, $d(x_2) = d(x_3) = 3$ and $d(x_5) = d(x_6) = 5$. It follows that, any three of $\{x_1, x_4, x_5, x_6\}$ must be a center vertex of stars in the decomposition D. Let $S_4^1 = (x_1; x_3, x_4, x_5, x_6)$ be a star in H_1. Then $H_2 = H_1 - E(S_4^1)$, we have $d(x_1) = 0$, $d(x_2) = d(x_4) = 3$, $d(x_5) = d(x_6) = 4$ and $d(x_3) = 2$. It follows that x_5 and x_6 must be center vertices of stars in the decomposition D. Let $S_4^2 = (x_5; x_2, x_3, x_4, x_6)$ in H_2. Then $H_3 = H_2 - E(S_4^2)$, we have $d(x_1) = d(x_5) = 0$, $d(x_2) = d(x_4) = 2$, $d(x_3) = 1$ and $d(x_6) = 3$. Hence H_3 can not have a S_4-decomposition, which is a contradiction. Hence $(p,q) \neq (1,3)$.
2. $(5,0)$: By Theorem 1, we get the required paths.

Hence, $M(K_6) = I(6) = (5,0)$. □

Lemma 4. *There exists a $\{pP_4, qS_4\}$-decomposition of K_7.*

Proof. From the definition of $I(n)$, we get $I(7) = \{(3,3),(7,0)\}$. Let $V(K_7) = \{x_i \mid 1 \leq i \leq 7\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_7 as follows:

1. For $p = 3$ and $q = 3$:
   $[x_1x_2x_3x_4]$, $[x_4x_5x_6x_7]$, $[x_5x_7x_4x_6]$, $(x_3; x_1, x_5, x_6, x_7)$, $(x_1, x_2; x_4, x_5, x_6, x_7)$.
2. For $p = 7$ and $q = 0$:
   By Theorem 1, we get the required paths.

Hence, $M(K_7) = I(7) = \{(3,3),(7,0)\}$. □

Lemma 5. *There exists a $\{pP_4, qS_4\}$-decomposition of K_8.*

Proof. From the definition of $I(n)$, we get $I(8) = \{(0,7),(4,4),(8,1)\}$. Let $V(K_8) = \{x_i \mid 1 \leq i \leq 8\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_8 as follows:

1. For $p = 0$ and $q = 7$:
   By Theorem 2, we get the required stars.

2. $p = 4$ and $q = 4$:
   $[x_1x_2x_8x_3], [x_2x_5x_3x_4], [x_6x_3x_1x_8], [x_1x_4x_2x_3], (x_4; x_5, x_6, x_7, x_8), (x_5; x_1, x_6, x_7, x_8), (x_6; x_1, x_2, x_7, x_8), (x_7; x_1, x_2, x_3, x_8)$.
3. For $p = 8$ and $q = 1$:
   The $4P_4$ along with $[x_1x_5x_4x_8], [x_2x_6x_7x_4], [x_1x_6x_5x_7]$, $[x_4x_6x_8x_5]$ gives the required paths and the $1S_4$ is $(x_7; x_1, x_2, x_3, x_8)$.

Hence, $M(K_8) = I(8) = \{(0, 7), (4, 4), (8, 1)\}$. □

Lemma 6. *There exists a $\{pP_4, qS_4\}$-decomposition of K_9.*

Proof. From the definition of $I(n)$, we get $I(9) = \{(0, 9), (4, 6), (8, 3), (12, 0)\}$. Let $V(K_9) = \{x_i \mid 1 \leq i \leq 9\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_9 as follows:

1. For $p = 0$ and $q = 9$:
   By Theorem 2, we get the required stars.
2. For $p = 4$ and $q = 6$:
   $[x_1x_2x_5x_4], [x_2x_4x_1x_5], [x_6x_8x_7x_9], [x_7x_6x_9x_8], (x_3; x_1, x_2, x_4, x_5)$,
   $(x_1, x_2, x_3, x_4, x_5; x_6, x_7, x_8, x_9)$.
3. For $p = 8$ and $q = 3$:
   The $4P_4$ with $[x_1x_6x_2x_9], [x_1x_7x_3x_6], [x_3x_9x_1x_8]$, $[x_3x_8x_2x_7]$ gives the required paths and $3S_4$ are $(x_3; x_1, x_2, x_4, x_5)$, $(x_4, x_5; x_6, x_7, x_8, x_9)$.
4. $p = 12$ and $q = 0$:
   By Theorem 1, we get the required paths.

Hence, $M(K_9) = I(9) = \{(0, 9), (4, 6), (8, 3), (12, 0)\}$. □

Lemma 7. *There exists a $\{pP_4, qS_4\}$-decomposition of K_{10}.*

Proof. From the definition of $I(n)$, we get $I(10) = \{(3, 9), (7, 6), (11, 3), (15, 0)\}$. Let $V(K_{10}) = \{x_i \mid 1 \leq i \leq 10\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_{10} as follows:

1. For $p = 3$ and $q = 9$:
   $[x_1x_2x_3x_4], [x_2x_4x_1x_3], [x_4x_5x_6x_7], (x_5; x_1, x_2, x_3, x_7), (x_6, x_7, x_8, x_9, x_{10}; x_1, x_2, x_3, x_4), (x_8; x_5, x_6, x_7, x_9)$, $(x_9; x_5, x_6, x_7, x_{10}), (x_{10}; x_5, x_6, x_7, x_8)$.
2. For $p = 7$ and $q = 6$:
   The $3P_4$ along with $[x_1x_{10}x_2x_8], [x_3x_9x_1x_8], [x_2x_9x_4x_{10}]$, $[x_4x_8x_3x_{10}]$ gives the required paths and $6S_4$ are $(x_6, x_7; x_1, x_2, x_3, x_4)$, $(x_5; x_1, x_2, x_3, x_7), (x_8; x_5, x_6, x_7, x_9), (x_9; x_5, x_6, x_7, x_{10})$, $(x_{10}; x_5, x_6, x_7, x_8)$.
3. For $p = 11$ and $q = 3$:
   The $7P_4$ along with $[x_1x_5x_2x_6], [x_1x_7x_3x_6], [x_1x_6x_4x_7]$, $[x_2x_7x_5x_3]$ gives the required paths and $3S_4$ are $(x_8; x_5, x_6, x_7, x_9)$, $(x_9; x_5, x_6, x_7, x_{10})$, $(x_{10}; x_5, x_6, x_7, x_8)$.
4. For $p = 15$ and $q = 0$:
   By Theorem 1, we get the required paths.

Hence, $M(K_{10}) = I(10) = \{(3, 9), (7, 6), (11, 3), (15, 0)\}$. □

Lemma 8. *There exists a $\{pP_4, qS_4\}$-decomposition of K_{11}.*

Proof. From the definition of $I(n)$, we get $I(11) = \{(1, 13), (5, 10), (9, 7), (13, 4), (17, 1)\}$. Let $V(K_{11}) = \{x_i \mid 1 \leq i \leq 11\}$. We exhibit the $\{pP_4, qS_4\}$-decomposition of K_{11} as follows:

1. For $p = 1$ and $q = 13$:
   $[x_3x_1x_{10}x_9], (x_1, x_2; x_4, x_5, x_6, x_7), (x_3, x_4, x_5, x_7; x_8, x_9, x_{10}, x_{11}), (x_1; x_2, x_8, x_9, x_{11}), (x_2; x_3, x_9, x_{10}, x_{11}), (x_3; x_1, x_5, x_6, x_7), (x_4; x_1, x_2, x_3, x_5), (x_6; x_5, x_7, x_9, x_{10}), (x_8; x_2, x_6, x_9, x_{10}), (x_{11}; x_6, x_8, x_9, x_{10})$.

2. For $p = 5$ and $q = 10$:
   $[x_1x_2x_3x_4], [x_4x_5x_6x_7], [x_5x_7x_4x_6], [x_8x_9x_{10}x_{11}], [x_9x_{11}x_8x_{10}], (x_1, x_2, x_3, x_4, x_5, x_6, x_7; x_8, x_9, x_{10}, x_{11}), (x_1, x_2; x_4, x_5, x_6, x_7), (x_3; x_1, x_5, x_6, x_7)$.
3. For $p = 9$ and $q = 7$:
   The $5P_4$ along with $[x_2x_8x_3x_9], [x_2x_{10}x_3x_{11}], [x_2x_{11}x_1x_{10}], [x_2x_9x_1x_8]$ gives the required paths and last $7S_4$ gives the required stars.
4. For $p = 13$ and $q = 4$:
   The $9P_4$ along with $[x_5x_8x_6x_9], [x_5x_{10}x_6x_{11}], [x_5x_9x_4x_8], [x_5x_{11}x_4x_{10}]$ gives the required paths and $4S_4$ are $(x_1, x_2; x_4, x_5, x_6, x_7), (x_3; x_1, x_5, x_6, x_7), (x_7; x_8, x_9, x_{10}, x_{11})$.
5. For $p = 17$ and $q = 1$:
   The $13P_4$ along with $[x_5x_1x_3x_7], [x_1x_7x_2x_6], [x_3x_5x_2x_4], [x_3x_6x_1x_4]$ gives the required paths and the $1S_4$ is $(x_7; x_8, x_9, x_{10}, x_{11})$.

Hence, $M(K_{11}) = I(11) = \{(1, 13), (5, 10), (9, 7), (13, 4), (17, 1)\}$. □

Lemma 9. *There exists a $\{pP_4, qS_4\}$-decomposition of K_{12}.*

Proof. From the definition of $I(n)$, we get $I(12) = \{(2, 15), (6, 12), (10, 9), (14, 6), (18, 3), (22, 0)\}$. We can write $K_{12} = 2K_6 \oplus K_{6,6}$. By Remark 1, and Lemmas 1, 3, we have $M(K_{12}) \supseteq 2M(K_6) + M(K_{6,6}) \supseteq (10, 0) + \{(0, 9), (4, 6), (8, 3), (12, 0)\} = \{(10, 9), (14, 6), (18, 3), (22, 0)\} = I(12) - \{(2, 15), (6, 12)\}$. We can write $K_{12} = K_4 \oplus K_8 \oplus K_{4,8}$. Then by Theorems 1 and 3, the graphs K_4 and $K_{4,8}$ have $2P_4$ and $8S_4$ respectively, and by Lemma 5 the graph K_8 has a decomposition for the case $(p, q) \in \{(0, 7), (4, 4)\}$. Hence $M(K_{12}) = I(12) = \{(2, 15), (6, 12), (10, 9), (14, 6), (18, 3), (22, 0)\}$. □

Lemma 10. *There exists a $\{pP_4, qS_4\}$-decomposition of K_{14}.*

Proof. From the definition of $I(n)$, we get $I(14) = \{(1, 22), (5, 19), \ldots, (29, 1)\}$. We can write $K_{14} = K_8 \oplus K_6 \oplus 2K_{4,6}$. Then by Remark 1, and Lemmas 1, 3 and 5, we have $M(K_{14}) \supseteq M(K_8) + M(K_6) + 2M(K_{4,6}) = \{(0, 7), (4, 4), (8, 1)\} + (5, 0) + 2\{(0, 6), (4, 3), (8, 0)\} = \{(5, 19), (9, 16), \ldots, (29, 1)\} = I(14) - (1, 22)$. Let $V(K_{14}) = \{x_i \mid 1 \leq i \leq 14\}$. Then the required decomposition for the case $(p, q) = (1, 22)$ is given as follows: $[x_7, x_6, x_{14}, x_{11}], (x_1; x_2, x_{11}, x_{12}, x_{14}), (x_3; x_1, x_2, x_{11}, x_{14}), (x_4; x_1, x_2, x_3, x_5), (x_5; x_1, x_2, x_3, x_7), (x_6; x_5, x_{11}, x_{12}, x_{13}), (x_8; x_5, x_6, x_7, x_9), (x_9; x_5, x_6, x_7, x_{10}), (x_{10}; x_5, x_6, x_7, x_8), (x_{12}; x_3, x_{11}, x_{13}, x_{14}), (x_{13}; x_1, x_3, x_{11}, x_{14}), (x_2, x_4, x_5, x_7, x_8, x_9, x_{10}; x_{11}, x_{12}, x_{13}, x_{14}), (x_6, x_7, x_8, x_9, x_{10}; x_1, x_2, x_3, x_4)$.

Hence, $M(K_{14}) = I(14) = \{(1, 22), \ldots, (29, 1)\}$. □

3. Main Result

In this section, we prove that K_n can be decomposed into p copies of P_4 and q copies of S_4 for all positive integer $n \geq 4$.

Lemma 11. *Let $p, q \in \mathbb{Z}_+ \cup \{0\}$ and $n \equiv 0 \pmod 6$. There exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p + 4q = \binom{n}{2}$, and $n \geq 6$. That is, $M(K_{6s}) = I(6s)$, where $s \in \mathbb{Z}_+$.*

Proof. Necessity: The conditions $3p+4q = \binom{n}{2}$ and $n \geq 6$ are trivial. That is, $M(K_{6s}) \subseteq I(6s)$. Sufficiency: We have to prove $M(K_{6s}) \supseteq I(6s)$. The proof is by induction on s. If $s = 1$, then $M(K_6) = I(6)$, by Lemma 3. Since $K_{6k+6} = K_{6k} \oplus K_6 \oplus K_{6k,6} = K_{6k} \oplus K_6 \oplus kK_{6,6}$. From the definition of $I(n)$, we have

$$I(24r) = \left\{ (p, q) \middle| p = \frac{(24r)(24r-1)}{6} - 4i,\ q = \frac{(24r)(24r-1)}{8} - \frac{3p}{4}, \right.$$

$$\begin{aligned}
& \quad 0 \le i \le \left\lfloor \frac{(24r)(24r-1)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x,3y)|0 \le x \le 24r^2 - r,\ y = (24r^2 - r) - x\}, \\
I(24r+6) &= \Bigg\{ (p,q) \Big| p = \frac{(24r+6)(24r+5)}{6} - 4i,\ q = \frac{(24r+6)(24r+5)}{8} - \frac{3p}{4}, \\
& \quad 0 \le i \le \left\lfloor \frac{(24r+6)(24r+5)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+1,3y)|0 \le x \le 24r^2 + 11r + 1,\ y = (24r^2 + 11r + 1) - x\}, \\
I(24r+12) &= \Bigg\{ (p,q) \Big| p = \frac{(24r+12)(24r+11)}{6} - 4i,\ q = \frac{(24r+12)(24r+11)}{8} \\
& \quad - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+12)(24r+11)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+2,3y)|0 \le x \le 24r^2 + 23r + 5,\ y = (24r^2 + 23r + 5) - x\}, \\
I(24r+18) &= \Bigg\{ (p,q) \Big| p = \frac{(24r+18)(24r+17)}{6} - 4i,\ q = \frac{(24r+18)(24r+17)}{8} \\
& \quad - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+18)(24r+17)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+3,3y)|0 \le x \le 24r^2 + 35r + 12,\ y = (24r^2 + 35r + 12) - x\}, \\
I(24r+24) &= \Bigg\{ (p,q) \Big| p = \frac{(24r+24)(24r+23)}{6} - 4i,\ q = \frac{(24r+24)(24r+23)}{8} \\
& \quad - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+24)(24r+23)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x,3y)|0 \le x \le 24r^2 + 47r + 23,\ y = (24r^2 + 47r + 23) - x\}.
\end{aligned}$$

Case 1. If $k = 4r$, then we can write $K_{24r+6} = K_{24r} \oplus K_6 \oplus (4r)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+6}) \supseteq M(K_{24r}) + M(K_6) + (4r)M(K_{6,6}) = \{(4x,3y)|0 \le x \le 24r^2 - r,\ y = (24r^2 - r) - x\} + (5,0) + (4r)\{(0,9),(4,6),(8,3),(12,0)\} = \{(4u+1,3v)|1 \le u \le 24r^2+11r+1,\ v = (24r^2+11r+1)-u\} = I(24r+6) - \big(1, 3(24r^2 + 11r + 1)\big)$. If $r = 1$, then $K_{30} = K_{16} \oplus K_{14} \oplus K_{16,14}$. The graph K_{14} can be decomposed into $1P_4$ and $22S_4$, by Lemma 10, and the graphs K_{16} and $K_{16,14}$ have an S_4-decomposition, by Theorems 2 and 3. Hence the graph K_{30} has a decomposition into $1P_4$ and $108S_4$. For $r \ge 2$, we can write $K_{24r+6} = K_{24r-8} \oplus K_{14} \oplus K_{24r-8,14}$. Then by Lemma 10, the graph K_{14} can be decomposed into $1P_4$ and $22S_4$, and by Theorems 2 and 3, the graphs K_{24r-8} and $K_{24r-8,14}$ have an S_4-decomposition. Hence the graph K_{24r+6} has a decomposition into $1P_4$ and $3(24r^2 + 11r + 1)S_4$. Therefore $M(K_{24r+6}) = \{(4x+1,3y)|0 \le x \le 24r^2 + 11r + 1,\ y = (24r^2 + 11r + 1) - x\} = I(24r+6)$.

Case 2. If $k = 4r + 1$, then we can write $K_{24r+12} = K_{24r+6} \oplus K_6 \oplus (4r+1)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+12}) \supseteq M(K_{24r+6}) + M(K_6) + (4r+1)M(K_{6,6}) = \{(4x+1,3y)|0 \le x \le 24r^2 + 11r + 1,\ y = (24r^2 + 11r + 1) - x\} + (5,0) + (4r+1)\{(0,9),(4,6),(8,3),(12,0)\} = \{(4u+2,3v)|1 \le u \le 24r^2 + 23r + 5,\ v = (24r^2 + 23r + 5) - u\} = I(24r+12) - \big(2, 3(24r^2 + 23r + 5)\big)$. Let $K_{24r+12} = K_{24r} \oplus K_{12} \oplus K_{24r,12}$. The graph K_{12} can be decomposed into $2P_4$ and $15S_4$, by Lemma 9, and by Theorems 2 and 3, the graphs K_{24r} and $K_{24r,12}$ have an S_4-decomposition. Hence the graph K_{24r+12} has a decomposition into $2P_4$ and $3(24r^2+23r+5)S_4$. Therefore $M(K_{24r+12}) = \{(4x+2, 3y)|0 \leq x \leq 24r^2+23r+5,\ y = (24r^2+23r+5)-x\} = I(24r+12)$.

Case 3. If $k = 4r+2$, then we can write $K_{24r+18} = K_{24r+12} \oplus K_6 \oplus (4r+2)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+18}) \supseteq M(K_{24r+12}) + M(K_6) + (4r+2)M(K_{6,6}) = \{(4x+2, 3y)|0 \leq x \leq 24r^2+23r+5,\ y = (24r^2+23r+5)-x\} + (5,0) + (4r+2)\{(0,9),(4,6),(8,3),(12,0)\} = \{(4u+3, 3v)|1 \leq u \leq 24r^2+35r+12,\ v = (24r^2+35r+12)-u\} = I(24r+18) - \big(3, 3(24r^2+35r+12)\big)$. Let $K_{24r+18} = K_{24r+8} \oplus K_{10} \oplus K_{24r+8,10}$. The graph K_{10} can be decomposed into $3P_4$ and $9S_4$, by Lemma 7, and by Theorems 2 and 3, the graphs K_{24r+8} and $K_{24r+8,10}$ have an S_4-decomposition. Hence the graph K_{24r+18} has a decomposition into $3P_4$ and $3(24r^2+35r+12)S_4$. Therefore $M(K_{24r+18}) = \{(4x+3, 3y)|0 \leq x \leq 24r^2+35r+12,\ y = (24r^2+35r+12)-x\} = I(24r+18)$.

Case 4. If $k = 4r+3$, then we can write $K_{24r+24} = K_{24r+18} \oplus K_6 \oplus (4r+3)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+24}) \supseteq M(K_{24r+18}) + M(K_6) + (4r+3)M(K_{6,6}) = \{(4x+3, 3y)|0 \leq x \leq 24r^2+35r+12,\ y = (24r^2+35r+12)-x\} + (5,0) + (4r+3)\{(0,9),(4,6),(8,3),(12,0)\} = \{(4u, 3v)|2 \leq u \leq 24r^2+47r+23,\ v = (24r^2+47r+23)-u\} = I(24r+24) - \Big\{\big(0, 3(24r^2+47r+23)\big), \big(4, 3(24r^2+47r+22)\big)\Big\}$. The graph K_{24r+24} has $3(24r^2+47r+23)S_4$, by Theorem 3, and hence $M(K_{24r+24}) = I(24r+24) - \big(4, 324r^2+47r+23\big)$. Let $K_{24r+24} = K_{24r+16} \oplus K_8 \oplus K_{24r+16,8}$. Then by Lemma 5, the graph K_8 can be decomposed into $4P_4$ and $4S_4$, and by Theorems 2 and 3, the graphs K_{24r+16} and $K_{24r+16,8}$ have an S_4-decomposition. Hence the graph K_{24r+24} has a decomposition into $4P_4$ and $3(24r^2+47r+22)S_4$. Therefore $M(K_{24r+24}) = \{(4x, 3y)|0 \leq x \leq 24r^2+47r+23,\ y = (24r^2+47r+23)-x\} = I(24r+24)$.

Thus $M(K_{6s}) = I(6s)$, for each $s \in \mathbb{Z}_+$. □

Lemma 12. *Let $p, q \in \mathbb{Z}_+ \cup \{0\}$ and $n \equiv 1 \pmod 6$. There exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p+4q = \binom{n}{2}$ and $n \geq 7$. That is, $M(K_{6s+1}) = I(6s+1)$, where $s \in \mathbb{Z}_+$.*

Proof. Necessity: The conditions $3p+4q = \binom{n}{2}$ and $n \geq 6$ are trivial. That is, $M(K_{6s+1}) \subseteq I(6s+1)$. Sufficiency: We have to prove $M(K_{6s+1}) \supseteq I(6s+1)$. The proof is by induction on s. If $s = 1$, then $M(K_7) = I(7)$, by Lemma 4. Since $K_{6k+7} = K_{6k+1} \oplus K_7 \oplus K_{6k,6} = K_{6k+1} \oplus K_7 \oplus kK_{6,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24r+1) &= \left\{(p,q)\middle| p = \frac{(24r+1)(24r)}{6} - 4i,\ q = \frac{(24r+1)(24r)}{8} - \frac{3p}{4},\right. \\
&\qquad \left. 0 \leq i \leq \left\lfloor \frac{(24r+1)(24r)}{24} \right\rfloor \right\}, \\
&= \{(4x, 3y)|0 \leq x \leq 24r^2+r,\ y = (24r^2+r)-x\}, \\
I(24r+7) &= \left\{(p,q)\middle| p = \frac{(24r+7)(24r+6)}{6} - 4i,\ q = \frac{(24r+7)(24r+6)}{8} - \frac{3p}{4},\right. \\
&\qquad \left. 0 \leq i \leq \left\lfloor \frac{(24r+7)(24r+6)}{24} \right\rfloor \right\}, \\
&= \{(4x+3, 3y)|0 \leq x \leq 24r^2+13r+1,\ y = (24r^2+13r+1)-x\}, \\
I(24r+13) &= \left\{(p,q)\middle| p = \frac{(24r+13)(24r+12)}{6} - 4i,\ q = \frac{(24r+13)(24r+12)}{8}\right. \\
&\qquad \left. -\frac{3p}{4},\ 0 \leq i \leq \left\lfloor \frac{(24r+13)(24r+12)}{24} \right\rfloor \right\},
\end{aligned}$$

$$= \{(4x+2,3y)|0 \le x \le 24r^2+25r+6,\ y=(24r^2+25r+6)-x\},$$

$$I(24r+19) = \left\{(p,q)\Big|p=\frac{(24r+19)(24r+18)}{6}-4i,\ q=\frac{(24r+19)(24r+18)}{8} -\frac{3p}{4},\ 0\le i\le \left\lfloor\frac{(24r+19)(24r+18)}{24}\right\rfloor\right\},$$

$$= \{(4x+1,3y)|0 \le x \le 24r^2+37r+14,\ y=(24r^2+37r+14)-x\},$$

$$I(24r+25) = \left\{(p,q)\Big|p=\frac{(24r+25)(24r+24)}{6}-4i,\ q=\frac{(24r+25)(24r+24)}{8} -\frac{3p}{4},\ 0\le i\le \left\lfloor\frac{(24r+25)(24r+24)}{24}\right\rfloor\right\},$$

$$= \{(4x,3y)|0 \le x \le 24r^2+49r+25,\ y=(24r^2+49r+25)-x\}.$$

Case 1. If $k=4r$, then we can write $K_{24r+7}=K_{24r+1}\oplus K_7\oplus(4r)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+7})\supseteq M(K_{24r+1})+M(K_7)+(4r)M(K_{6,6})=\{(4x,3y)|0\le x\le 24r^2+r,\ y=(24r^2+r)-x\}+\{(3,3),(7,0)\}+(4r)\{(0,9),(4,6),(8,3),(12,0)\}$
$=\{(4u+3,3v)|0\le u\le 24r^2+13r+1,\ v=(24r^2+13r+1)-u\}=I(24r+7)$.

Case 2. If $k=4r+1$, then we can write $K_{24r+13}=K_{24r+7}\oplus K_7\oplus(4r+1)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+13})\supseteq M(K_{24r+7})+M(K_7)+(4r+1)M(K_{6,6})=\{(4x+3,3y)|0\le x\le 24r^2+13r+1,\ y=(24r^2+13r+1)-x\}+\{(3,3),(7,0)\}+(4r+1)\{(0,9),(4,6),(8,3),(12,0)\}=\{(4u+2,3v)|1\le u\le 24r^2+25r+6,\ v=(24r^2+25r+6)-u\}=I(24r+13)-\Big(2,3(24r^2+25r+6)\Big)$. Let $K_{24r+13}=K_{24r+9}\oplus K_4\oplus(8r+3)K_{3,4}$. Then the graphs K_{24r+9} and $K_{3,4}$ have an S_4-decomposition, by Theorems 2 and 3, the graph K_4 has $2P_4$. Hence the graph K_{24r+13} has a decomposition into $2P_4$ and $3(24r^2+25r+6)S_4$. Therefore $M(K_{24r+13})=\{(4x+2,3y)|0\le x\le 24r^2+25r+6,\ y=(24r^2+25r+6)-x\}=I(24r+13)$.

Case 3. If $k=4r+2$, then we can write $K_{24r+19}=K_{24r+13}\oplus K_7\oplus(4r+2)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+19})\supseteq M(K_{24r+13})+M(K_7)+(4r+2)M(K_{6,6})=\{(4x+2,3y)|0\le x\le 24r^2+25r+6,\ y=(24r^2+25r+6)-x\}+\{(3,3),(7,0)\}+(4r+2)\{(0,9),(4,6),(8,3),(12,0)\}=\{(4u+1,3v)|1\le u\le 24r^2+37r+14,\ v=(24r^2+37r+14)-u\}=I(24r+19)-\Big(1,3(24r^2+37r+14)\Big)$. Let $K_{24r+19}=K_{24r+8}\oplus K_{11}\oplus K_{24r+8,11}$. Then the graph K_{11} can be decomposed into $1P_4$ and $13S_4$, by Lemma 8, and Theorems 2 and 3, the graphs K_{24r+8} and $K_{24r+8,11}$ have an S_4-decomposition. Hence the graph K_{24r+19} has a decomposition into $1P_4$ and $3(24r^2+37r+14)S_4$. Therefore $M(K_{24r+19})=\{(4x+1,3y)|0\le x\le 24r^2+37r+14,\ y=(24r^2+37r+14)-x\}=I(24r+19)$.

Case 4. If $k=4r+3$, then we can write $K_{24r+25}=K_{24r+19}\oplus K_7\oplus(4r+3)K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+25})\supseteq M(K_{24r+19})+M(K_7)+(4r+3)M(K_{6,6})=\{(4x+1,3y)|0\le x\le 24r^2+37r+14,\ y=(24r^2+37r+14)-x\}+\{(3,3),(7,0)\}+(4r+3)\{(0,9),(4,6),(8,3),(12,0)\}=\{(4u,3v)|1\le u\le 24r^2+49r+25,\ v=(24r^2+49r+25)-u\}=I(24r+25)-\Big(0,3(24r^2+49r+25)\Big)$. The graph K_{24r+25} has $3(24r^2+49r+25)S_4$, by Theorem 3. Hence $M(K_{24r+25})=\{(4x,3y)|0\le x\le 24r^2+49r+25,\ y=(24r^2+49r+25)-x\}=I(24r+25)$.

Thus $M(K_{6s+1})=I(6s+1)$, for each $s\in\mathbb{Z}_+$. □

Lemma 13. *Let $p, q \in \mathbb{Z}_+ \cup \{0\}$ and $n \equiv 2 \pmod 6$. There exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p + 4q = \binom{n}{2}$, $n \geq 8$ and $q \geq 1$. That is, $M(K_{6s+2}) = I(6s+2)$, where $s \in \mathbb{Z}_+$.*

Proof. Necessity: The conditions $3p + 4q = \binom{n}{2}$ and $n \geq 6$ are trivial. That is, $M(K_{6s+2}) \subseteq I(6s+2)$. Then $q \geq 1$, since by Theorem 1, the graph K_{6s+2} can not have a P_4-decomposition. Sufficiency: We have to prove $M(K_{6s+2}) \supseteq I(6s+2)$. The proof is by induction on s. If $s = 1$, then $M(K_8) = I(8)$, by Lemma 5. Since $K_{6k+8} = K_{6k+2} \oplus K_6 \oplus K_{6k+2,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24r+2) &= \Bigg\{(p,q)\Bigg|p = \frac{(24r+2)(24r+1)-8}{6} - 4i,\ q = \frac{(24r+2)(24r+1)}{8} \\
&\quad -\frac{3p}{4},\ 0 \leq i \leq \left\lfloor \frac{(24r+2)(24r+1)-8}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+3, 3y+1) | 0 \leq x \leq 24r^2+3r-1,\ y = (24r^2+3r-1)-x\}, \\
I(24r+8) &= \Bigg\{(p,q)\Bigg|p = \frac{(24r+8)(24r+7)-8}{6} - 4i,\ q = \frac{(24r+8)(24r+7)}{8} \\
&\quad -\frac{3p}{4},\ 0 \leq i \leq \left\lfloor \frac{(24r+8)(24r+7)-8}{24} \right\rfloor \Bigg\}, \\
&= \{(4x, 3y+1) | 0 \leq x \leq 24r^2+15r+2,\ y = (24r^2+15r+2)-x\}, \\
I(24r+14) &= \Bigg\{(p,q)\Bigg|p = \frac{(24r+14)(24r+13)-8}{6} - 4i,\ q = \frac{(24r+14)(24r+13)}{8} \\
&\quad -\frac{3p}{4},\ 0 \leq i \leq \left\lfloor \frac{(24r+14)(24r+13)-8}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+1, 3y+1) | 0 \leq x \leq 24r^2+27r+7,\ y = (24r^2+27r+7)-x\}, \\
I(24r+20) &= \Bigg\{(p,q)\Bigg|p = \frac{(24r+20)(24r+19)-8}{6} - 4i,\ q = \frac{(24r+20)(24r+19)}{8} \\
&\quad -\frac{3p}{4},\ 0 \leq i \leq \left\lfloor \frac{(24r+20)(24r+19)-8}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+2, 3y+1) | 0 \leq x \leq 24r^2+39r+15,\ y = (24r^2+39r+15)-x\}, \\
I(24r+26) &= \Bigg\{(p,q)\Bigg|p = \frac{(24r+26)(24r+25)-8}{6} - 4i,\ q = \frac{(24r+26)(24r+25)}{8} \\
&\quad -\frac{3p}{4},\ 0 \leq i \leq \left\lfloor \frac{(24r+26)(24r+25)-8}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+3, 3y+1) | 0 \leq x \leq 24r^2+51r+26,\ y = (24r^2+51r+26)-x\}.
\end{aligned}$$

Case 1. If $k = 4r$, then we can write $K_{24r+8} = K_{24r+2} \oplus K_6 \oplus K_{24r+2,6}$. Since $K_{24r+2,6} = K_{24r,6} \oplus K_{2,6} = (4r)K_{6,6} \oplus K_{2,6}$. Then $K_{24r+8} = K_{24r+2} \oplus K_6 \oplus (4r)K_{6,6} \oplus K_{2,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+8}) \supseteq M(K_{24r+2}) + M(K_6) + (4r)M(K_{6,6}) + M(K_{2,6}) = \{(4x+3, 3y+1)|0 \leq x \leq 24r^2+3r-1,\ y = (24r^2+3r-1)-x\} + (5,0) + (4r)\{(0,9),(4,6),(8,3),(12,0)\} + (4,0) = \{(4u, 3v+1)|3 \leq u \leq 24r^2+15r+2,\ v = (24r^2+15r+2)-u\} = I(24r+8) - \Big\{\Big(0, 3(24r^2+15r+2)\Big), \Big(4, 3(24r^2+

$15r+1)\Big),\Big(8,3(24r^2+15r)\Big)\Big\}$. The graph K_{24r+8} has $3(24r^2+15r+2)S_4$, by Theorem 2, we have $M(K_{24r+8}) = I(24r+8) - \Big\{\Big(4,3(24r^2+15r+1)\Big),\Big(8,3(24r^2+15r)\Big)\Big\}$. Let $K_{24r+8} = K_{24r} \oplus K_8 \oplus K_{24r,8}$. Then by Lemma 5, the graph K_8 can be decomposed into $8P_4$ or $4P_4$ and $4S_4$, and by Theorems 2 and 3, the graphs K_{24r} and $K_{24r,8}$ have an S_4-decomposition. Hence the graph K_{24r+8} has a decomposition into p copies of P_4 and q copies of S_4, where $(p,q) \in \Big\{\Big(0,3(24r^2+15r+2)\Big),\Big(4,3(24r^2+15r+1)\Big),\Big(8,3(24r^2+15r)\Big)\Big\}$. Therefore $M(K_{24r+8}) = \{(4x,3y+1)|0 \le x \le 24r^2+15r+2,\ y = (24r^2+15r+2)-x\} = I(24r+8)$.

Case 2. If $k = 4r+1$, then we can write $K_{24r+14} = K_{24r+8} \oplus K_6 \oplus K_{24r+8,6}$. Since $K_{24r+8,6} = (6r+2)K_{4,6}$. Then $K_{24r+14} = K_{24r+8} \oplus K_6 \oplus (6r+2)K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+14}) \supseteq M(K_{24r+8}) + M(K_6) + (6r+2)M(K_{4,6}) = \{(4x,3y+1)|0 \le x \le 24r^2+15r+2,\ y = (24r^2+15r+2)-x\} + (5,0) + (6r+2)\{(0,6),(4,3),(8,0)\} = \{(4u+1,3v+1)|1 \le u \le 24r^2+27r+7,\ v = (24r^2+27r+7)-u\} = I(24r+14) - \Big(1,3(24r^2+27r+7)\Big)$. If $r=0$, then $M(K_{14}) = I(14)$, by Lemma 10 . If $r \ge 1$, we can write $K_{24r+14} = K_{24r} \oplus K_{14} \oplus K_{24r,14}$. Then by Lemma 10, the graph K_{14} can be decomposed into $1P_4$ and $22S_4$, and by Theorems 2 and 3, the graphs K_{24r} and $K_{24r,14}$ have an S_4-decomposition. Hence the graph K_{24r+14} has a decomposition into $1P_4$ and $3(24r^2+27r+7)S_4$. Therefore $M(K_{24r+14}) = \{(4x+1,3y+1)|0 \le x \le 24r^2+27r+7,\ y = (24r^2+27r+7)-x\} = I(24r+14)$.

Case 3. If $k = 4r+2$, then we can write $K_{24r+20} = K_{24r+14} \oplus K_6 \oplus K_{24r+14,6}$. Since $K_{24r+14,6} = (6r+2)K_{4,6} \oplus K_{6,6}$. Then $K_{24r+20} = K_{24r+14} \oplus K_6 \oplus (6r+2)K_{4,6} \oplus K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+20}) \supseteq M(K_{24r+14}) + M(K_6) + (6r+2)M(K_{4,6}) = \{(4x+1,3y+1)|0 \le x \le 24r^2+27r+7,\ y = (24r^2+27r+7)-x\} + (5,0) + (6r+2)\{(0,6),(4,3),(8,0)\} = \{(4u+2,3v+1)|1 \le u \le 24r^2+39r+15,\ v = (24r^2+39r+15)-u\} = I(24r+20) - \Big(2,3(24r^2+39r+15)\Big)$. Let $K_{24r+20} = K_{24r+16} \oplus K_4 \oplus K_{24r+16,4}$. Then by Theorems 2 and 3, the graphs K_{24r+16} and $K_{24r+16,4}$ have an S_4-decomposition, and the graph K_4 has $2P_4$. Hence the graph K_{24r+20} has a decomposition into $2P_4$ and $3(24r^2+39r+15)S_4$. Therefore $M(K_{24r+20}) = \{(4x+2,3y+1)|0 \le x \le 24r^2+39r+15,\ y = (24r^2+39r+15)-x\} = I(24r+20)$.

Case 4. If $k = 4r+3$, then we can write $K_{24r+26} = K_{24r+20} \oplus K_6 \oplus K_{24r+16,4}$. Since $K_{24r+20,6} = (6r+5)K_{4,6}$. Then $K_{24r+26} = K_{24r+20} \oplus K_6 \oplus (6r+5)K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+26}) \supseteq M(K_{24r+20}) + M(K_6) + (6r+5)M(K_{4,6}) = \{(4x+2,3y+1)|0 \le x \le 24r^2+39r+15,\ y = (24r^2+39r+15)-x\} + (5,0) + (6r+5)\{(0,6),(4,3),(8,0)\} = \{(4u+3,3v+1)|1 \le u \le 24r^2+51r+26,\ v = (24r^2+51r+26)-u\} = I(24r+26) - \Big(3,3(24r^2+51r+26)\Big)$. Let $K_{24r+26} = K_{24r+16} \oplus K_{10} \oplus K_{24r+16,10}$. Then by Lemma 7, the graph K_{10} can be decomposed into $3P_4$ and $7S_4$, and by Theorems 2 and 3, the graphs K_{24r+16} and $K_{24r+16,10}$ have an S_4-decomposition. Hence the graph K_{24r+26} has a decomposition into $3P_4$ and $3(24r^2+51r+26)S_4$. Therefore $M(K_{24r+26}) = \{(4x+3,3y+1)|0 \le x \le 24r^2+51r+26,\ y = (24r^2+51r+26)-x\} = I(24r+26)$.

Thus $M(K_{6s+2}) = I(6s+2)$, for each $s \in \mathbb{Z}_+$. □

Lemma 14. *Let $p,q \in \mathbb{Z}_+ \cup \{0\}$ and $n \equiv 3 \pmod 6$. There exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p+4q = \binom{n}{2}$ and $n \ge 9$. That is, $M(K_{6s+3}) = I(6s+3)$, where $s \in \mathbb{Z}_+$.*

Proof. Necessity: The conditions $3p+4q = \binom{n}{2}$ and $n \ge 6$ are trivial. That is, $M(K_{6s+3}) \subseteq I(6s+3)$. Sufficiency: We have to prove $M(K_{6s+3}) \supseteq I(6s+3)$. The proof is by induction on s. If $s=1$, then $M(K_9) = I(9)$, by Lemma 6. Since $K_{6k+9} = K_{6k+3} \oplus K_7 \oplus K_{6k+2,6}$. From the definition of $I(n)$, we have

$$I(24r+3) \quad = \quad \left\{(p,q)\middle| p = \frac{(24r+3)(24r+2)}{6} - 4i,\ q = \frac{(24r+3)(24r+2)}{8} - \frac{3p}{4},\right.$$

$$\begin{aligned}
& 0 \le i \le \left\lfloor \frac{(24r+3)(24r+2)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+1, 3y) | 0 \le x \le 24r^2+5r,\ y = (24r^2+5r) - x\}, \\
I(24r+9) &= \left\{ (p,q) \Big| p = \frac{(24r+9)(24r+8)}{6} - 4i,\ q = \frac{(24r+9)(24r+8)}{8} - \frac{3p}{4}, \right. \\
& 0 \le i \le \left\lfloor \frac{(24r+9)(24r+8)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x, 3y) | 0 \le x \le 24r^2+17r+3,\ y = (24r^2+17r+3) - x\}, \\
I(24r+15) &= \left\{ (p,q) \Big| p = \frac{(24r+15)(24r+14)}{6} - 4i,\ q = \frac{(24r+15)(24r+14)}{8} \right. \\
& -\frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+15)(24r+14)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+3, 3y) | 0 \le x \le 24r^2+29r+8,\ y = (24r^2+29r+8) - x\},
\end{aligned}$$

$$\begin{aligned}
I(24r+21) &= \left\{ (p,q) \Big| p = \frac{(24r+21)(24r+20)}{6} - 4i,\ q = \frac{(24r+21)(24r+20)}{8} \right. \\
& -\frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+21)(24r+20)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+2, 3y) | 0 \le x \le 24r^2+41r+17,\ y = (24r^2+41r+17) - x\}, \\
I(24r+27) &= \left\{ (p,q) \Big| p = \frac{(24r+27)(24r+26)}{6} - 4i,\ q = \frac{(24r+27)(24r+26)}{8} \right. \\
& -\frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+27)(24r+26)}{24} \right\rfloor \Bigg\}, \\
&= \{(4x+1, 3y) | 0 \le x \le 24r^2+53r+29,\ y = (24r^2+53r+29) - x\}.
\end{aligned}$$

Case 1. If $k = 4r$, then we can write $K_{24r+9} = K_{24r+3} \oplus K_7 \oplus K_{24r+2,6}$. Since $K_{24r+2,6} = (6r)K_{4,6} \oplus K_{2,6}$. Then $K_{24r+9} = K_{24r+3} \oplus K_7 \oplus (6r)K_{4,6} \oplus K_{2,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+9}) \supseteq M(K_{24r+3}) + M(K_7) + (6r)M(K_{4,6}) + M(K_{2,6}) = \{(4x+1, 3y) | 0 \le x \le 24r^2+5r,\ y = (24r^2+5r)-x\} + \{(3,3),(7,0)\} + (6r)\{(0,6),(4,3),(8,0)\} + (4,0) = \{(4u, 3v) | 2 \le u \le 24r^2+17r+3,\ v = (24r^2+17r+3) - u\} = I(24r+9) - \Big\{\big(0, 3(24r^2+17r+3)\big), \big(4, 3(24r^2+17r+2)\big)\Big\}$. The graph K_{24r+9} has $3(24r^2+17r+3)S_4$, by Theorem 2, we have $M(K_{24r+8}) = I(24r+8) - \big(4, 3(24r^2+17r+2)\big)$. Let $K_{24r+9} = K_{24r+1} \oplus K_8 \oplus K_{24r+1,8}$. Then by Lemma 5, the graph K_8 can be decomposed into $4P_4$ and $4S_4$, and by Theorems 2 and 3, the graphs K_{24r+1} and $K_{24r+1,8}$ have an S_4-decomposition. Hence the graph K_{24r+9} has a decomposition into $4P_4$ and $3(24r^2+17r+2)S_4$. Therefore $M(K_{24r+9}) = \{(4x, 3y) | 0 \le x \le 24r^2+17r+3,\ y = (24r^2+17r+3)-x\} = I(24r+9)$.

Case 2. If $k = 4r+1$, then we can write $K_{24r+15} = K_{24r+9} \oplus K_7 \oplus K_{24r+8,6}$. Since $K_{24r+8,6} = (6r+2)K_{4,6}$. Then $K_{24r+15} = K_{24r+9} \oplus K_7 \oplus (6r+2)K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+15}) \supseteq M(K_{24r+9}) + M(K_7) + (6r+2)M(K_{4,6}) = \{(4x, 3y) | 0 \le x \le 24r^2+17r+3,\ y = (24r^2+17r+3) - x\} + \{(3,3),(7,0)\} + (6r+2)\{(0,6),(4,3),(8,0)\} = \{(4u+3, 3v) | 0 \le u \le 24r^2+29r+8,\ v = (24r^2+29r+8) - u\} = I(24r+15)$.

Case 3. If $k = 4r+2$, then we can write $K_{24r+21} = K_{24r+15} \oplus K_7 \oplus K_{24r+15,6}$. Since $K_{24r+15,6} = (6r+2)K_{4,6} \oplus K_{6,6}$, we have $K_{24r+21} = K_{24r+15} \oplus K_7 \oplus (6r+2)K_{4,6} \oplus K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+21}) \supseteq M(K_{24r+15}) + M(K_7) + (6r+2)M(K_{4,6}) + M(K_{6,6}) = \{(4x+3, 3y)|0 \le x \le 24r^2+29r+8,\ y = (24r^2+29r+8) - x\} + \{(3,3),(7,0)\} + (6r+2)\{(0,6),(4,3),(8,0)\} + \{(0,9),(4,6),(8,3),(12,0)\} = \{(4u+2, 3v)|1 \le u \le 24r^2+41r+17,\ v = (24r^2+41r+17) - u\} = I(24r+21) - \Big(2, 3(24r^2+41r+17)\Big)$. Let $K_{24r+21} = K_{24r+17} \oplus K_4 \oplus K_{24r+17,4}$. Then the graphs K_{24r+17} and $K_{24r+17,4}$ have an S_4-decomposition, by Theorems 2 and 3, the graph K_4 has $2P_4$. Hence the graph K_{24r+21} has a decomposition into $2P_4$ and $3(24r^2+41r+17)S_4$. Therefore $M(K_{24r+21}) = \{(4x+2, 3y)|0 \le x \le 24r^2+41r+17,\ y = (24r^2+41r+17) - x\} = I(24r+21)$.

Case 4. If $k = 4r+3$, then we can write $K_{24r+27} = K_{24r+21} \oplus K_7 \oplus K_{24r+20,6}$. Since $K_{24r+20,6} = (6r+5)K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+27}) \supseteq M(K_{24r+21}) + M(K_7) + (6r+5)M(K_{4,6}) = \{(4x+2, 3y)|0 \le x \le 24r^2+41r+17,\ y = (24r^2+41r+17) - x\} + \{(3,3),(7,0)\} + (6r+5)\{(0,6),(4,3),(8,0)\} = \{(4u+1, 3v)|1 \le u \le 24r^2+53r+29,\ v = (24r^2+53r+29) - u\} = I(24r+27) - \Big(1, 3(24r^2+53r+29)\Big)$. Let $K_{24r+27} = K_{24r+16} \oplus K_{11} \oplus K_{24r+16,11}$. Then by Lemma 8, the graph K_{11} can be decomposed into $1P_4$ and $13S_4$, and by Theorems 2 and 3, the graphs K_{24r+16} and $K_{24r+16,11}$ have an S_4-decomposition. Hence the graph K_{24r+27} has a decomposition into $1P_4$ and $3(24r^2+53r+29)S_4$. Therefore $M(K_{24r+27}) = \{(4x+1, 3y)|0 \le x \le 24r^2+53r+29,\ y = (24r^2+53r+29) - x\} = I(24r+27)$.

Thus $M(K_{6s+3}) = I(6s+3)$, for each $s \in \mathbb{Z}_+$. □

Lemma 15. *Let* $p, q \in \mathbb{Z}_+ \cup \{0\}$ *and* $n \equiv 4 \pmod 6$. *There exists a* $\{pP_4, qS_4\}$*-decomposition of* K_n *if and only if* $3p + 4q = \binom{n}{2}$ *and* $n \ge 4$. *That is,* $M(K_{6s+4}) = I(6s+4)$*, where* $s \in \mathbb{Z}_+$.

Proof. Necessity: The conditions $3p + 4q = \binom{n}{2}$ and $n \ge 4$ are trivial. That is, $M(K_{6s+4}) \subseteq I(6s+4)$. Sufficiency: We have to prove $M(K_{6s+4}) \supseteq I(6s+4)$. The proof is by induction on s. If $s = 0$, then $M(K_4) = I(4)$, by Theorem 1. If $s = 1$, then $M(K_{10}) = I(10)$, by Lemma 7. Since $K_{6k+10} = K_{6k+4} \oplus K_6 \oplus K_{6k+4,6}$. From the definition of $I(n)$, we have

$$\begin{aligned}
I(24r+4) &= \Bigg\{(p,q)\Bigg| p = \frac{(24r+4)(24r+3)}{6} - 4i,\ q = \frac{(24r+4)(24r+3)}{8} - \frac{3p}{4},\\
&\qquad 0 \le i \le \left\lfloor \frac{(24r+4)(24r+3)}{24} \right\rfloor \Bigg\},\\
&= \{(4x+2, 3y)|0 \le x \le 24r^2+7r,\ y = (24r^2+7r) - x\},\\
I(24r+10) &= \Bigg\{(p,q)\Bigg| p = \frac{(24r+10)(24r+9)}{6} - 4i,\ q = \frac{(24r+10)(24r+9)}{8}\\
&\qquad - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+10)(24r+9)}{24} \right\rfloor \Bigg\},\\
&= \{(4x+3, 3y)|0 \le x \le 24r^2+19r+3,\ y = (24r^2+19r+3) - x\},\\
I(24r+16) &= \Bigg\{(p,q)\Bigg| p = \frac{(24r+16)(24r+15)}{6} - 4i,\ q = \frac{(24r+16)(24r+15)}{8}\\
&\qquad - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+16)(24r+15)}{24} \right\rfloor \Bigg\},
\end{aligned}$$

$$= \{(4x,3y)|0 \le x \le 24r^2+31r+10,\ y=(24r^2+31r+10)-x\},$$

$$I(24r+22) = \left\{(p,q)\middle|p=\frac{(24r+22)(24r+21)}{6}-4i,\ q=\frac{(24r+22)(24r+21)}{8}-\frac{3p}{4},\ 0\le i\le\left\lfloor\frac{(24r+22)(24r+21)}{24}\right\rfloor\right\},$$

$$= \{(4x+1,3y)|0 \le x \le 24r^2+43r+19,\ y=(24r^2+43r+19)-x\},$$

$$I(24r+28) = \left\{(p,q)\middle|p=\frac{(24r+28)(24r+27)}{6}-4i,\ q=\frac{(24r+28)(24r+27)}{8}-\frac{3p}{4},\ 0\le i\le\left\lfloor\frac{(24r+28)(24r+27)}{24}\right\rfloor\right\},$$

$$= \{(4x+2,3y)|0 \le x \le 24r^2+55r+31,\ y=(24r^2+55r+31)-x\}.$$

Case 1. If $k=4r$, then we can write $K_{24r+10}=K_{24r+4}\oplus K_6\oplus K_{24r+4,6}$. Since $K_{24r+4,6}=4rK_{6,6}\oplus K_{4,6}$. Then $K_{24r+10}=K_{24r+4}\oplus K_6\oplus 4rK_{6,6}\oplus K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+10})\supseteq M(K_{24r+4})+M(K_6)+(4r)M(K_{6,6})+M(K_{4,6})=\{(4x+2,3y)|0\le x\le 24r^2+7r,\ y=(24r^2+7r)-x\}+(5,0)+(4r)\{(0,9),(4,6),(8,3),(12,0)\}+\{(0,6),(4,3),(8,0)\}=\{(4u+3,3v)|1\le u\le 24r^2+19r+3,\ v=(24r^2+19r+3)-u\}=I(24r+10)-\Big(3,3(24r^2+19r+3)\Big)$. Let $K_{24r+10}=K_{24r}\oplus K_{10}\oplus K_{24r,10}$. Then by Lemma 7, the graph K_{10} can be decomposed into $3P_4$ and $9S_4$, and by Theorems 2 and 3, the graphs K_{24r} and $K_{24r,10}$ have an S_4-decomposition. Hence the graph K_{24r+10} has a decomposition into $3P_4$ and $3(24r^2+19r+3)S_4$. Therefore $M(K_{24r+10})=\{(4x+3,3y)|0\le x\le 24r^2+19r+3,\ y=(24r^2+19r+3)-x\}=I(24r+10)$.

Case 2. If $k=4r+1$, then we can write $K_{24r+16}=K_{24r+10}\oplus K_6\oplus K_{24r+10,6}$. Since $K_{24r+10,6}=(4r+1)K_{6,6}\oplus K_{4,6}$. Then $K_{24r+16}=K_{24r+10}\oplus K_6\oplus(4r+1)K_{6,6}\oplus K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+16})\supseteq M(K_{24r+10})+M(K_6)+(4r+1)M(K_{6,6})+M(K_{4,6})=\{(4x+3,3y)|0\le x\le 24r^2+19r+3,\ y=(24r^2+19r+3)-x\}+(5,0)+(4r+1)\{(0,9),(4,6),(8,3),(12,0)\}+\{(0,6),(4,3),(8,0)\}=\{(4u,3v)|2\le u\le 24r^2+31r+10,\ v=(24r^2+31r+10)-u\}=I(24r+16)-\Big\{\Big(0,3(24r^2+31r+10)\Big),\Big(4,3(24r^2+31r+9)\Big)\Big\}$. The graph K_{24r+16} has $3(24r^2+31r+10)S_4$, by Theorem 2. Hence $K_{24r+16}=I(24r+16)-\Big(4,3(24r^2+31r+9)\Big)$. Let $K_{24r+16}=K_{24r+8}\oplus K_8\oplus K_{24r+8,8}$. Then by Lemma 5, graph K_8 can be decomposed into $4P_4$ and $6S_4$, and by Theorems 2 and 3, the graphs K_{24r+8} and $K_{24r+8,8}$ have an S_4-decomposition. Hence the graph K_{24r+16} has a decomposition into $4P_4$ and $3(24r^2+31r+9)S_4$. Therefore $M(K_{24r+16})=\{(4x,3y)|0\le x\le 24r^2+31r+10,\ y=(24r^2+31r+10)-x\}=I(24r+16)$.

Case 3. If $k=4r+2$, then we can write $K_{24r+22}=K_{24r+16}\oplus K_6\oplus K_{24r+16,6}$. Since $K_{24r+16,6}=(4r+2)K_{6,6}\oplus K_{4,6}$. Then $K_{24r+22}=K_{24r+16}\oplus K_6\oplus(4r+2)K_{6,6}\oplus K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+22})\supseteq M(K_{24r+16})+M(K_6)+(4r+2)M(K_{6,6})+M(K_{4,6})=\{(4x,3y)|0\le x\le 24r^2+31r+10,\ y=(24r^2+31r+10)-x\}+(5,0)+(4r+2)\{(0,9),(4,6),(8,3),(12,0)\}+\{(0,6),(4,3),(8,0)\}=\{(4u+1,3v)|1\le u\le 24r^2+43r+19,\ v=(24r^2+43r+19)-u\}=I(24r+22)-\Big(1,3(24r^2+43r+19)\Big)$.

Let $K_{24r+22}=K_{24r+8}\oplus K_{14}\oplus K_{24r+8,14}$. Then by Lemma 10, the graph K_{14} can be decomposed into $1P_4$ and $22S_4$, and by Theorems 2 and 3, the graphs K_{24r+8} and $K_{24r+8,14}$ have an S_4-decomposition. Hence the graph K_{24r+22} has a decomposition into $1P_4$ and $3(24r^2+43r+19)S_4$. Therefore $M(K_{24r+22})=\{(4x+1,3y)|0\le x\le 24r^2+43r+19,\ y=(24r^2+43r+19)-x\}=I(24r+22)$.

Case 4. If $k=4r+3$, then we can write $K_{24r+28}=K_{24r+22}\oplus K_6\oplus K_{24r+22,6}$. Since

$K_{24r+22,6} = (4r+3)K_{6,6} \oplus K_{4,6}$. Then $K_{24r+28} = K_{24r+22} \oplus K_6 \oplus (4r+3)K_{6,6} \oplus K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 3, we have $M(K_{24r+28}) \supseteq M(K_{24r+22}) + M(K_6) + (4r+3)M(K_{6,6}) + M(K_{4,6}) = \{(4x+1, 3y) | 0 \le x \le 24r^2+43r+19,\ y = (24r^2+43r+19)-x\} + (5,0) + (4r+3)\{(0,9),(4,6),(8,3),(12,0)\} + \{(0,6),(4,3),(8,0)\} = \{(4u+2, 3v) | 1 \le u \le 24r^2+55r+31,\ v = (24r^2+55r+31)-u\} = I(24r+28) - \big(2, 3(24r^2+55r+31)\big)$. Let $K_{24r+28} = K_{24r+24} \oplus K_4 \oplus K_{24r+24,4}$. Then by Theorems 2 and 3, the graphs K_{24r+24} and $K_{24r+24,4}$ have an S_4-decomposition, the graph K_4 has $2P_4$. Hence the graph K_{24r+28} has a decomposition into $2P_4$ and $3(24r^2+55r+31)S_4$. Therefore $M(K_{24r+28}) = \{(4x+2, 3y) | 0 \le x \le 24r^2+55r+31,\ y = (24r^2+55r+31)-x\} = I(24r+28)$.

Thus $M(K_{6s+4}) = I(6s+4)$, for each $s \in \mathbb{Z}_+$. □

Lemma 16. *Let $p, q \in \mathbb{Z}_+ \cup \{0\}$ and $n \equiv 5 \pmod 6$. There exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p+4q = \binom{n}{2}$, $n \ge 5$ and $q \ge 1$. That is, $M(K_{6s+5}) = I(6s+5)$, where $s \in \mathbb{Z}_+ \cup \{0\}$.*

Proof. Necessity: The conditions $3p+4q = \binom{n}{2}$ and $n \ge 5$ are trivial. That is, $M(K_{6s+5}) \subseteq I(6s+5)$. Then $q \ge 1$, since by Theorem 1, the graph K_{6s+5} can not have a P_4-decomposition. Sufficiency: We have to prove $M(K_{6s+5}) \supseteq I(6s+5)$. The proof is by induction on s. If $s = 0$, then $M(K_5) = I(5)$, by Lemma 2. Since $K_{6k+11} = K_{6k+5} \oplus K_7 \oplus K_{6k+4,6}$. From the definition of $I(n)$, we have

$$
\begin{aligned}
I(24r+5) &= \left\{(p,q) \middle| p = \frac{(24r+5)(24r+4)-8}{6} - 4i,\ q = \frac{(24r+5)(24r+4)}{8} \right.\\
&\qquad \left. - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+5)(24r+4)-8}{24} \right\rfloor \right\},\\
&= \{(4x+2, 3y+1) | 0 \le x \le 24r^2+9r,\ y = (24r^2+9r)-x\},\\
I(24r+11) &= \left\{(p,q) \middle| p = \frac{(24r+11)(24r+10)-8}{6} - 4i,\ q = \frac{(24r+11)(24r+10)}{8} \right.\\
&\qquad \left. - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+11)(24r+10)-8}{24} \right\rfloor \right\},\\
&= \{(4x+1, 3y+1) | 0 \le x \le 24r^2+21r+4,\ y = (24r^2+21r+4)-x\},\\
I(24r+17) &= \left\{(p,q) \middle| p = \frac{(24r+17)(24r+16)-8}{6} - 4i,\ q = \frac{(24r+17)(24r+16)}{8} \right.\\
&\qquad \left. - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+17)(24r+16)-8}{24} \right\rfloor \right\},\\
&= \{(4x, 3y+1) | 0 \le x \le 24r^2+33r+11,\ y = (24r^2+33r+11)-x\},\\
I(24r+23) &= \left\{(p,q) \middle| p = \frac{(24r+23)(24r+22)-8}{6} - 4i,\ q = \frac{(24r+23)(24r+22)}{8} \right.\\
&\qquad \left. - \frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+23)(24r+22)-8}{24} \right\rfloor \right\},\\
&= \{(4x+3, 3y+1) | 0 \le x \le 24r^2+45r+20,\ y = (24r^2+45r+20)-x\},\\
I(24r+29) &= \left\{(p,q) \middle| p = \frac{(24r+29)(24r+28)-8}{6} - 4i,\ q = \frac{(24r+29)(24r+28)}{8} \right.
\end{aligned}
$$

$$-\frac{3p}{4},\ 0 \le i \le \left\lfloor \frac{(24r+29)(24r+28)-8}{24} \right\rfloor \Bigg\},$$
$$= \{(4x+2, 3y+1) | 0 \le x \le 24r^2+57r+33,\ y = (24r^2+57r+33)-x\}.$$

Case 1. If $k = 4r$, then we can write $K_{24r+11} = K_{24r+5} \oplus K_7 \oplus K_{24r+4,6}$. Since $K_{24r+4,6} = (6r+1)K_{4,6}$. Then $K_{24r+11} = K_{24r+5} \oplus K_7 \oplus (6r+1)K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+11}) \supseteq M(K_{24r+5}) + M(K_7) + (6r+1)M(K_{4,6}) = \{(4x+2, 3y+1) | 0 \le x \le 24r^2+9r,\ y = (24r^2+9r)-x\} + \{(3,3),(7,0)\} + (6r+1)\{(0,6),(4,3),(8,0)\} = \{(4u+1, 3v+1) | 1 \le u \le 24r^2+21r+4,\ v = (24r^2+21r+4)-u\} = I(24r+11) - \Big(1, 3(24r^2+21r+4)\Big)$. Let $K_{24r+11} = K_{24r} \oplus K_{11} \oplus K_{24r,11}$. Then by Lemma 8, the graph K_{11} can be decomposed into $1P_4$ and $13S_4$, and by Theorems 2 and 3, the graphs K_{24r} and $K_{24r,11}$ have an S_4-decomposition. Hence the graph K_{24r+11} has a decomposition into $1P_4$ and $3(24r^2+21r+4)S_4$. Therefore $M(K_{24r+11}) = \{(4x+1, 3y+1) | 0 \le x \le 24r^2+21r+4,\ y = (24r^2+21r+4)-x\} = I(24r+11)$.

Case 2. If $k = 4r+1$, then we can write $K_{24r+17} = K_{24r+11} \oplus K_7 \oplus K_{24r+10,6}$. Since $K_{24r+10,6} = (6r+1)K_{4,6} \oplus K_{6,6}$. Then $K_{24r+17} = K_{24r+11} \oplus K_7 \oplus (6r+1)K_{4,6} \oplus K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+17}) \supseteq M(K_{24r+11}) + M(K_7) + (6r+1)M(K_{4,6}) + M(K_{6,6}) = \{(4x+1, 3y+1) | 0 \le x \le 24r^2+21r+4,\ y = (24r^2+21r+4)-x\} + \{(3,3),(7,0)\} + (6r+1)\{(0,6),(4,3),(8,0)\} + \{(0,9),(4,6),(8,3),(12,0)\} = \{(4u, 3v+1) | 1 \le u \le 24r^2+33r+11,\ v = (24r^2+33r+11)-u\} = I(24r+17) - \Big(0, 3(24r^2+33r+11)\Big)$. The graph K_{24r+17} has an S_4-decomposition, by Theorem 2. Hence $M(K_{24r+17}) = \{(4x, 3y+1) | 0 \le x \le 24r^2+33r+11,\ y = (24r^2+33r+11)-x\} = I(24r+17)$

Case 3. If $k = 4r+2$, then we can write $K_{24r+23} = K_{24r+17} \oplus K_7 \oplus K_{24r+16,6}$. Since $K_{24r+16,6} = (6r+4)K_{4,6}$. Then $K_{24r+23} = K_{24r+17} \oplus K_7 \oplus (6r+4)K_{4,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+23}) \supseteq M(K_{24r+17}) + M(K_7) + (6r+4)M(K_{4,6}) = \{(4x, 3y+1) | 0 \le x \le 24r^2+33r+11,\ y = (24r^2+33r+11)-x\} + \{(3,3),(7,0)\} + (6r+4)\{(0,6),(4,3),(8,0)\} = \{(4u+3, 3v+1) | 0 \le u \le 24r^2+45r+20,\ v = (24r^2+45r+20)-u\} = I(24r+23)$.

Case 4. If $k = 4r+3$, then we can write $K_{24r+29} = K_{24r+23} \oplus K_7 \oplus K_{24r+22,6}$. Since $K_{24r+22,6} = (6r+4)K_{4,6} \oplus K_{6,6}$. By the induction hypothesis, Remark 1.1, and Lemmas 1, 4, we have $M(K_{24r+29}) \supseteq M(K_{24r+23}) + M(K_7) + (6r+4)M(K_{4,6}) + M(K_{6,6}) = \{(4x+3, 3y+1) | 0 \le x \le 24r^2+45r+20,\ y = (24r^2+45r+20)-x\} + \{(3,3),(7,0)\} + (6r+4)\{(0,6),(4,3),(8,0)\} + \{(0,9),(4,6),(8,3),(12,0)\} = \{(4u+2, 3v+1) | 1 \le u \le 24r^2+57r+33,\ v = (24r^2+57r+33)-u\} = I(24r+29) - \Big(2, 3(24r^2+57r+33)\Big)$. Let $K_{24r+29} = K_{24r+25} \oplus K_4 \oplus K_{24r+25,4}$. Then by Theorems 2 and 3, the graphs K_{24r+25} and $K_{24r+25,4}$ have an S_4-decomposition, the graph K_4 has $2P_4$. Hence the graph K_{24r+29} has a decomposition into $2P_4$ and $3(24r^2+57r+33)S_4$. Therefore $M(K_{24r+29}) = \{(4u+2, 3v+1) | 0 \le u \le 24r^2+57r+33,\ v = (24r^2+57r+33)-u\} = I(24r+29)$.

Thus $M(K_{6s+5}) = I(6s+5)$, for each $s \in \mathbb{Z}_+ \cup \{0\}$. □

The consequences of Lemmas 11 to 16 implies our main result as follows.

Theorem 4. *Let p and q are nonnegative integers, and $n \ge 4$ be a positive integer. There exists a $\{pP_4, qS_4\}$-decomposition of K_n if and only if $3p + 4q = \binom{n}{2}$. That is, $M(K_n) = I(n)$, where $4 \le n \in \mathbb{Z}_+$.*

4. Acknowledgments

Author's thank the University Grants Commission, Government of India, New Delhi for its support through the Grant No.F.510/7/DRS-I/2016(SAP-I) and the anonymous referees whose suggestion improved the presentation of the paper.

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