

Article

Efficient Domination in Grid Graphs

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Abstract: Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ of vertices is an *efficient dominating set* if every vertex $v \in V$ is adjacent to exactly one vertex in S , where a vertex $u \in S$ is considered to be adjacent to itself. Efficient domination is highly desirable in many real world applications, and yet, in general, graphs are often not efficient. It is of value, therefore, to determine optimum ways in which inefficient graphs can be changed in order to make them efficient. It is well known, for example, that almost no $m \times n$ grid graphs have efficient dominating sets. In this paper we consider the minimum number of vertices which can be removed from an $m \times n$ grid graph so that the remaining graph has an efficient dominating set.

Keywords: Efficient dominating sets, Grid graphs, Minimum vertex removal

1. Introduction

Let $G = (V, E) = (V(G), E(G))$ be a graph with vertex set $V = \{v_1, v_2, \dots, v_n\}$ and order $n = |V|$. The *open neighborhood* of a vertex v is the set $N(v) = \{u | uv \in E\}$ of vertices u that are adjacent to v ; the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. Similarly, the *closed neighborhood of a set S* is the set $N[S] = \bigcup_{v \in S} N[v]$. A set $S \subseteq V$ is a *dominating set* of G if every vertex in $V - S$ is adjacent to at least one vertex in S , or equivalently if $N[S] = V$. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S of G ; we say that such a set S is a γ -set. For more on domination theory, we refer the reader to [1–3].

A set $S \subseteq V$ of vertices is called *independent* if no two vertices in S are adjacent. A set $S \subseteq V$ of vertices is called a *2-packing*, or simply a *packing*, if for every vertex $v \in V$, $|N[v] \cap S| \leq 1$, that is, no vertex $w \in V - S$ has two or more neighbors in S , and no vertex $u \in S$ has a neighbor $v \in S$. Let $S \subset V$ be any set of vertices in a graph G , and let $G - S$ denote the graph obtained from G by deleting all vertices in S and deleting all edges $uv \in E$ which are incident with a vertex in S .

An $m \times n$ grid graph $G_{m,n}$ has vertex set $V = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$ with (i, j) adjacent to (k, l) if $i = k$ and $|j - l| = 1$ or $j = l$ and $|i - k| = 1$. Several papers have been published on the domination numbers of classes of grid graphs, including [4–9]. In [4, 7] the authors prove an exact formula for the domination number of $G_{m,n}$, for all $m, n \geq 1$. In [10, 11]

the authors give methods for constructing γ -sets of $G_{m,n}$.

2. Efficient and Total Efficient Domination

Let $G = (V, E)$ be a graph. A subset $S \subseteq V$ of vertices is an *efficient dominating set* if every vertex $v \in V$ is adjacent to exactly one vertex in S , where a vertex $u \in S$ is considered to be adjacent to itself. Thus, a set S is an efficient dominating set if for every vertex $v \in V$, $|N[v] \cap S| = 1$. A graph is said to be *efficient* if it has an efficient dominating set. Notice that every efficient dominating set S in a graph G is both an independent set and a 2-packing. Not every graph G has an efficient dominating set. Two simple examples are the cycles C_4 and C_5 , although it is easy to see that all cycles C_{3k} , for $k \geq 1$, have efficient dominating sets.

A dominating set S is called *total efficient* if for every vertex $v \in V$, $|N(v) \cap S| = 1$, that is, the *open* neighborhood $N(v)$ of every vertex $v \in V$ contains exactly one vertex in S . Not every graph G has a total efficient dominating set. A simple example, again, is the cycle C_5 , although it is easy to see that all cycles C_{4k} , for $k \geq 1$, have total efficient dominating sets. We say that graphs having a total efficient dominating set are *total efficient*. Figure 1 illustrates an efficient dominating set of the 2×7 grid graph $G_{2,7}$. Figure 2 illustrates a total efficient dominating set of the 4×7 grid graph $G_{4,7}$.

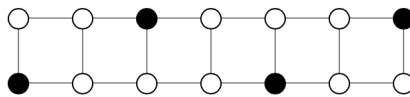


Figure 1. An Efficient Dominating Set in the 2×7 Grid Graph $G_{2,7}$

Efficient domination is of considerable importance in almost all real-world applications of domination. Efficient dominating sets have minimum cardinality of all dominating sets in a graph, but even more, they dominate the vertices of a graph with no redundancy or overlap. And thus, they provide domination at absolute minimum cost. In coding theory efficient dominating sets are called *perfect codes*, which means that they correspond to single-error-correcting codes. This means that every vertex, or code word, is either a member of a dominating set, or is adjacent to a unique member of the dominating set, which represents the single-error-corrected code word.

Although efficiency in graphs is highly desirable, in general, graphs are usually not efficient. It is of value, therefore, to determine optimum ways in which inefficient graphs can be changed in order to make them efficient. One way to make an inefficient graph efficient is to remove a minimum number S of vertices such that the remaining graph $G - S$ is efficient. Thus, in

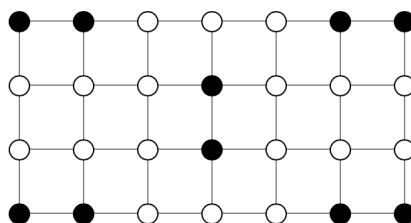


Figure 2. A Total Efficient Dominating Set in the 4×7 Grid $G_{4,7}$

this paper we will consider the following efficiency parameter. The *(vertex) efficiency deletion number*, $\varepsilon_v^-(G)$, equals the minimum number of vertices in a set S such that the graph $G - S$ is efficient.

2.1. Efficiency in Grid Graphs and Generalized Grid Graphs

As discussed in Chapter 9 of [3], the current theory of efficiency in graphs is not very extensive and has focused primarily on families of graphs such as circulants, also called Cayley graphs, vertex-transitive graphs, and cube-connected cycles (cf. [3]). In 1990 Livingston and Stout [12] showed that very few grid graphs are efficient.

Theorem 1 ([12]). *A grid graph $G_{m,n}$, for $m, n \geq 2$, has an efficient domination set if and only if either (i) $m = n = 4$, or (ii) $m = 2$ and $n = 2k + 1$, for any $k \geq 1$.*

Although Theorem 1 points out that there are very few efficient grid graphs, this family of graphs is but a small subfamily of a much larger family of grid-like graphs we can call *semigrid graphs*, which can be defined as 2-connected, plane graphs, every interior face of which is a 4-cycle, as illustrated in Figure 3. It is easy to see that there are far more efficient semigrid graphs than rectangular grid graphs, as in Figure 3.

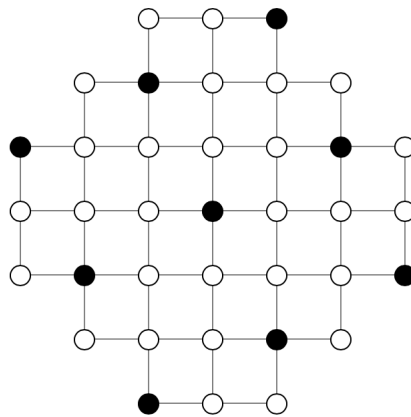


Figure 3. An Efficient Dominating Set in a Semigrid Graph

Notice in Figure 4 that the 5×5 grid graph $G_{5,5}$, which is not an efficient graph, is an induced subgraph of an efficient semigrid graph.

From Figure 4, the following result can easily be proved.

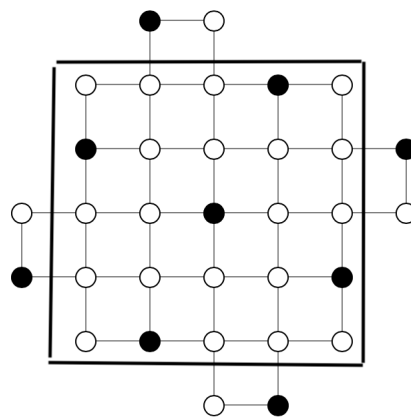


Figure 4. A 5×5 Grid $G_{5,5}$ as an Induced Subgraph of an Efficient Semigrid

Theorem 2. Every $m \times n$ grid graph $G_{m,n}$ is an induced subgraph of an efficient semigrid graph.

Proof sketch. As illustrated in Figure 4, given an $m \times n$ grid graph $G_{m,n}$, select a set of vertices which creates what is called a *star pattern*, in which vertices are distanced from each other according to a knight's move in chess. This pattern will efficiently dominate all vertices inside the grid, with the exception of a few vertices on the exterior which are not dominated. But these vertices can easily be efficiently dominated by augmenting the grid as indicated in Figure 4, the result of which is an efficient semigrid graph. \square

Notice in Figure 5 that the 10×10 grid graph $G_{10,10}$ is an induced subgraph of an efficient semigrid graph. However, in comparison to the semigrid which has $G_{5,5}$ as an induced subgraph, there are twice as many vertices exterior to the grid in Figure 5 to create an efficient domination. Further, these exterior vertices occur either five columns or five rows away from the previous set of exterior vertices. This indicates that the star pattern repeats every five rows or five columns.

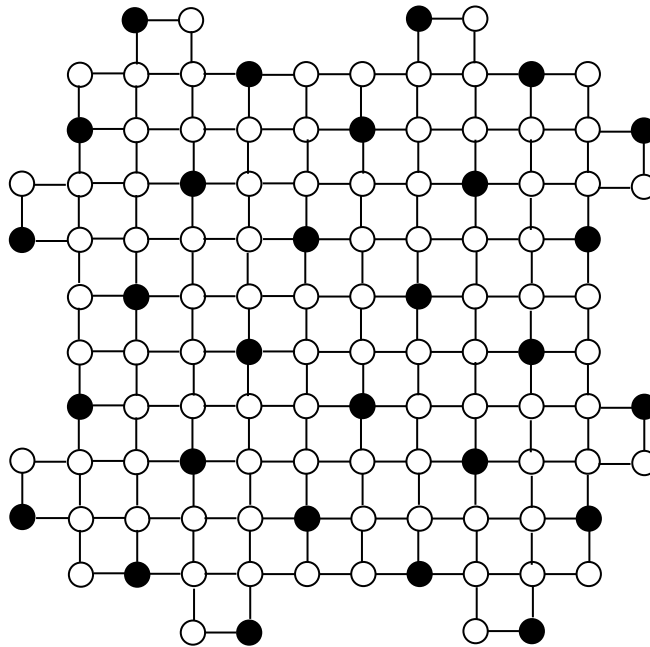


Figure 5. The 10×10 Grid $G_{10,10}$ as an Induced Subgraph of an Efficient Semigrid

3. Slant Grids

As previously defined, an $m \times n$ grid graph $G_{m,n}$ has vertex set $V = \{(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$ with (i, j) adjacent to (k, l) if $i = k$ and $|j - l| = 1$ or $j = l$ and $|i - k| = 1$. We define the *rectilinear distance*, $dist$, between vertices (i, j) and (k, l) as

$$dist((i, j), (k, l)) = |i - k| + |j - l|.$$

We note that $dist$ equals the number of edges in a shortest path in $G_{m,n}$ between (i, j) and (k, l) .

We extend the idea of a semigrid to look at the star pattern created on a minimally larger grid to $G_{m,n}$ formed by adding one row to both the top and bottom and one column to both the

left and right. Let $G_{m+2,n+2}$ be an $(m + 2) \times (n + 2)$ grid graph, where $(i, j) \in V(G_{m+2,n+2})$ if $0 \leq i \leq m + 1$ and $0 \leq j \leq n + 1$, and $G_{m,n}$ is a subgraph of $G_{m+2,n+2}$, such that the (i, j) entry of $G_{m,n}$ is the $(i + 1, j + 1)$ entry of $G_{m+2,n+2}$. We call any vertex in $V(G_{m+2,n+2}) - V(G_{m,n})$ an *exterior vertex*.

To mimic the star pattern created by an efficient dominating set on a semigrid, as in [11], we define a $(2,1)$ -slant grid $S(2,1)$ as an infinite graph with vertex set $V(S(2,1)) = \{(i, j) : i \in \mathbb{Z}, j \in \mathbb{Z} \text{ with the property that if } (i, j) \in V(S(2,1)) \text{ then so are } (i + 2, j + 1), (i + 1, j - 2), (i - 2, j - 1), \text{ and } (i - 1, j + 2)\}$. The edges that connect vertices $(i + 2, j + 1), (i + 1, j - 2), (i - 2, j - 1),$ and $(i - 1, j + 2)$ to (i, j) are in the edge set $E(S(2,1))$. Figure 6 shows a slant grid overlaid on $G_{m+2,n+2}$ so that the coordinate system created by $G_{m+2,n+2}$ is consistent with the slant grid. To highlight $G_{m,n}$ as a subgraph of $G_{m+2,n+2}$, we have removed the edges connecting exterior vertices of $G_{m+2,n+2}$ to vertices in $G_{m,n}$. We note that there are 5 isomorphic graphs of $S(2,1)$ depending on the smallest value of $k, 0 \leq k \leq 4$ for which the vertex $(0, k)$ of $G_{m+2,n+2}$ is a vertex of $S(2,1)$. We denote these five slant grids as $S^k(2,1), 0 \leq k \leq 4$. Figure 6 shows a partial slant grid with $k = 0$, because the slant grid intersects with vertex $(0, 0)$ in the upper left corner.

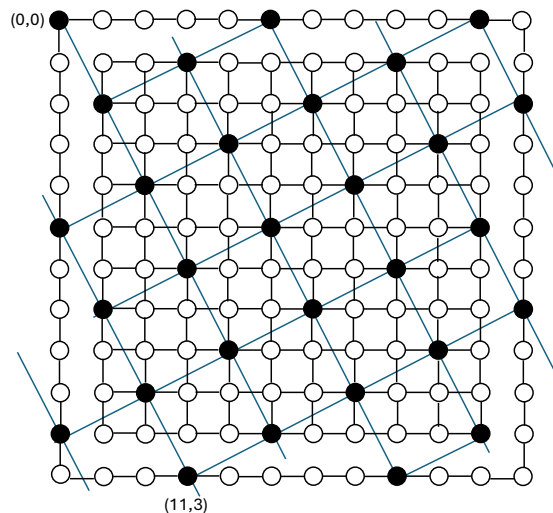


Figure 6. $G_{m+2,n+2}$ and $S^0(2,1)$ when $m = 10, n = 10, k = 0$

Let $V^k = V(S^k(2,1)) \cap V(G_{m+2,n+2})$. In [11] the following theorem is shown.

Theorem 3. For all $0 \leq k \leq 4$ and all $m, n \geq 1, V^k = V(S^k(2,1)) \cap V(G_{m+2,n+2})$ is a dominating set of $V(G_{m,n})$.

Proof. Let $(x, y) \in V(G_{m,n})$, and let $k \in \{0, 1, 2, 3, 4\}$. Let $(i, j) \in V^k$ so that (i, j) minimizes the rectilinear distance $dist$ to (x, y) . If $dist((i, j), (x, y)) = 0$, then $(x, y) = (i, j) \in V^k$. Otherwise, we wish to show that $dist((i, j), (x, y)) = 1$. Assume that $dist((i, j), (x, y)) \geq 2$. By definition of $S^k(2,1)$, if $(i, j) \in V(S^k(2,1))$ then so are $(i + 2, j + 1), (i + 1, j - 2), (i - 2, j - 1),$ and $(i - 1, j + 2)$. Note that each of these 4 vertices are rectilinear distance 3 from (i, j) . Further, any vertex of rectilinear distance 2 to (i, j) is adjacent to one of these four vertices. Hence, if $dist((i, j), (x, y)) \geq 2, (x, y)$ is closer to one of $(i + 2, j + 1), (i + 1, j - 2), (i - 2, j - 1),$ and $(i - 1, j + 2)$ than it is to (i, j) , a contradiction. Hence, $dist((i, j), (x, y)) \leq 1$ and (x, y) is dominated by $(i, j) \in V^k$. \square

Corollary 1. V^k dominates $G_{m,n}$ so that no vertex in $V(G_{m,n})$ is dominated more than once.

Proof. Suppose $(x, y) \in V(G_{m,n})$ is dominated by $v = (i, j) \in V^k$ and $w = (k, l) \in V^k$. Then $\text{dist}(v, w) = 2$, contradicting the fact that vertices in $S^k(2, 1)$ have minimum dist of 3. \square

This corollary implies that vertices in $V^k \cap V(G_{m,n})$ efficiently dominate some subgraph of $G_{m,n}$. The only vertices not dominated are on the boundary of $G_{m,n}$. These vertices could be removed in order to form an efficient dominating set for a new graph $G'_{m,n}$, the subgraph of $G_{m,n}$ efficiently dominated by $V^k \cap V(G_{m,n})$. Since there are five possible slant grids, $k = 0, \dots, 4$, we seek the value of k which minimizes the number of vertices that we need to remove to form $G'_{m,n}$.

4. Exterior Domination Adjustments

In [11], the authors show that we can move the exterior vertices in V^k to an adjacent vertex in $G_{m,n}$ to construct a minimum dominating set for $G_{m,n}$ composed entirely of vertices in $V(G_{m,n})$. However, moving such vertices to $G_{m,n}$ could create a dominating set that is not efficient because a vertex is dominated more than once. Consider the vertex at location $(11, 3)$ in Figure 6. If we move the exterior vertex in V^k , $(11, 3)$, to the vertex it is dominating in $G_{m,n}$, $(10, 3)$, we will create double-dominated vertices at locations $(9, 3)$ and $(10, 4)$. Hence, it is more efficient just to remove vertex $(10, 3)$ from the grid in creating $G'_{10,10}$. This is also true for $(0, 5), (5, 0), (7, 12) \in V^k$. In fact, noncorner adjustments will always create two or more double-dominated vertices when moved to $G_{m,n}$ by the nature of the slant grid. However, there are cases that occur at the corners of the grid graph where we actually can remove more exterior vertices than the number of double-dominated vertices created by moving the exterior vertex to $G_{m,n}$.

The corner adjustment procedures in [11] will create minimum dominating sets for $G_{m,n}$, but those dominating sets will not be efficient since they create possibly multiple double-dominated vertices which would need to be removed from the grid. Below, we look at how these corner adjustments lead to grid vertex removals so that there is an efficient dominating set of the resulting graph. We break the cases into whether k is even or odd.

We begin with the odd cases. Figure 7 shows that when $k = 1$, there are dominators on the boundary at locations $(0, 1)$ and $(3, 0)$. To move these dominators to $G_{m,n}$ in locations $(1, 1)$ and $(3, 1)$ creates double-dominated vertices at $(1, 2)$, $(2, 1)$, and $(3, 2)$. It is more efficient just to remove vertices $(1, 1)$ and $(3, 1)$, shown in gray, from $G_{m,n}$ to create efficient domination around that corner. When $k = 3$, Figure 7 shows that there are boundary dominators at $(0, 3)$ and $(4, 0)$. As before moving those vertices to $G_{m,n}$ creates double-dominated vertices at $(1, 2)$, $(1, 4)$, $(3, 1)$ and $(4, 2)$. It is more efficient to just remove the vertices $(4, 1)$ and $(1, 3)$, shown in gray, to create an efficient domination of the resulting graph.

When k is even, Figure 8 shows that some efficiencies can be realized by moving boundary dominators to $G_{m,n}$. When $k = 0$ we simply remove the boundary dominator at $(0, 0)$ and $G_{m,n}$ is still efficiently dominated around that corner. When $k = 2$, we combine the boundary dominators $(1, 0)$ and $(0, 2)$ to form a dominator at $(1, 1)$ to create one double-dominated vertex $(2, 1)$. Hence we remove two boundary dominators and only have to remove one vertex $(2, 1)$ from the $G_{m,n}$ to have efficient domination around that corner. Finally, when $k = 4$, we have two boundary dominators at $(2, 0)$ and $(0, 4)$. In making the adjustment from [11] to form a minimum dominating set, we move dominators at $(2, 0)$ and $(1, 2)$ to $(2, 1)$ and $(1, 3)$ respectively while removing the dominator at $(0, 4)$. This removes two boundary dominators, $(2, 0)$ and $(0, 4)$, while creating a dominating set in $G_{m,n}$ that double-dominates the vertices

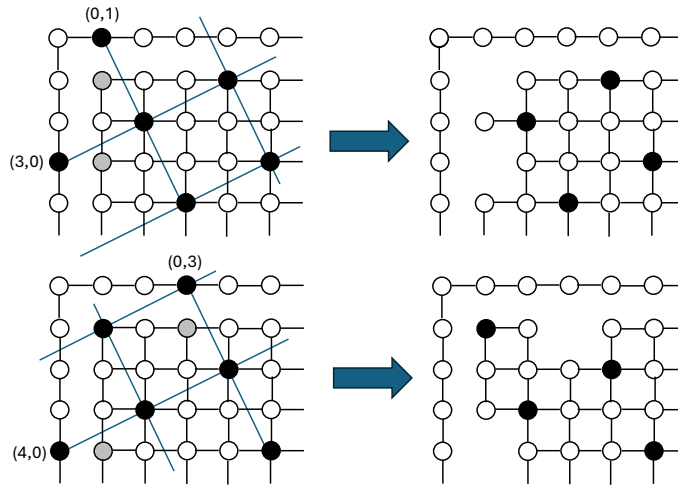


Figure 7. Corner Adjustments for Cases Where $k = 1$ and $k = 3$

$(3, 1)$ and $(2, 3)$. Hence, we remove vertices $(3, 1)$ and $(2, 3)$ from $G_{m,n}$ to create efficient corner domination. Note that we could just as well have removed vertices $(2, 1)$ and $(1, 4)$ and achieved the same net result.

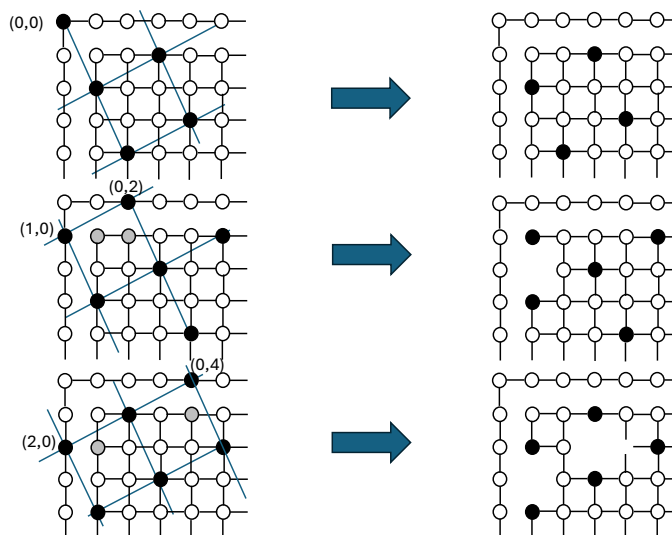


Figure 8. Corner Adjustments for Cases where $k = 0, 2, 4$

5. Upper Bounds for $\varepsilon_v^-(G_{m,n})$ using Slant Grids

In this section we use the results from the previous section to determine an upper bound for the minimum number of vertex removals from $G_{m,n}$ needed to achieve an efficient dominating set of $G'_{m,n}$. Given $G_{m,n}$, we overlay a slant grid $S^k(2, 1)$, for some k , over $G_{m+2,n+2}$ to form a set of vertices that dominate the vertices in $G_{m,n}$, but not all of them are contained in $G_{m,n}$. Any vertex in $v \in V(G_{m,n})$ that is dominated by an exterior vertex in $V(G_{m+2,n+2}) - V(G_{m,n})$ will be a potential vertex to be removed from $G_{m,n}$. In fact, if v is not located around a corner of $G_{m,n}$, then it is most efficient to remove v from $V(G_{m,n})$ (along with its incident edges). Around the corners of $G_{m,n}$, when k is odd or $k = 4$, we will also just remove the vertex v . However, when $k = 0$ or $k = 2$, we actually remove more vertices from the boundary for every double-dominated vertex created. Hence, we would like to maximize the number of times we

have the opportunity to create the scenarios at a corner where $k = 0$ or $k = 2$.

Before that discussion, if we could count the number of dominators that are on the exterior and subtract off the number of dominators not on the exterior, this would give us an upper bound on the number of vertices needed to be removed from $G_{m,n}$ in order for the dominators in $V^k \cap V(G_{m,n})$ to form an efficient dominating set. This upper bound can be lowered by four vertices if each corner adjustment results in cases where $k = 0$ or $k = 2$. In [11], we show that

$$\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor \leq |V^k| \leq \left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor + 1.$$

By extension of this idea, it follows that

$$\left\lfloor \frac{\binom{m}{5} \binom{n}{5}}{5} \right\rfloor \leq |V^k \cap V(G_{m,n})| \leq \left\lfloor \frac{\binom{m}{5} \binom{n}{5}}{5} \right\rfloor + 1.$$

Combining these two ideas above, we have the following.

Lemma 1. *If $\varepsilon_v^-(G_{m,n})$ is the minimum number of vertices that need to be removed from $V(G_{m,n})$ in order to create an efficient graph, then*

$$\varepsilon_v^-(G_{m,n}) \leq \left(\left\lfloor \frac{(m+2)(n+2)}{5} \right\rfloor + 1 - \left\lfloor \frac{\binom{m}{5} \binom{n}{5}}{5} \right\rfloor \right).$$

This bound can be improved further by maximizing the number of adjustments at the corners that lead to more removals from the boundary of $G_{m+2,n+2}$ than would need to be removed from $G_{m,n}$, i.e. cases where the corners have an abundance of $k = 0$ or $k = 2$ cases. In what follows, we show, for each $4 \leq m \leq 8$ and $n \geq m$, the configuration of the slant grid (choice of k) that allows for the minimum number of vertices to be removed from the grid graph to create an efficient subgraph in this way.

When $m = 4$, in Figure 9, we show that $k = 0$ allows for the same case to show up in the lower left portion of the grid as well. For the grid graph $G_{4,4}$ this scenario shows up in all four corners. By inspecting all five possibilities ($k = 0, 1, 2, 3, 4$), we can see that this case minimizes the number of grid vertices needed to be removed in order to achieve efficient domination in the resulting graph. When $n = 4$, we note that we remove zero vertices to achieve efficient domination, as was noted in [12]. As n progresses from 4 to 8, we see that the corners on the right-hand-side of the figures have different configurations. Upon removing the vertices as dictated by the specific type of corner adjustment, we create efficient augmented grids.

Note that in the 4×9 case (not shown), that the corner removals present in the 4×4 case reappear, allowing for only the removal of two vertices (those present in the interior of the 4×7 and 4×8 cases). Likewise, the 4×10 case will have the same corner removal pattern as the 4×5 case, and so on. However, the 4×10 case will have two more vertices removed compared to the 4×5 case, namely vertices at locations $(1, 5)$ and $(4, 5)$. So, the same corner removal pattern will appear from these five starting positions for every increase in n by 5. For each increase in n by 5, we will need to remove two extra interior vertices of the grid graph as compared to $4 \times (n - 5)$. To illustrate this more concretely, compare the removal pattern for $G_{4,12}$ in Figure 10 to the removal pattern of $G_{4,7}$ in Figure 9 to see that the corner adjustments are the same but two extra vertices at $(1, 10)$ and $(4, 10)$ must be removed in forming $G'_{4,12}$ because of the cyclical pattern in slant grids. We have indicated the vertices $(1, 5)$, $(4, 5)$, $(1, 10)$, and $(4, 10)$ in gray.

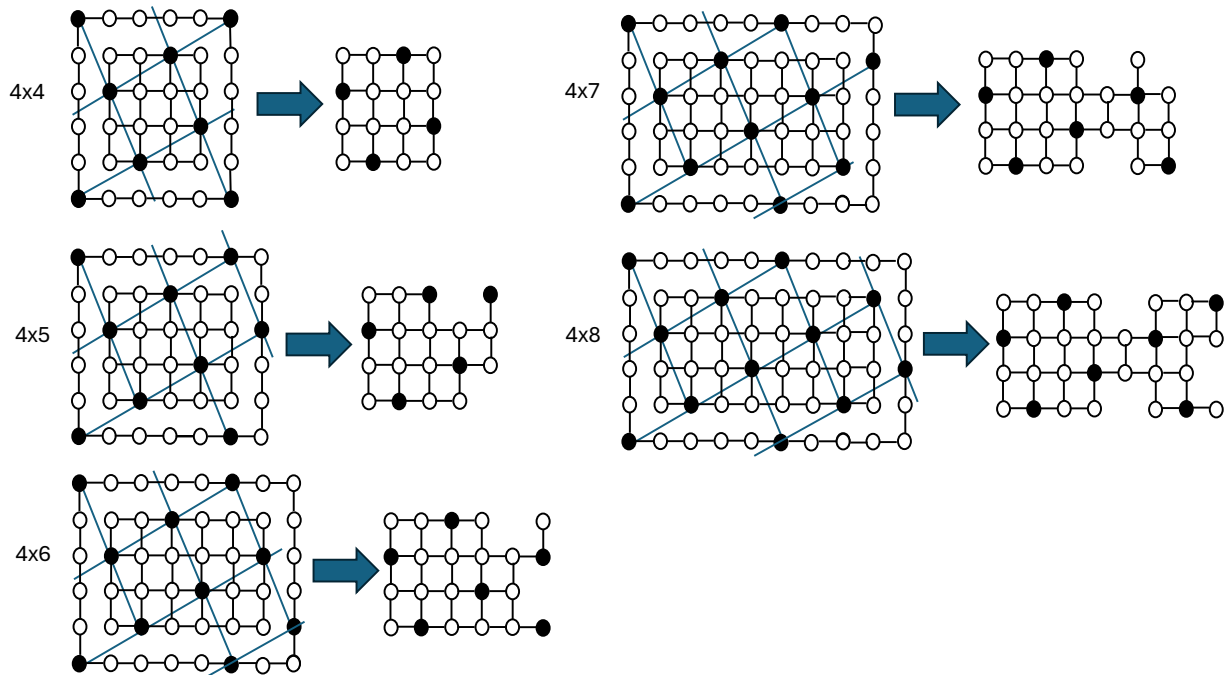


Figure 9. Augmented Grid Graphs with Efficient Domination for $4 \times k$ Grids, where $4 \leq k \leq 8$

When $m = 5$, the corner configuration in the upper left corner that works best for $n = 5, 6, 7, 8$ is when $k = 2$ as shown in Figure 11. By inspection, every other value of k either removes the same number of vertices or more.

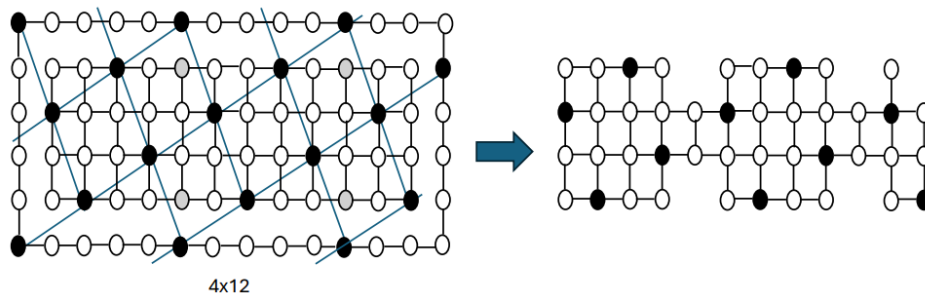


Figure 10. Removal Pattern for $G_{4,12}$

It is interesting to note that we do not show the ideal case for $G_{5,9}$. This is because a dimension of nine is five units more than a dimension of four. If we look at $G_{9,5}$ the corner configurations that works best are the same as the case for $G_{4,5}$. To get the best corner configuration for $G_{5,9}$ we would flip the grid along the $m = n$ axis. We show this in Figure 12.

When $m = 6$, we only show the best corner configurations for $G_{6,6}$, $G_{6,7}$, and $G_{6,8}$ as configurations of $G_{6,9}$ and $G_{6,10}$ are the same as $G_{4,6}$ and $G_{5,6}$, respectively. Further, corner configurations for $G_{6,11}$, $G_{6,12}$, and $G_{6,13}$ are the same as $G_{6,6}$, $G_{6,7}$, and $G_{6,8}$ respectively. The patterns then repeat for an increase in m or n by five. The minimum number of vertex removals from these corner configurations are shown in Figure 13.

Finally, as before, we only need to show the optimal corner configurations for $G_{7,7}$, $G_{7,8}$ and $G_{8,8}$, since other configurations for these values of m can be found as extensions of cases where $m = 4, 5, 6$. These corner configurations inspire vertex removals as shown in Figure 14.

From the base cases shown in Figures 9 through 14 we can show how vertex removals grow

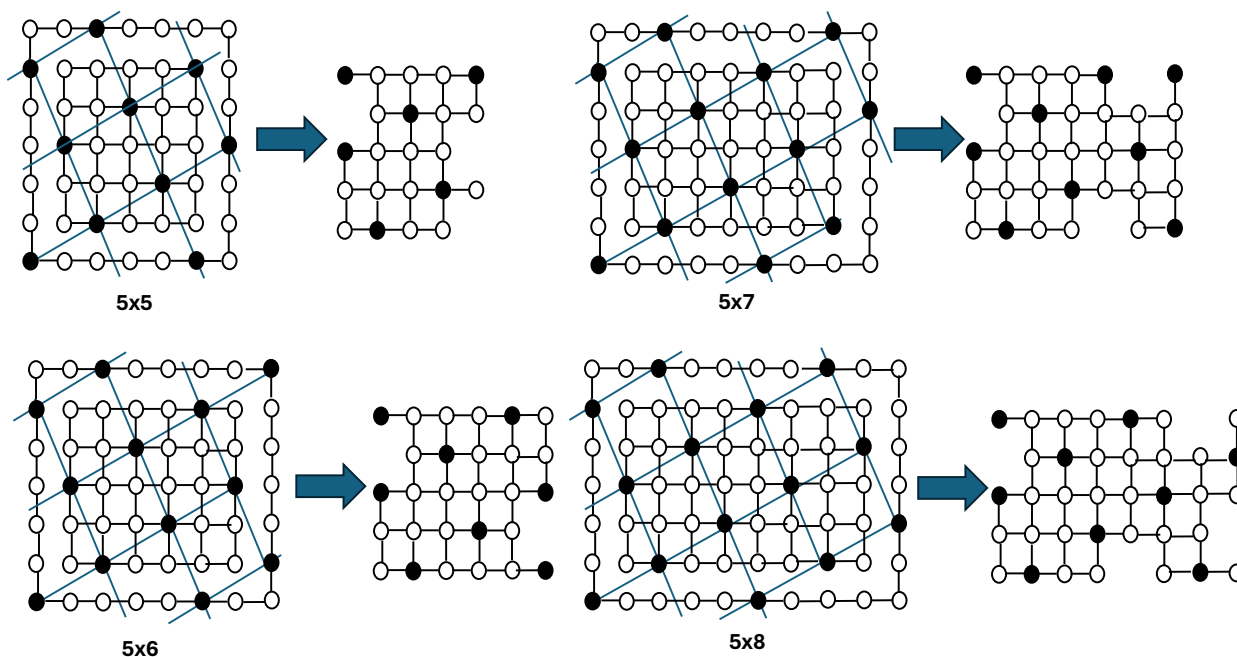


Figure 11. Augmented Grid Graphs with Efficient Domination for $5 \times k$ Grids, where $5 \leq k \leq 8$

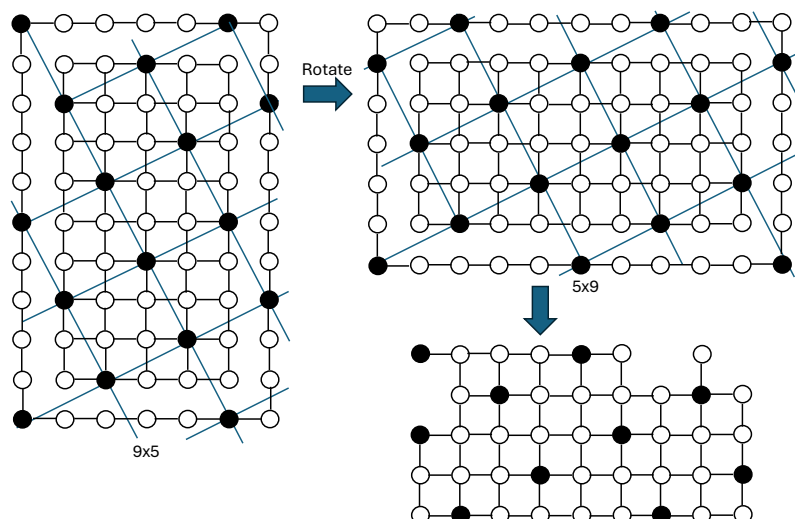


Figure 12. Example Showing that Corner Adjustments for $G_{9,5}$ are the Same as for $G_{5,9}$

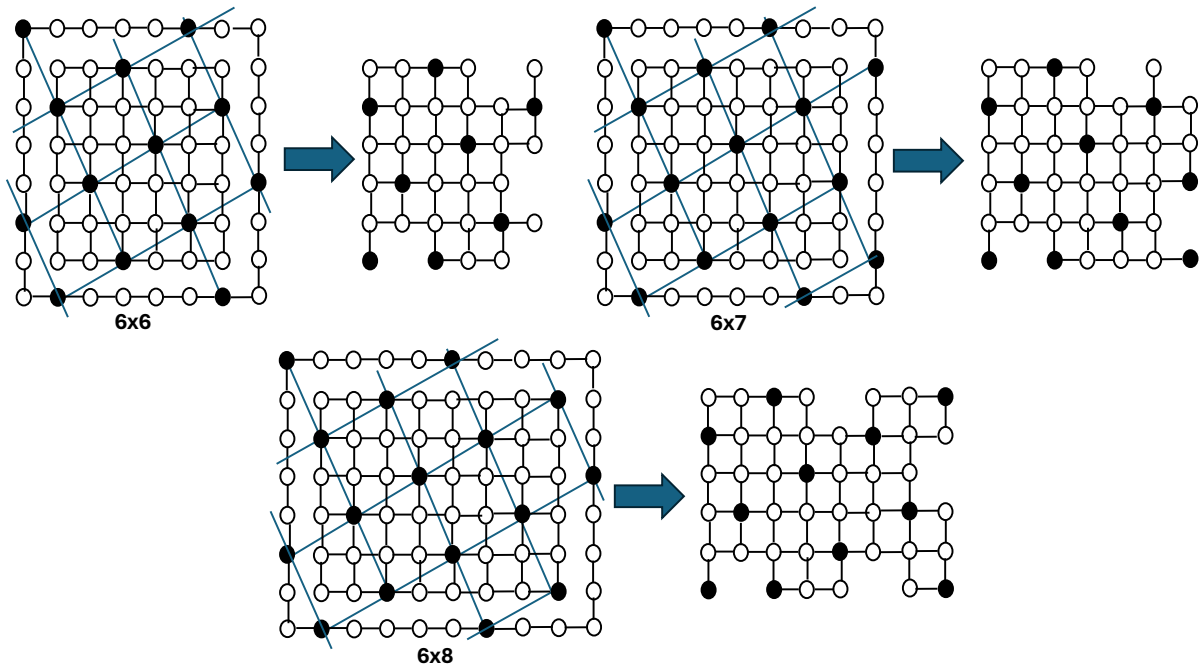


Figure 13. Augmented Grid Graphs with Efficient Domination for $6 \times k$ Grids, where $6 \leq k \leq 8$

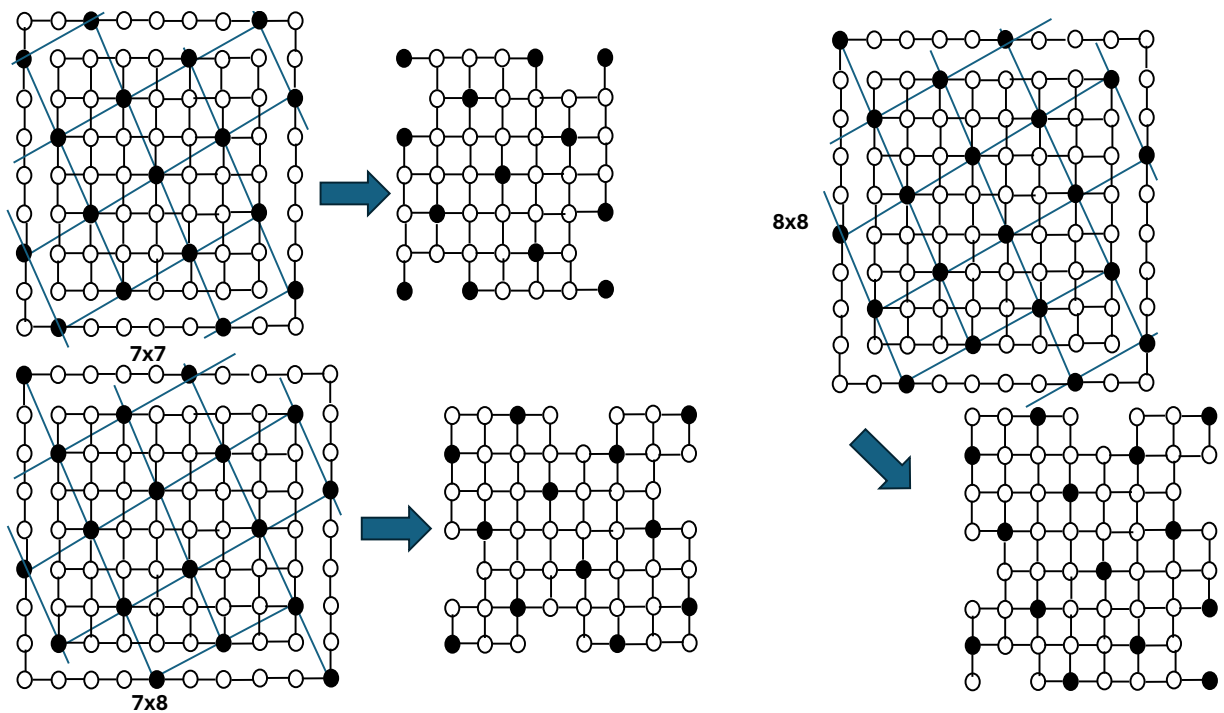


Figure 14. Augmented Grid Graphs with Efficient Domination for 7×7 , 7×8 , and 8×8 Grids

as m and n increase by five. Table 1 shows the minimum number of vertices which need to be removed from $G_{m,n}$ using this slant grid method. This will serve as an upper bound for $\varepsilon_v^-(G_{m,n})$ in these cases.

Grid Graph	No. Vertices Removed	Grid Graph	No. Vertices Removed
$G_{4,4}$	0	$G_{5,8}$	4
$G_{4,5}$	2	$G_{6,6}$	4
$G_{4,6}$	2	$G_{6,7}$	4
$G_{4,7}$	3	$G_{6,8}$	4
$G_{4,8}$	3	$G_{7,7}$	4
$G_{5,5}$	3	$G_{7,8}$	4
$G_{5,6}$	2	$G_{8,8}$	5
$G_{5,7}$	3		

Table 1. Bounds on Minimum Number of Vertices to Remove (Base Cases)

For every increase of m or n by five we will have to remove two vertices from the grid graph since slant grid dominators appear on the boundary of $G_{m+2,n+2}$ every five columns or rows. Hence, we can summarize the upper bounds on vertex removals for $G_{m,n}$ with the following result.

Theorem 4. *Given $m, n \geq 4$, Table 2 gives upper bounds on the minimum number of vertices to be removed from $G_{m,n}$ so that the resulting graph has an efficient dominating set. This table assumes that i and j are non-negative integers.*

Grid Graph	UB for $\varepsilon_v^-(G_{m,n})$
$G_{5i+4,5j+4}$	$0 + 2i + 2j$
$G_{5i+4,5j+5}$	$2 + 2i + 2j$
$G_{5i+4,5j+6}$	$2 + 2i + 2j$
$G_{5i+4,5j+7}$	$3 + 2i + 2j$
$G_{5i+4,5j+8}$	$3 + 2i + 2j$
$G_{5i+5,5j+5}$	$3 + 2i + 2j$
$G_{5i+5,5j+6}$	$2 + 2i + 2j$
$G_{5i+5,5j+7}$	$3 + 2i + 2j$
$G_{5i+5,5j+8}$	$4 + 2i + 2j$
$G_{5i+6,5j+6}$	$4 + 2i + 2j$
$G_{5i+6,5j+7}$	$4 + 2i + 2j$
$G_{5i+6,5j+8}$	$4 + 2i + 2j$
$G_{5i+7,5j+7}$	$4 + 2i + 2j$
$G_{5i+7,5j+8}$	$4 + 2i + 2j$
$G_{5i+8,5j+8}$	$5 + 2i + 2j$

Table 2. Bounds on Minimum Number of Vertices to Remove

6. Case by Case Improvements to Upper Bounds

While Theorem 4 provides general upper bounds for cases when $m, n \geq 4$, we can improve those bounds in specific cases. As the authors in [11] point out, for some values of m , minimum

dominating sets are found by repeatedly flipping the dominating structure of a γ -set. For example, Figure 9 shows the minimum dominating set for a 4×4 grid. This γ -set is also efficient. In [11] the authors show that the optimal γ -set for a 4×7 grid is the γ -set for a $G_{4,4}$ reflected over column four of the grid, producing the γ -set seen in Figure 15. Note that this γ -set can be made efficient by removing the vertex $(1, 4)$ shown in gray in the figure.

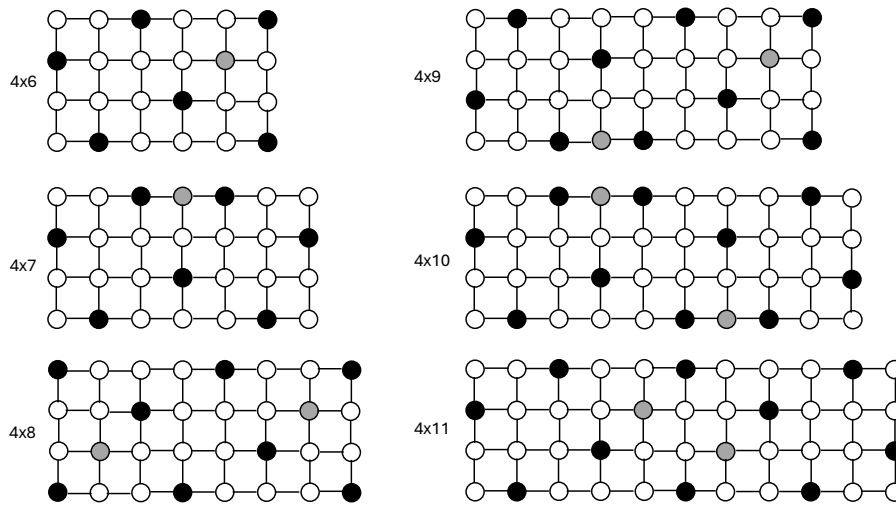


Figure 15. Efficient Domination of $G_{4,k}$, $6 \leq k \leq 11$

In fact, this idea of repeatedly flipping/reflecting the γ -set for $G_{4,4}$ creates an optimal γ -set for $G_{4,4+3k}$, $k \geq 0$. These γ -sets have a double-dominated vertex that needs to be removed for each reflection in order to make the γ -set produce an efficient dominating set. This is shown for $G_{4,7}$ and $G_{4,10}$ in Figure 15 and $G_{4,13}$ in Figure 16.

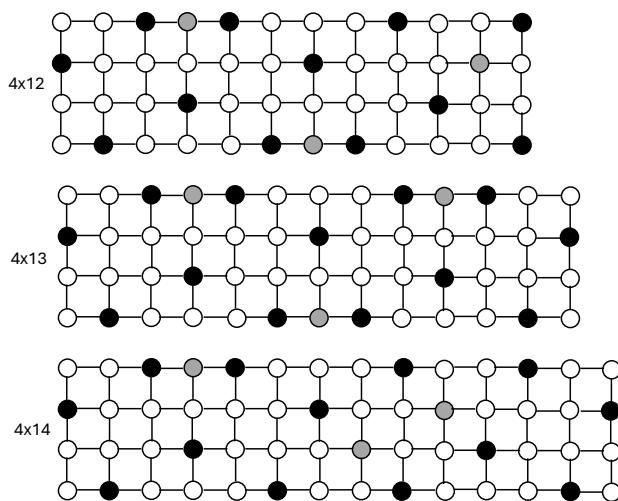


Figure 16. Efficient Domination of $G_{4,k}$, $12 \leq k \leq 14$

There is a simple pattern that emerges for creating efficient domination from $G_{4,k}$ when k is not of the form $4 + 3i$. Note that for $G_{4,3k}$ we see the same pattern of reflected dominators as with $G_{4,3k-2}$ with the last two columns the same as the last two columns of $G_{4,6}$. For $G_{4,3k+2}$, $k \geq 1$, while $G_{4,8}$ has a slightly different pattern, $G_{4,11}$'s domination pattern is to take the pattern for $G_{4,6}$ and reflect the pattern over the sixth column and then flip this pattern (the last five columns) again vertically. For $G_{4,14}$ we take the dominating pattern for $G_{4,6}$ and attach a reflected and flipped pattern from $G_{4,6}$ to account for the last five columns (as we did for

$G_{4,11}$). We can continue to do this for each of the grids $G_{4,3k+2}$. Note that the gray dots dominate the vertices that would need to be removed for efficient domination in these grids. Note that the removed vertices grow at a rate of one removal for every pattern “flip” and thus is a rate of one removal for every three columns, which is lower than the rate of $\frac{2}{3}$ as in Theorem 2. Table 3 summarizes these upper bound improvements.

Grid Graph	k	UB for $\varepsilon_v^-(G_{m,n})$
$G_{4,4+(3k-1)}$	$k \geq 1$	k
$G_{4,4+(3k)}$	$k \geq 1$	k
$G_{4,4+(3k+1)}$	$k \geq 1$	$k + 1$

Table 3. Bounds on Minimum Number of Vertices to Remove for Grids where $m = 4$ and $n \geq 6$

When $m = 5$, there are several cases where we can improve upon the bounds given in Theorem 4, namely when $n = 5k$ or $n = 5k + 3$. While $G_{5,5}$ is a special case, consider the dominating set for $G_{5,8}$ in Figure 17. This dominating set creates the need to only remove three vertices from the grid instead of four given by Theorem 4. Notice that in creating a dominating set from $G_{5,10}$, we take the dominating set for $G_{5,8}$ and attach two extra columns with a flipped dominating pattern of the first two columns of the dominating set of $G_{5,8}$. To create the dominating set for $G_{5,13}$, we take the dominating set for $G_{5,8}$ and flip columns three through seven over column eight to create the dominating pattern for columns nine through thirteen. In each of these we create dominating sets which become efficient when removing the gray vertices and require fewer vertex removals than Theorem 4 requires.

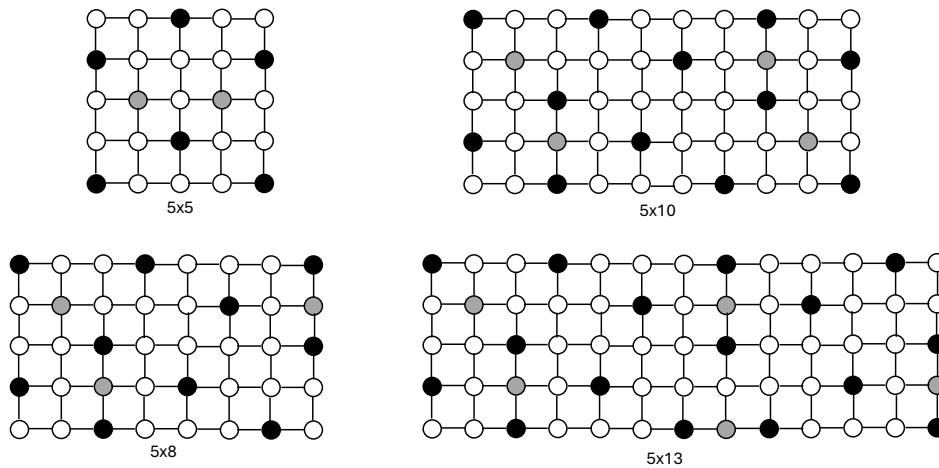


Figure 17. Efficient Domination of $G_{5,k}$, $k = 5, 8, 10, 13$

In Figure 18, the creation of a minimum dominating set for $G_{5,15}$ involves taking the dominating pattern for $G_{5,8}$ and reflecting the first seven columns over column eight. Further, the dominating set for $G_{5,18}$ takes the dominating set for $G_{5,13}$ and reflects columns eight through twelve over column thirteen to create the dominating pattern for $G_{5,18}$. This reflecting pattern continues for all grids of the form $G_{5,5k}$ and $G_{5,5k+3}$.

Note that the number of vertices that need to be removed grows at the same rate that is described in Theorem 4, which is two removals for every five columns. However, $G_{5,5}$ and $G_{5,8}$ has a dominating set which requires fewer removals than described by Theorem 4, namely two

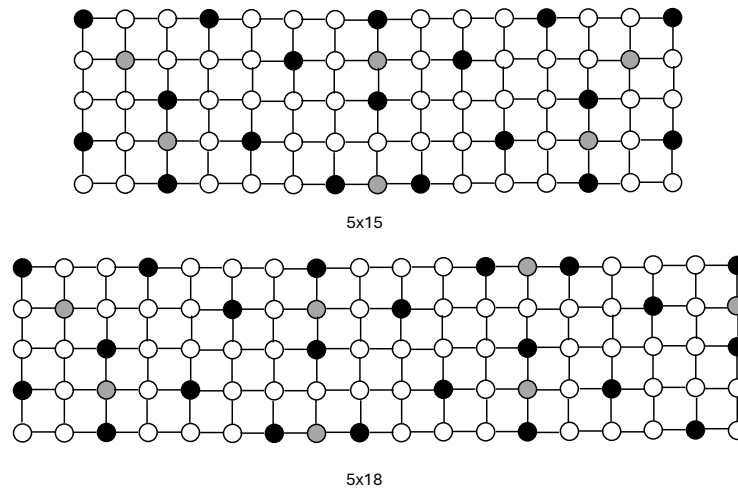


Figure 18. Efficient Domination of $G_{5,k}$, $k = 15, 18$

and three respectively rather than three and four. Table 4 describes the number of vertex removals needed per value of k when $m = 5$.

Grid Graph	k	UB for $\varepsilon_v^-(G_{m,n})$
$G_{5,5k}$	$k \geq 1$	$2k$
$G_{5,5k+3}$	$k \geq 1$	$2k + 1$

Table 4. Bounds on Minimum Number of Vertices to Remove for Grids where $m = 5$ and $n = 5k$ or $n = 5k + 3$

When $m = 6$, Figure 19 shows that we can do better than the bounds described in Theorem 4 for the cases where $n = 6$ and $n = 7$. For each of these two, we can create efficient dominating sets for that only require removing three vertices instead of four. Then, using a similar slant grid approach, we must remove two vertices for every five column we add from that point. So, in creating an efficient domination for $G_{6,11}$ and $G_{6,12}$, Figure 19 shows what those dominating sets would look like. This pattern will continue for grids of the form $G_{6,6+5k}$ and $G_{6,7+5k}$.

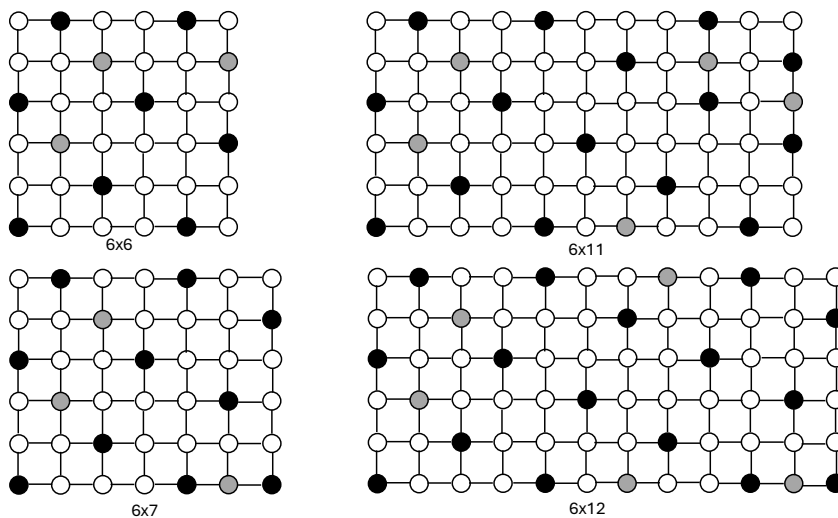


Figure 19. Efficient Domination of $G_{6,k}$, $k = 6, 7, 11, 12$

Table 5 summarizes these improvements to the upper bounds for $\varepsilon_v^-(G_{m,n})$ for certain cases where $m = 6$.

Grid Graph	k	UB for $\varepsilon_v^-(G_{m,n})$
$G_{6,6+5k}$	$k \geq 1$	$3 + 2k$
$G_{6,7+5k}$	$k \geq 1$	$3 + 2k$

Table 5. Bounds on Minimum Number of Vertices to Remove for Grids where $m = 6$ and $n = 6 + 5k$ or $n = 7 + 5k$

Finally, when $m = 7$, there is one case where we can achieve a better upper bound on $\varepsilon_v^-(G_{7,k})$ than described in Theorem 4. It is the case when $k = 7$. Figure 20 shows that we can flip the optimal domination pattern for $G_{4,7}$ over row four and create a γ -set for $G_{7,7}$ which requires only three vertices to be removed to achieve efficient domination.

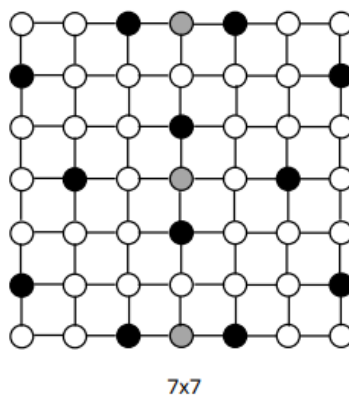


Figure 20. Efficient Domination of $G_{7,7}$

To check the optimality of these results, we created the following integer programming model and ran it through a solver (glpk) to find the minimum number of vertices which would need to be removed to achieve efficient domination for $G_{m,m}$, for all values of $4 \leq m, n \leq 16$. It verified that the upper bound either achieved by Theorem 4 or the tables in this section were indeed optimal. However, we do not currently have an approach to prove this result.

in formulating the integer program, we set up three types of decision variables. We let $x(i, j)$ be a binary variable indicating whether or not the vertex at (i, j) will be selected as a dominator for $G'_{m,n}$. We let $y(i, j)$ be a binary variable indicating whether or not the vertex at (i, j) will be removed because it is not being dominated. Finally, we let $z(i, j)$ be an integer variable indicating how many times the vertex at (i, j) will be dominated. If $z(i, j) > 1$ in the optimal solution, then we reduce the solution to one, since it could not be removed more than once. For each of these variables, $1 \leq i \leq m$ and $1 \leq j \leq n$.

Integer Programming Model for $\varepsilon_v^-(G_{m,n})$

min $\sum_{i=1}^m \sum_{j=1}^n (y(i, j) + z(i, j))$

s.t.

Constraints for the Corners

$x(1, 1) + x(2, 1) + x(1, 2) = 1 - y(1, 1) + z(1, 1),$
 $x(1, n) + x(1, n - 1) + x(2, n) = 1 - y(1, n) + z(1, n),$
 $x(m, 1) + x(m - 1, 1) + x(m, 2) = 1 - y(m, 1) + z(m, 1),$
 $x(m, n) + x(m - 1, n) + x(m, n - 1) = 1 - y(m, n) + z(m, n),$

Constraints for the Non-Corner Sides of the Grids

$$x(i, 1) + x(i - 1, 1) + x(i + 1, 1) + x(i, 2) = 1 - y(i, 1) + z(i, 1), \quad 2 \leq i \leq m - 1,$$

$$x(i, n) + x(i - 1, n) + x(i + 1, n) + x(i, n - 1) = 1 - y(i, n) + z(i, n), \quad 2 \leq i \leq m - 1,$$

$$x(1, j) + x(1, j - 1) + x(1, j + 1) + x(2, j) = 1 - y(1, j) + z(1, j), \quad 2 \leq j \leq n - 1,$$

$$x(m, j) + x(m, j - 1) + x(m, j + 1) + x(m - 1, j) = 1 - y(m, j) + z(m, j), \quad 2 \leq j \leq n - 1,$$

Constraints for the Interior of the Grid

$$x(i, j) + x(i - 1, j) + x(i + 1, j) + x(i, j - 1) + x(i, j + 1) = 1 - y(i, j) + z(i, j), \quad 2 \leq i \leq m - 1, 2 \leq j \leq n - 1,$$

$$x(i, j), y(i, j) \in \{0, 1\}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n,$$

$$z(i, j) \text{ integer}, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n.$$

7. Conclusions and Future Work

In this paper, we have established upper bounds on the vertex efficiency deletion number for grid graphs. These bounds were found using a construction approach in [11] designed to create γ -sets for grids. These approaches were modified to indicate which vertices in a grid graph need to be removed in order to create efficient domination from vertices that appear in a slant grid pattern within the grid. Although there are multiple slant grid overlays, we indicate the one for each value of $m \geq 4$ leading to the minimum number of vertices which need to be removed from $G_{m,n}$, for each $n \geq 4$, that leaves an efficient resulting graph.

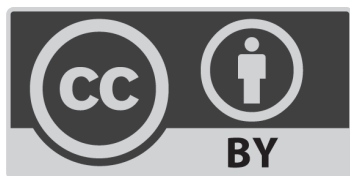
For certain values of m and n , the bounds given in Theorem 4 could be improved. Using an integer programming model as a guide, we found improvements to the upper bounds listed in Theorem 4. Although we believe the bounds listed in this paper are the exact value for $\varepsilon_v^-(G_{m,n})$ for $m, n \geq 4$, it is still an open question to find proofs for this.

Further, we have focused on removing vertices from the grid to produce efficient domination in the resulting graph. However, we could ask what is the smallest efficient semigrind, one having no leaves, that contains a given grid $G_{m,n}$ as an induced subgraph. In other words, what is the minimum addition to the grid that results in an efficient semigrind. In addition, our approach to create the upper bounds in Theorem 4 removed vertices on the exterior boundary of $G_{m,n}$. However, we found improvements for certain cases that allowed interior vertices to be removed. We might ask if there are ways to achieve efficient domination for grids where only interior vertices are removed. As a quick example, $G_{3,5}$ has an efficient domination when the interior vertex at location $(2, 3)$ is removed.

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