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Article

Fault Tolerant Metric Dimension of Arithmetic Graphs

Muhammad Shoaib Sardar[1,*], Komal Rasheed[2], Murat Cancan[3], Mohammad Reza Farahani[4], Mehdi Alaeiyan[4], and Shobha V. Patil[5]

[1] School of Mathematics and Statstics, Gansu Center for Applied Mathematics, Lanzhou University, Lanzhou, 730000, China

[2] School of Mathematics, Minhaj University, Lahore, Pakistan

[3] Faculty of Education, Yuzuncu Yil University, van, Turkey

[4] Department of Mathematics and Computer Science, Iran University of Science and Technology(IUST) Narmak Tehran 16844, Iran

[5] Department of Mathematics KLE Dr. MSSCET, Belagavi, Karnataka India

* **Correspondence:** shoaibsardar093@gmail.com

Abstract: For a graph G, two vertices $x, y \in G$ are said to be resolved by a vertex $s \in G$ if $d(x|s) \neq d(y|s)$, where $d(x|s)$ denotes the distance between x and s. The minimum cardinality of such a resolving set R in G is called the *metric dimension*. A resolving set R is said to be fault-tolerant if, for every $p \in R$, the set $R - p$ preserves the property of being a resolving set. The *fault-tolerant metric dimension* of G is the minimal possible order of a fault-tolerant resolving set. The concept of metric dimension has wide applications in areas where connection, distance, and network connectivity are critical. This includes understanding the structure and dynamics of complex networks, as well as addressing problems in robotic network design, navigation, optimization, and facility placement. By utilizing the concept of metric dimension, robots can optimize their methods for localization and navigation using a limited number of reference points. As a result, various applications in robotics, such as collaborative robotics, autonomous navigation, and environment mapping, have become more precise, efficient, and resilient. The arithmetic graph A_l is defined as the graph where the vertex set is the set of all divisors of a composite number l, where $l = p_1^{\gamma_1} p_2^{\eta_2} \cdots p_n^{\alpha_n}$ and the p_i's are distinct primes with $p_i \geq 2$. Two distinct divisors x and y of l are said to have the same parity if they share the same prime factors (e.g., $x = p_1 p_2$ and $y = p_1^2 p_2^3$ have the same parity). Furthermore, two distinct vertices $x, y \in A_l$ are adjacent if and only if they have different parity and $\gcd(x, y) = p_i$ (greatest common divisor) for some $i \in \{1, 2, \ldots, t\}$. This article focuses on the investigation of the arithmetic graph of a composite number l, referred to throughout as A_l. In this study, we compute the fault-tolerant resolving set and the fault-tolerant metric dimension of the arithmetic graph A_l, where l is a composite number.

Keywords: Arithmetic graphs, Simple connected graphs, Fault-tolerant metric dimension, Metric Dimension, Resolving set

1. Introduction

Graph theory is the study of graphs, a type of mathematical structure used to depict interactions between two entities. Graphs are one of the most ubiquitous data structures, used in fields as diverse as physics, sociology, architecture, chemistry, genetics, electrical engineering, operational research, and linguistics to model a wide range of linkages and processes. In this paper, we study graph resolvability, an essential concept in metric graph theory that is used in facility placement problems, networking, robot navigation, mathematical and pharmaceutical chemistry, and Mastermind games.

Over 3000 problems have been identified as NP-complete. Many practical issues, such as routing, fault tolerance, coding, embedding, resolvability, and coloring, lend themselves naturally to modeling and explanation using graph theoretical language [1]. Studies of the metric properties of graphs, such as the metric dimension and fault-tolerant metric dimension, have become increasingly popular over the past few decades due to their practical applications.

In 1975, Slater (and, independently, Harary and Melter) introduced the concept of the metric dimension. Recent progress has been made with the introduction of the fault-tolerant metric dimension [2, 3]. Censors are defined as metric base components in [4]. If some censors fail, we may not have enough data to handle the threat (e.g., fire, intruder). Hernando et al. developed the concept of the fault-tolerant metric dimension to address such issues [5]. A fault-tolerant resolving set continues to provide accurate results even if one of the censors fails. Consequently, the fault-tolerant metric dimension is useful in all scenarios where the traditional metric dimension has found applications.

Raza et al. investigated the fault-tolerant metric dimension of some families of convex polytopes [6]. Fault-tolerant resolvability in specific crystal structures was explored in [7]. Sharma and Bhat studied the fault-tolerant metric dimensions of a two-fold heptagonal-nonagonal circular ladder [8]. They demonstrated that such a ladder has the same metric dimension as a ladder with the same number of rungs. Additionally, they explored its fault-tolerant metric dimension, showing that the metric basis and the edge metric basis are distinct. For more information on fault-tolerant metric dimensions, we encourage interested readers to consult the literature [9–12]. In this paper, we only analyze basic graphs that are connected and undirected.

2. Notations and Preliminary Results

A graph, denoted by the letter G, consists of two sets: the vertex set, denoted $V(G)$, and the edge set, denoted $E(G)$. The components of $V(G)$ and $E(G)$, known as the vertices and edges of G, respectively, form the graph G. In a graph, the length of the path connecting two vertices is called the distance between those vertices. The distances between two vertices p and q with respect to the set R are represented by $d(p|R)$ and $d(q|R)$, respectively. A vertex set $R \subseteq V(G)$ is said to resolve the underlying graph G if, for any two vertices $p, q \in V(G)$, there exists a vertex $x \in R$ that resolves p and q. Two vertices $x, y \in G$ are resolved by a vertex $s \in G$ if $d(x|s) \neq d(y|s)$. There may be multiple resolving sets for a graph. The set $V(G)$ is trivially a resolving set. The minimum cardinality of a resolving set in G is called its metric dimension, denoted by $\beta(G)$ [13].

A resolving set R is fault-tolerant if, for every $p \in R$, $R - p$ preserves the property of being a resolving set. Analogous to the metric dimension, the fault-tolerant metric dimension is the minimum number $\beta'(G)$ such that there exists a fault-tolerant resolving set of order $\beta'(G)$. A fault-tolerant resolving set of minimal order is called the fault-tolerant metric dimension of G [13]. The objective of this study is to classify arithmetic graphs according to their fault-tolerant metric dimension. Arithmetic graphs are important in applications such as aircraft design, electronic circuit design, computer engineering, and transportation systems because of their ability to model complex networks.

The arithmetic graph A_l is defined as the graph whose vertex set is the set of all divisors of a composite number l, where $l = p_1^{\gamma_1} p_2^{\eta_2} \cdots p_n^{\alpha_n}$, and p_i's are distinct primes with $p_i \geq 2$. Two distinct divisors x and y of l are said to have the same parity if they have the same prime factors (e.g., $x = p_1p_2$ and $y = p_1^2p_2^3$ have the same parity). Furthermore, two distinct vertices $x, y \in A_l$ are adjacent if and only if they have different parity and $\gcd(x, y) = p_i$ (greatest common divisor) for some $i \in \{1, 2, \ldots, t\}$ (see Figure 1) [14].

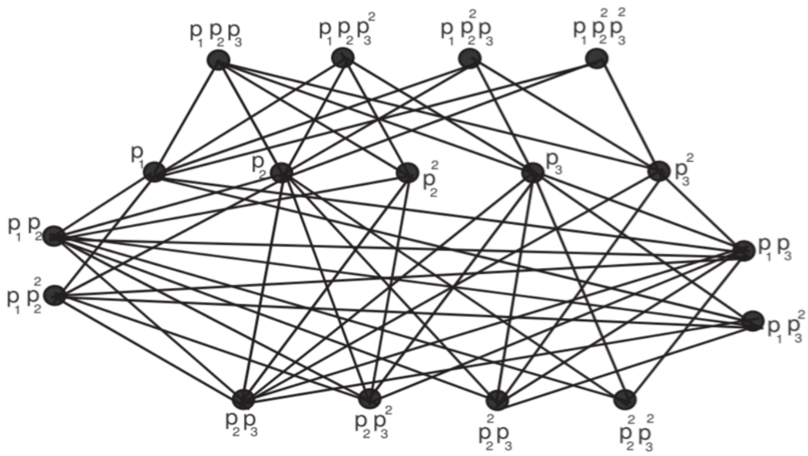

Figure 1. The arithmetic graph A_l, where $l = p_1p_1^2p_2^2$.

3. Main Results

The main results of this paper are discussed in this section.

Theorem 1. *Let A_l be an arithmetic graph, where l is a composite number with the canonical form $l = p_1^{\gamma}p_2^{\eta}$, and $\gamma, \eta \geq 1$. Then:*

1. *$\beta'(A_l) = 2$, for $\gamma = 1$, $\eta = 1$.*
2. *$\beta'(A_l) = 2\eta$, for $\gamma = 1$, $\eta > 1$.*
3. *$\beta'(A_l) = 6$, for $\gamma = 2$, $\eta = 2$.*
4. *$\beta'(A_l) = (\gamma + 1)(\eta + 1) - 2$, for $\gamma, \eta > 2$.*

Proof. The set of vertices for the arithmetic graph A_l is

$$V_l = \{p_1^{\gamma}, p_2^{\eta}, p_1^{\gamma}p_2, p_1p_2^{\eta}, p_1^{\gamma}p_2^{\gamma}\}$$

where $1 \leq \gamma, \eta \leq n$. We will differentiate the proof into four cases.

Case 1. For $\gamma = 1$ and $\eta = 1$, we have

$$V_l = \{p_1, p_2, p_1p_2\}.$$

Consider the set $R = \{p_1, p_2\} \subseteq V_l$. The distances are calculated as follows:

$$d(p_1 \mid R) = (0, 2), \quad d(p_2 \mid R) = (2, 0), \quad d(p_1p_2 \mid R) = (1, 1).$$

Since all the distances are distinct, R is a resolving set.

Next, we will prove that $R - \{v\} = R'$ is also a resolving set, where v is any vertex that is a member of R. Let $p_2 \in R$. Then,

$$R - \{p_2\} = \{p_1\} = R'.$$

Now, the distances are:

$$d\,(p_1 \mid R') = (0), \quad d(p_2 \mid R') = 2, \quad d(p_1p_2 \mid R') = 1.$$

Since all the distances are distinct, R' is also a resolving set. Therefore, R is a fault-tolerant resolving set with

$$\beta'(A_l) = 2.$$

Case 2. For $\gamma = 1$ and $\eta \geq 2$, the set of vertices is

$$V_l = \{p_1, p_2, p_2^2, \ldots, p_2^\eta, p_1p_2, p_1p_2^2, \ldots, p_1^\gamma p_2^\eta\}.$$

Consider the set

$$R = \{p_1, p_2, p_2^2, \ldots, p_2^\eta, p_1p_2^2, \ldots, p_1^\gamma p_2^\eta\} \subseteq V_l.$$

Now, we calculate the distances:

$$\begin{aligned}
d(p_1 \mid R) &= (0, 2, 2, \ldots, 2, 1, \ldots, 1),\\
d(p_2 \mid R) &= (2, 0, 2, \ldots, 2, 1, \ldots, 1),\\
d(p_2^2 \mid R) &= (2, 2, 0, 2, \ldots, 2, 3, \ldots, 3),\\
d(p_2^\eta \mid R) &= (2, 2, 2, \ldots, 2, 0, 3, 3, \ldots, 3),\\
d(p_1p_2 \mid R) &= (1, 1, \ldots, 1, 2, \ldots, 2),\\
d(p_1p_2^2 \mid R) &= (1, 1, 3, \ldots, 3, 0, 2, \ldots, 2),\\
d(p_1p_2^{\eta-1} \mid R) &= (1, 3, \ldots, 3, 2, \ldots, 2, 0, 2),\\
d(p_1p_2^\eta \mid R) &= (1, 1, 3, \ldots, 3, 2, \ldots, 2, 0).
\end{aligned}$$

Since all the distances are distinct, R is a resolving set. Now we will prove that $R - \{v\} = R'$ is also a resolving set, where v is any vertex that is a member of R. Let $p_1 \in R$. Then,

$$R - \{p_1\} = R' = \{p_2, p_2^2, \ldots, p_2^\eta, p_1p_2^2, \ldots, p_1p_2^\eta\}.$$

Now, we calculate the distances:

$$\begin{aligned}
d(p_1 \mid R') &= (2, 2, \ldots, 2, 1, \ldots, 1),\\
d(p_2 \mid R') &= (0, 2, \ldots, 2, 1, \ldots, 1),\\
d(p_2^2 \mid R') &= (2, 0, 2, \ldots, 2, 3, \ldots, 3),\\
d(p_1p_2 \mid R') &= (1, \ldots, 1, 2, \ldots, 2),\\
d(p_1p_2^2 \mid R') &= (1, 3, \ldots, 3, 0, 2, \ldots, 2),\\
d(p_1p_2^{\eta-1} \mid R') &= (1, 3, \ldots, 3, 2, \ldots, 2, 0, 2),\\
d(p_1p_2^\eta \mid R') &= (1, 3, \ldots, 3, 2, \ldots, 2, 0).
\end{aligned}$$

Since all the distances are distinct, R' is also a resolving set. Therefore, the fault-tolerant metric dimension of A_l is 2η.

Next, we will prove the minimality of the fault-tolerant resolving set R. Consider the set $R' - \{v\} = R''$, where v is any vertex that is a member of R'. Let $p_2 \in R'$. Then,

$$R' - \{p_2\} = R'' = \{p_2^2, \ldots, p_2^{\eta}, p_1 p_2^2, \ldots, p_1^{\gamma} p_2^{\eta}\}.$$

Now, we calculate the distances:

$$\begin{aligned} d(p_1 \mid R'') &= (2, 2, \ldots, 2, 1, \ldots, 1), \\ d(p_2 \mid R'') &= (2, 2, \ldots, 2, 1, \ldots, 1). \end{aligned}$$

Since the distances are the same, the set R is a minimal fault-tolerant resolving set. Hence, the minimum cardinality of the fault-tolerant resolving set is 2η.

Case 3. For $\gamma,\ \eta = 2,\ V_l = \{p_1,\ p_2,\ p_1^2,\ p_2^2,\ p_1p_2,\ p_1p_2^2,\ p_1^2p_2,\ p_1^2p_2^2\}$. Consider the resolving set

$$R = \{p_1,\ p_2,\ p_1^2,\ p_2^2,\ p_1p_2^2,\ p_1^2p_2\} \subseteq V_l.$$

Now, we compute the distances:

$$\begin{aligned} d(p_1|R) &= (0, 2, 2, 2, 1, 1), \\ d(p_2|R) &= (2, 0, 2, 2, 1, 1), \\ d(p_1^2|R) &= (2, 2, 0, 2, 1, 3), \\ d(p_2^2|R) &= (2, 2, 2, 0, 3, 1), \\ d(p_1p_2|R) &= (1, 1, 1, 1, 2, 2), \\ d(p_1^2p_2|R) &= (1, 1, 3, 1, 2, 0), \\ d(p_1p_2^2|R) &= (1, 1, 1, 3, 0, 2), \\ d(p_1^2p_2^2|R) &= (1, 1, 3, 3, 2, 2). \end{aligned}$$

Since all the distances are distinct, R is a resolving set. Now we will prove that $R - \{v\} = R'$ is also a resolving set, where v is any vertex that is a member of R. Let $p_1 \in R$. Then, $R - \{p_1\} = R'$.

$$R' = \{p_2,\ p_1^2,\ p_2^2,\ p_1p_2^2,\ p_1^2p_2\}.$$

$$\begin{aligned} d(p_1|R') &= (2, 2, 2, 1, 1), \\ d(p_2|R') &= (0, 2, 2, 1, 1), \\ d(p_1^2|R') &= (2, 0, 2, 1, 3), \\ d(p_2^2|R') &= (2, 2, 0, 3, 1), \\ d(p_1p_2|R') &= (1, 1, 1, 2, 2), \\ d(p_1^2p_2|R') &= (1, 3, 1, 2, 0), \\ d(p_1p_2^2|R') &= (1, 1, 3, 0, 2), \\ d(p_1^2p_2^2|R') &= (1, 3, 3, 2, 2). \end{aligned}$$

Since all the distances are distinct, R' is also a resolving set. Thus, the fault-tolerant metric dimension of $A_l = 6$.

Now we will prove the minimality of the fault-tolerant resolving set R. Consider the set $R' - \{v\} = R''$, where v is any vertex that is a member of R'. Let $p_2 \in R'$. Then, $R' - \{p_2\} = R'' = \{p_1^2,\ p_2^2,\ p_1p_2^2,\ p_1^2p_2\}$.

Now, we compute the distances:

$$\begin{aligned} d(p_1|R'') &= (2,2,1,1), \\ d(p_2|R'') &= (2,2,1,1). \end{aligned}$$

Since the distances are the same, the set R is a minimal fault-tolerant resolving set. Hence, the minimum cardinality of the fault-tolerant resolving set is 6.

Case 4. For $\gamma, \eta \geq 2$, let

$$\begin{aligned} V_l = &\left\{p_1, p_2, p_1^2, \ldots, p_1^\gamma, p_2^2, \ldots, p_2^\eta, p_1p_2, p_1^2p_2, \ldots, p_1^\gamma p_2\right\}, \\ &\left\{p_1p_2^2, \ldots, p_1p_2^\eta, p_1^2p_2^2, \ldots, p_1^\gamma p_2^\eta\right\}. \end{aligned}$$

Consider the set

$$R = \{p_1, p_2, p_1^2, \ldots, p_1^\gamma, p_2^2, \ldots, p_2^\eta, p_1^2p_2, \ldots, p_1^\gamma p_2, p_1p_2^2, \ldots, p_1p_2^\eta, p_1^2p_2^2, \ldots, p_1^\gamma p_2^\eta\} \subseteq V_l.$$

Now, we compute the distances:

$$\begin{aligned} d(p_1|R) &= (0,2,2,\ldots,2,2,\ldots,2,1,\ldots,1,1,\ldots,1,1,\ldots,1,1), \\ d(p_2|R) &= (2,0,2,\ldots,2,2,\ldots,2,1,\ldots,1,1,\ldots,1,1,\ldots,1), \\ d(p_1^2|R) &= (2,2,0,2,\ldots,2,3,\ldots,3,1,\ldots,1,3,\ldots,3,3,\ldots,3), \\ &\vdots \\ d(p_1^\gamma|R) &= (2,2,2,\ldots,2,0,3,\ldots,3,1,\ldots,1,3,\ldots,3,3,\ldots,3), \\ d(p_2^2|R) &= (2,2,2,\ldots,2,0,2,\ldots,2,1,\ldots,1,3,\ldots,3,3,\ldots,3), \\ d(p_2^3|R) &= (2,2,2,\ldots,2,2,0,2,\ldots,2), \\ &\vdots \\ d(p_2^\eta|R) &= (2,2,2,\ldots,2,2,\ldots,2,0,1,\ldots,1,3,\ldots,3,3,\ldots,3), \\ d(p_1p_2|R) &= (1,1,\ldots,1,1,\ldots,1,2,\ldots,2,2,\ldots,2,2,\ldots,2), \\ d(p_1^2p_2|R) &= (1,1,3,\ldots,3,1,\ldots,1,0,2,\ldots,2,2,\ldots,2,2,\ldots,2), \\ &\vdots \\ d(p_1^\gamma p_2|R) &= (1,1,3,\ldots,3,1,\ldots,1,2,\ldots,2,0,2,\ldots,2,2,\ldots,2), \\ d(p_1p_2^2|R) &= (1,1,1,\ldots,1,3,\ldots,3,2,\ldots,2,0,2,\ldots,2,2,\ldots,2), \\ d(p_1p_2^3|R) &= (1,1,1,\ldots,1,3,\ldots,3,2,\ldots,2,2,0,2,\ldots,2,2,\ldots,2), \\ &\vdots \\ d(p_1p_2^\eta|R) &= (1,1,1,\ldots,1,3,\ldots,3,2,\ldots,2,2,2,\ldots,2,0,2,\ldots,2), \\ d(p_1^2p_2^2|R) &= (1,1,3,\ldots,3,3,\ldots,3,2,\ldots,2,2,\ldots,2,0,2,\ldots,2), \\ d(p_1^2p_2^3|R) &= (1,1,3,\ldots,3,3,\ldots,3,2,\ldots,2,2,\ldots,2,2,0,2,\ldots,2), \\ &\vdots \\ d(p_1^\gamma p_2^\eta|R) &= (1,1,3,\ldots,3,3,\ldots,3,2,\ldots,2,2,\ldots,2,2,\ldots,2,0). \end{aligned}$$

Since all distances are distinct, R' is a resolving set. This concludes the proof for this case. □

Theorem 2. *Let l be a composite number expressed in its canonical form as $l = p_1^{\gamma} p_2^{\eta} p_3^{\xi}$, where $\gamma, \eta, \xi \geq 1$. Then the following holds:*

1. $\beta'(A_l) = 4$ *for* $\gamma = 1, \eta = 1, \xi = 1$.
2. $\beta'(A_l) = 6$ *for* $\gamma = 2, \eta = 1, \xi = 1$.
3. $\beta'(A_l) = 4(\gamma - 1)$ *for* $\gamma > 2, \eta = 1, \xi = 1$.
4. $\beta'(A_l) = 9$ *for* $\gamma = 2, \eta = 2, \xi = 1$.
5. $\beta'(A_l) = 2(\gamma + 1)(\eta + 1) - 8$ *for* $\gamma > 2, \eta > 2, \xi = 1$.
6. $\beta'(A_l) = 15$ *for* $\gamma = 2, \eta = 2, \xi = 2$.
7. $\beta'(A_l) = (\gamma + 1)(\eta + 1)(\xi + 1) - 8$ *for* $\gamma > 2, \eta > 2, \xi > 2$.

Proof. The set of vertices for the arithmetic graph A_l is given by

$$V_l = \Big\{p_1^{\gamma}, p_2^{\eta}, p_3^{\xi}, p_1^{\gamma} p_2, p_1^{\gamma} p_3, p_1 p_2^{\eta}, p_2 p_3^{\xi}, p_1 p_3^{\xi}, p_1^{\gamma} p_2^{\eta}, p_1^{\gamma} p_3^{\xi}, p_2^{\eta} p_3^{\xi}, p_1^{\gamma} p_2 p_3, p_1 p_2^{\eta} p_3, p_1 p_2 p_3^{\xi}, p_1^{\gamma} p_2^{\eta} p_3,$$
$$p_1^{\gamma} p_2 p_3^{\xi}, p_1 p_2^{\eta} p_3^{\xi}, p_1^{\gamma} p_2^{\eta} p_3^{\xi}\Big\}$$

where $1 \leq \gamma, \eta, \xi \leq n$. We will proceed with the proof by considering seven distinct cases.

Case 1. For $\gamma = 1, \eta = 1, \xi = 1$, we have

$$V_l = \{p_1, p_2, p_3, p_1 p_2, p_1 p_3, p_2 p_3, p_1 p_2 p_3\}.$$

Let us consider the subset

$$R = \{p_1 p_2, p_1 p_3, p_2 p_3, p_1 p_2 p_3\} \subseteq V_l.$$

The distances from the vertices to the set R are calculated as follows:

$$\begin{aligned}
d\,(p_1|R) &= (1, 1, 2, 1),\\
d\,(p_2|R) &= (1, 2, 1, 1),\\
d\,(p_3|R) &= (2, 1, 1, 1),\\
d\,(p_1 p_2|R) &= (0, 1, 1, 2),\\
d\,(p_1 p_3|R) &= (1, 0, 1, 2),\\
d\,(p_2 p_3|R) &= (1, 1, 0, 2),\\
d\,(p_1 p_2 p_3|R) &= (2, 2, 2, 0).
\end{aligned}$$

Since all the distances are distinct, the set R is a resolving set. Now, we will show that the set $R' = R \setminus \{v\}$ is also a resolving set, where v is any vertex in R.

Assume $p_1 p_2 \in R$. Then,

$$R' = \{p_1 p_3, p_2 p_3, p_1 p_2 p_3\}.$$

Calculating the distances for the vertices with respect to R':

$$\begin{aligned}
d\,(p_1|R') &= (1, 2, 1),\\
d\,(p_2|R') &= (2, 1, 1),\\
d\,(p_3|R') &= (1, 1, 1),
\end{aligned}$$

$$\begin{aligned} d\left(p_1p_2|R'\right) &= (1,1,2),\\ d\left(p_1p_3|R'\right) &= (0,1,2),\\ d\left(p_2p_3|R'\right) &= (1,0,2),\\ d\left(p_1p_2p_3|R'\right) &= (2,2,0). \end{aligned}$$

Since all the distances are distinct, R' is also a resolving set.

Next, we will establish the minimality of the fault-tolerant resolving set R. Consider the set $R'' = R' \setminus \{v\}$, where v is any vertex in R'. Let $p_1p_3 \in R'$. Then,

$$R'' = \{p_2p_3, p_1p_2p_3\}.$$

Calculating the distances gives us:

$$\begin{aligned} d\left(p_1p_2|R''\right) &= (1,2),\\ d\left(p_1p_3|R''\right) &= (1,2). \end{aligned}$$

Since the distances are not distinct, R'' is not a resolving set. Therefore, the minimum cardinality of the fault-tolerant resolving set is 4.

Case 2. For $\gamma = 2,\ \eta,\ \xi = 1$,

$$V_l = \{p_1, p_2, p_3, p_1^2, p_1p_2, p_1^2p_2, p_1p_3, p_1^2p_3, p_2p_3, p_1p_2p_3, p_1^2p_2p_3\}$$

Let the set $R = \{p_1^2p_2, p_1p_3, p_1^2p_3, p_2p_3, p_1p_2p_3, p_1^2p_2p_3\} \subseteq V_l$.

Now, we calculate the distances:

$$\begin{aligned} d(p_1|R) &= (1,1,1,2,1,1),\\ d(p_2|R) &= (1,2,2,1,1,1),\\ d(p_3|R) &= (2,1,1,1,1,1),\\ d(p_1^2|R) &= (2,1,2,2,1,3),\\ d(p_1p_2|R) &= (2,1,1,1,2,2),\\ d(p_1^2p_2|R) &= (0,1,2,2,2,2),\\ d(p_1p_3|R) &= (1,0,2,1,2,2),\\ d(p_1^2p_3|R) &= (2,2,0,1,2,2),\\ d(p_2p_3|R) &= (1,1,1,0,2,2),\\ d(p_1p_2p_3|R) &= (2,2,2,2,0,2),\\ d(p_1^2p_2p_3|R) &= (2,2,2,2,2,0). \end{aligned}$$

Since all the distances are distinct, R is a resolving set.

Next, we will prove that $R' = R - \{v\}$ is also a resolving set, where v is any vertex in R. Let $p_1p_2p_3 \in R$. Then, $R' = R - \{p_1p_2p_3\}$.

Now, $R' = \{p_1^2p_2, p_1p_3, p_1^2p_3, p_2p_3, p_1^2p_2p_3\}$. The distances with respect to R' are:

$$\begin{aligned} d(p_1|R') &= (1,1,1,2,1),\\ d(p_2|R') &= (1,2,2,1,1),\\ d(p_3|R') &= (2,1,1,1,1),\\ d(p_1^2|R') &= (2,1,2,2,3),\\ d(p_1p_2|R') &= (2,1,1,1,2), \end{aligned}$$

$$d(p_1^2p_2|R') = (0,1,2,2,2),$$
$$d(p_1p_3|R') = (1,0,2,1,2),$$
$$d(p_1^2p_3|R') = (2,2,0,1,2),$$
$$d(p_2p_3|R') = (1,1,1,0,2),$$
$$d(p_1p_2p_3|R') = (2,2,2,2,2),$$
$$d(p_1^2p_2p_3|R') = (2,2,2,2,0).$$

Since all the distances are distinct, R' is also a resolving set.

Finally, we will prove the minimality of the fault-tolerant resolving set R. Consider the set $R'' = R' - \{v\}$, where v is any vertex in R'. Let $p_1^2p_2p_3 \in R'$. Then, $R'' = R' - \{p_1^2, p_2p_3\}$.

Now, $R'' = \{p_1^2p_2, p_1p_3, p_1^2p_3, p_2p_3\}$. The distances with respect to R'' are:

$$d(p_1p_2p_3|R'') = (2,2,2,2), \quad d(p_1^2p_2p_3|R'') = (2,2,2,2).$$

Since the distances are not distinct, R'' is not a resolving set. Thus, the minimum cardinality of the fault-tolerant resolving set is 6.

Case 3. For $\gamma > 2,\ \eta,\ \xi = 1$, the set V_l is defined as:

$$V_l = \Big\{p_1, p_2, p_3, p_1^2, \ldots, p_1^\gamma, p_1p_2, p_1^2p_2, \ldots, p_1^\gamma p_2, p_1p_3, p_1^2p_3, \ldots, p_1^\gamma p_3, p_2p_3, p_1p_2p_3, p_1^2p_2p_3, \ldots, p_1^\gamma p_2p_3\Big\}.$$

Let the set

$$R = \{p_1^2, \ldots, p_1^\gamma, p_1^2p_2, \ldots, p_1^\gamma p_2, p_1^2p_3, \ldots, p_1^\gamma p_3, p_1^2p_2p_3, \ldots, p_1^\gamma p_2p_3\} \subseteq V_l$$

be the resolving set.

Now we compute the distances:

$$d(p_1|R) = (2,\ldots,2,1,\ldots,1,1,\ldots,1,1,\ldots,1),$$
$$d(p_2|R) = (2,\ldots,2,1,\ldots,1,2,\ldots,2,1,\ldots,1),$$
$$d(p_3|R) = (2,\ldots,2,2,\ldots,2,1,\ldots,1,1,\ldots,1),$$
$$d(p_1^2|R) = (0,2,\ldots,2,2,\ldots,2,2,\ldots,2,2,3,\ldots,3),$$
$$d(p_1^3|R) = (2,0,2,\ldots,2,2,\ldots,2,2,\ldots,2,2,3,\ldots,3),$$
$$\vdots$$
$$d(p_1^\gamma|R) = (2,\ldots,2,0,2,\ldots,2,2,\ldots,2,2,3,\ldots,3),$$
$$d(p_1p_2|R) = (1,\ldots,1,2,\ldots,2,1,\ldots,1,2,\ldots,2),$$
$$d(p_1^2p_2|R) = (2,\ldots,2,0,2,\ldots,2,2,\ldots,2,2,\ldots,2),$$
$$d(p_1^3p_2|R) = (2,\ldots,2,2,0,2,\ldots,2,2,\ldots,2),$$
$$\vdots$$
$$d(p_1^\gamma p_2|R) = (2,\ldots,2,2,\ldots,2,0,2,\ldots,2),$$
$$d(p_1p_3|R) = (1,\ldots,1,1,\ldots,1,2,\ldots,2,2,\ldots,2),$$
$$d(p_1^2p_3|R) = (2,\ldots,2,2,\ldots,2,0,2,\ldots,2),$$
$$d(p_1^3p_3|R) = (2,\ldots,2,2,\ldots,2,0,2,\ldots,2),$$
$$\vdots$$
$$d(p_1^\gamma p_3|R) = (2,\ldots,2,2,\ldots,2,2,\ldots,2,0,2,\ldots,2),$$
$$d(p_2p_3|R) = (2,\ldots,2,1,\ldots,1,2,\ldots,2,2,\ldots,2),$$

$$
\begin{aligned}
d(p_1p_2p_3|R) &= (1,\ldots,1,2,\ldots,2,2,\ldots,2,2,\ldots,2),\\
d(p_1^2p_2p_3|R) &= (3,\ldots,3,2,\ldots,2,2,2,0,2,\ldots,2),\\
&\vdots\\
d(p_1^{\gamma}p_2^{\eta}p_3^{\xi}|R) &= (3,\ldots,3,2,\ldots,2,2,\ldots,2,2,\ldots,2,0).
\end{aligned}
$$

Since all distances are unique, R is a resolving set.

Now, we will prove that $R-\{v\}=R'$ is also a resolving set, where v is any vertex that is a member of R. Let $p_1^2\in R$. Then, $R-\{p_1^2\}=R'$.

$$R'=\{p_1^3,\ldots,p_1^{\gamma},p_1^2p_2,\ldots,p_1^{\gamma}p_2,p_1^2p_3,\ldots,p_1^{\gamma}p_3,p_1^2p_2p_3,\ldots,p_1^{\gamma}p_2p_3\}.$$

Now we compute the distances in R':

$$
\begin{aligned}
d(p_1|R') &= (2,\ldots,2,1,\ldots,1,1,\ldots,1,1,\ldots,1),\\
d(p_2|R') &= (2,\ldots,2,1,\ldots,1,2,\ldots,2,1,\ldots,1),\\
d(p_3|R') &= (2,\ldots,2,2,\ldots,2,1,\ldots,1,1,\ldots,1),\\
d(p_1^2|R') &= (2,\ldots,2,2,\ldots,2,2,\ldots,2,2,3,\ldots,3),\\
d(p_1^3|R') &= (0,2,\ldots,2,2,\ldots,2,2,\ldots,2,2,3,\ldots,3),\\
&\vdots\\
d(p_1^{\gamma}|R') &= (2,\ldots,2,0,2,\ldots,2,2,\ldots,2,2,3,\ldots,3),\\
d(p_1p_2|R') &= (1,\ldots,1,2,\ldots,2,1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_2|R') &= (2,\ldots,2,0,2,\ldots,2,2,\ldots,2,2,\ldots,2),\\
d(p_1^3p_2|R') &= (2,\ldots,2,2,0,2,\ldots,2,2,\ldots,2),\\
&\vdots\\
d(p_1^{\gamma}p_2|R') &= (2,\ldots,2,2,\ldots,2,0,2,\ldots,2),\\
d(p_1p_3|R') &= (1,\ldots,1,1,\ldots,1,2,\ldots,2,2,\ldots,2),\\
d(p_1^2p_3|R') &= (2,\ldots,2,2,\ldots,2,0,2,\ldots,2),\\
d(p_1^3p_3|R') &= (2,\ldots,2,2,\ldots,2,0,2,\ldots,2),\\
&\vdots\\
d(p_1^{\gamma}p_3|R') &= (2,\ldots,2,2,\ldots,2,2,\ldots,2,0,2,\ldots,2),\\
d(p_2p_3|R') &= (2,\ldots,2,1,\ldots,1,2,\ldots,2,2,\ldots,2),\\
d(p_1p_2p_3|R') &= (1,\ldots,1,2,\ldots,2,2,\ldots,2,2,\ldots,2),\\
d(p_1^2p_2p_3|R') &= (3,\ldots,3,2,\ldots,2,2,2,0,2,\ldots,2),\\
&\vdots\\
d(p_1^{\gamma}p_2^{\eta}p_3^{\xi}|R') &= (3,\ldots,3,2,\ldots,2,2,\ldots,2,2,\ldots,2,0).
\end{aligned}
$$

Thus, since R' generates unique distances, it is also a resolving set.

Case 4. For $\gamma,\eta=2,\xi=1$.

Let

$$
\begin{aligned}
V_l = \Big\{&p_1,p_2,p_3,p_1^2,p_2^2,p_1p_2,p_1^2p_2,p_1p_3,p_1^2p_3,p_1^2p_2^2,p_1p_2^2,p_2p_3,\\
&p_2^2p_3,p_1p_2p_3,p_1^2p_2p_3,p_1p_2^2p_3,p_1^2p_2^2p_3\Big\}.
\end{aligned}
$$

Let the set

$$R = \{p_1^2p_2, p_1^2p_3, p_1^2p_2^2, p_1p_2^2, p_2^2p_3, p_1p_2p_3, p_1^2p_2p_3, p_1p_2^2p_3, p_1^2p_2^2p_3\} \subseteq V_l.$$

Now, we compute the distances:

$$\begin{aligned}
d(p_1|R) &= (1,1,1,1,2,1,1,1,1),\\
d(p_2|R) &= (1,2,1,1,1,1,1,1,1),\\
d(p_3|R) &= (2,1,2,2,1,1,1,1,1),\\
d(p_1^2|R) &= (1,1,2,2,2,1,3,1,3),\\
d(p_2^2|R) &= (1,2,2,2,2,1,1,3,3),\\
d(p_1p_2|R) &= (2,1,2,2,1,2,2,2,2),\\
d(p_1^2p_2|R) &= (0,2,2,2,1,2,2,2,2),\\
d(p_1p_3|R) &= (1,2,1,1,1,2,2,2,2),\\
d(p_1^2p_3|R) &= (2,0,2,1,1,2,2,2,2),\\
d(p_1^2p_2^2|R) &= (2,2,0,2,2,2,2,2,2),\\
d(p_1p_2^2|R) &= (2,1,2,0,2,2,2,2,2),\\
d(p_2p_3|R) &= (1,1,1,1,2,2,2,2,2),\\
d(p_2^2p_3|R) &= (1,1,2,2,0,2,2,2,2),\\
d(p_1p_2p_3|R) &= (2,2,2,2,0,2,2,2,2),\\
d(p_1^2p_2p_3|R) &= (2,2,2,2,2,2,0,2,2).
\end{aligned}$$

Since all the distances are distinct, R is a resolving set.

Now, we will prove that $R - \{v\} = R'$ is also a resolving set, where v is any vertex that is a member of R.

Let $p_1p_2p_3 \in R$. Then $R - \{p_1p_2p_3\} = R'$ such that

$$R' = \{p_1^2p_2, p_1^2p_3, p_1^2p_2^2, p_1p_2^2, p_2^2p_3, p_1p_2p_3, p_1^2p_2p_3, p_1p_2^2p_3, p_1^2p_2^2p_3\}.$$

Now, we compute the distances for R':

$$\begin{aligned}
d(p_1|R') &= (1,1,1,1,2,1,1,1,1),\\
d(p_2|R') &= (1,2,1,1,1,1,1,1),\\
d(p_3|R') &= (2,1,2,2,1,1,1,1),\\
d(p_1^2|R') &= (1,1,2,2,2,3,1,3),\\
d(p_2^2|R') &= (1,2,2,2,2,1,3,3),\\
d(p_1p_2|R') &= (2,1,2,2,1,2,2,2),\\
d(p_1^2p_2|R') &= (0,2,2,2,1,2,2,2),\\
d(p_1p_3|R') &= (1,2,1,1,1,2,2,2),\\
d(p_1^2p_3|R') &= (2,0,2,1,1,2,2,2),\\
d(p_1^2p_2^2|R') &= (2,2,0,2,2,2,2,2),\\
d(p_1p_2^2|R') &= (2,1,2,0,2,2,2,2),\\
d(p_2p_3|R') &= (1,1,1,1,2,2,2,2),\\
d(p_2^2p_3|R') &= (1,1,2,2,0,2,2,2),\\
d(p_1p_2p_3|R') &= (2,2,2,2,2,2,2,2),\\
d(p_1^2p_2p_3|R') &= (2,2,2,2,2,2,0,2).
\end{aligned}$$

Since all the distances are distinct, R' is also a resolving set.

Next, we will prove the minimality of the fault-tolerant resolving set R. Consider the set $R' - \{v\} = R''$ which is also a resolving set, where v is any vertex that is a member of R'. Let $p_1^2p_2p_3 \in R'$. Then

$$R'' = \{p_1^2p_2, p_1^2p_3, p_1^2p_2^2, p_1p_2^2, p_2^2p_3, p_1p_2p_3, p_1^2p_2p_3\}.$$

Now we compute the distances for R'':

$$\begin{aligned}
d(p_1|R'') &= (1,1,1,1,2,1,1),\\
d(p_2|R'') &= (1,2,1,1,1,1,1),\\
d(p_3|R'') &= (2,1,2,2,1,1,1),\\
d(p_1^2|R'') &= (1,1,2,2,2,2,1),\\
d(p_2^2|R'') &= (1,2,2,2,2,1,1),\\
d(p_1p_2|R'') &= (2,1,2,2,1,2,2),\\
d(p_1^2p_2|R'') &= (0,2,2,2,1,2,2),\\
d(p_1p_3|R'') &= (1,2,1,1,1,2,2),\\
d(p_1^2p_3|R'') &= (2,0,2,1,1,2,2),\\
d(p_1^2p_2^2|R'') &= (2,2,0,2,2,2,2),\\
d(p_1p_2^2|R'') &= (2,1,2,0,2,2,2),\\
d(p_2p_3|R'') &= (1,1,1,1,2,2,2),\\
d(p_2^2p_3|R'') &= (1,1,2,2,0,2,2),\\
d(p_1p_2p_3|R'') &= (2,2,2,2,2,2,2),\\
d(p_1^2p_2p_3|R'') &= (2,2,2,2,2,2,0).
\end{aligned}$$

We can see that the distance $d(p_1^2p_2|R'') = (0,2,2,2,1,2)$ implies that R'' is not a resolving set.

Thus, R is minimal and $m(R) = 9$.

In the case of $\gamma, \eta = 2, \xi = 1$, we find that the fault-tolerant resolving set R is indeed minimal with $m(R) = 9$.

Case 5. For $\gamma, \eta > 2, \xi = 1$,

$$\begin{gathered}
V_l = \{p_1, p_2, p_3, p_1^2, \ldots, p_1^\gamma, p_2^2, \ldots, p_2^\eta, p_1p_2, p_1^2p_2, \ldots, p_1^\gamma p_2, p_1p_3, p_1^2p_3, \ldots, p_1^\gamma p_3,\\
p_1^2p_2, \ldots, p_1^\gamma p_2, p_1p_2^2, \ldots, p_1p_2^\eta, p_2p_3, p_2^2p_3, \ldots, p_2^\eta p_3, p_1p_2p_3, p_1^2p_2p_3, \ldots, p_1^\gamma p_2p_3,\\
p_1p_2^2p_3, \ldots, p_1p_2^\gamma p_3, p_1^2p_2^2p_3, \ldots, p_1^\gamma p_2^\eta p_3\}
\end{gathered}$$

Let the set

$$\begin{gathered}
R = \{p_1^2, \ldots, p_1^\gamma, p_2^2, \ldots, p_2^\eta, p_1^2p_2, \ldots, p_1^\gamma p_2, p_1^2p_3, \ldots, p_1^\gamma p_3, p_1^2p_2^2, \ldots,\\
p_1^\gamma p_2^\eta, p_1p_2^2, \ldots, p_1p_2^\eta, p_2p_3, p_2^2p_3, \ldots, p_2^\eta p_3, p_1p_2p_3, p_1^2p_2p_3, \ldots, p_1^\gamma p_2p_3, p_1p_2^2p_3, \ldots, p_1p_2^\gamma p_3, p_1^2p_2^2p_3, \ldots,\\
p_1^\gamma p_2^\eta p_3\} \subseteq V_l.
\end{gathered}$$

Now, the distance sets are defined as follows:

$$\begin{aligned}
d(p_1|R) &= (2,\ldots,2,1,\ldots,1),\\
d(p_2|R) &= (2,\ldots,2,1,\ldots,1,2,\ldots,2),\\
d(p_3|R) &= (2,\ldots,2,1,\ldots,1),
\end{aligned}$$

$$
\begin{aligned}
d(p_1^2|R) &= (0,2,\ldots,2,2,\ldots,2,3,\ldots,3),\\
d(p_1^3|R) &= (2,0,2,\ldots,2,3,\ldots,3),\\
&\vdots\\
d(p_1^\gamma|R) &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_2^2|R) &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_2^3|R) &= (2,\ldots,2,0,2,\ldots,2),\\
&\vdots\\
d(p_2^\eta|R) &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_1p_2|R) &= (1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_2|R) &= (2,\ldots,2,1,\ldots,1,0,2,\ldots,2),\\
d(p_1p_3|R) &= (1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_3|R) &= (2,\ldots,2,0,2,\ldots,2),\\
&\vdots\\
d(p_1^2p_2^2|R) &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_1^3p_2^2|R) &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_1p_2^2|R) &= (1,\ldots,1,2,\ldots,2),\\
d(p_1p_3|R) &= (1,\ldots,1,2,\ldots,2),\\
&\vdots\\
d(p_2p_3|R) &= (2,\ldots,2,1,\ldots,1),\\
d(p_2^2p_3|R) &= (2,\ldots,2,1,\ldots,1,0,2,\ldots,2),\\
d(p_1p_2p_3|R) &= (1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_2p_3|R) &= (3,\ldots,3,1,\ldots,1,2,\ldots,2),\\
d(p_1^3p_2p_3|R) &= (3,\ldots,3,1,\ldots,1,2,\ldots,2).
\end{aligned}
$$

Since all the distances are unique, R is a resolving set. Now we will prove $R-\{v\}=R'$ is also a resolving set, where v is any vertex that is a member of R. Let $p_1^2 \in R$. Then,

$$
R-\{p_1^2\} = R' = \{p_1^3,\ldots,p_1^\gamma,p_2^2,\ldots,p_2^\eta,p_1p_2,p_1^2p_2,\ldots,p_1^\gamma p_2,p_1p_3,\ldots,p_1^\gamma p_3,p_1^2p_2^2,\ldots,
$$

$$
p_1^\gamma p_2^\eta,p_1p_2^2,\ldots,p_1p_2^\eta,p_2p_3,p_2^2p_3,\ldots,p_2^\eta p_3,p_1p_2p_3,p_1^2p_2p_3,\ldots,p_1^\gamma p_2p_3,p_1p_2^2p_3,\ldots,p_1p_2^\gamma p_3,p_1^2p_2^2p_3,\ldots,
$$

$$
p_1^\gamma p_2^\eta p_3\}.
$$

Now, the distances are defined as follows:

$$
\begin{aligned}
d(p_1|R') &= (2,\ldots,2,1,\ldots,1),\\
d(p_2|R') &= (2,\ldots,2,1,\ldots,1,2,\ldots,2),\\
d(p_3|R') &= (2,\ldots,2,1,\ldots,1),\\
d(p_1^2|R') &= (2,\ldots,2,1,\ldots,1,0,2,\ldots,2),\\
d(p_1^3|R') &= (0,2,\ldots,2,3,\ldots,3),\\
&\vdots\\
d(p_1^\gamma|R') &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_2^2|R') &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_2^3|R') &= (2,\ldots,2,0,2,\ldots,2),
\end{aligned}
$$

$$\begin{aligned}
&\vdots\\
d(p_2^{\eta}|R') &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_1p_2|R') &= (1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_2|R') &= (2,\ldots,2,1,\ldots,1,0,2,\ldots,2),\\
d(p_1p_3|R') &= (1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_3|R') &= (2,\ldots,2,0,2,\ldots,2),\\
&\vdots\\
d(p_1^2p_2^2|R') &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_1^3p_2^2|R') &= (2,\ldots,2,0,2,\ldots,2),\\
d(p_1p_2^2|R') &= (1,\ldots,1,2,\ldots,2),\\
d(p_1p_3|R') &= (1,\ldots,1,2,\ldots,2),\\
&\vdots\\
d(p_2p_3|R') &= (2,\ldots,2,1,\ldots,1),\\
d(p_2^2p_3|R') &= (2,\ldots,2,1,\ldots,1,0,2,\ldots,2),\\
d(p_1p_2p_3|R') &= (1,\ldots,1,2,\ldots,2),\\
d(p_1^2p_2p_3|R') &= (3,\ldots,3,1,\ldots,1,2,\ldots,2),\\
d(p_1^3p_2p_3|R') &= (3,\ldots,3,1,\ldots,1,2,\ldots,2).
\end{aligned}$$

Since all the distances are unique, R' is also a resolving set.
Case 6. For $\gamma,\eta,\xi=2$, we have:

$$\begin{aligned}
V_l = \Big\{ & p_1, p_1^2, p_2, p_2^2, p_3, p_3^2, p_1p_2, p_1p_2^2, p_1^2p_2, p_1p_3, p_1p_3^2, p_1^2p_3, p_2p_3, p_2p_3^2, p_2^2p_3,\\
& p_1^2p_2^2, p_1^2p_2p_3, p_1p_2^2p_3, p_1p_2p_3^2, p_1^2p_2^2p_3, p_1^2p_2p_3^2, p_1p_2^2p_3^2, p_1^2p_2^2p_3^2\Big\}.
\end{aligned}$$

Let the set

$$\begin{aligned}
R = \Big\{ & p_1p_2^2, p_1^2p_2, p_1p_3^2, p_1^2p_3, p_1^2p_2^2, p_1^2p_2p_3, p_2^2p_3, p_1p_2p_3, p_1^2p_2p_3, p_1p_2^2p_3,\\
& p_1p_2p_3^2, p_1^2p_2^2p_3, p_1^2p_2p_3^2, p_1p_2^2p_3^2, p_1^2p_2^2p_3^2\Big\} \subseteq V_l.
\end{aligned}$$

Now we calculate the distances:

$$\begin{aligned}
d(p_1 \mid R) &= (1,1,1,1,1,1,2,1,1,1,1,1,1,1,1),\\
d(p_1^2 \mid R) &= (1,2,1,2,2,2,2,1,3,1,1,3,3,1,3),\\
d(p_2 \mid R) &= (1,1,2,2,1,2,1,1,1,1,1,1,1,1,1),\\
d(p_2^2 \mid R) &= (2,1,2,2,2,2,2,1,1,3,1,3,1,3,3),\\
d(p_3 \mid R) &= (2,2,1,1,2,1,1,1,1,1,1,1,1,1,1),\\
d(p_3^2 \mid R) &= (2,2,2,1,2,2,2,1,1,1,3,1,3,3,3),\\
d(p_1p_2 \mid R) &= (2,2,1,1,2,1,1,2,2,2,2,2,2,2,2),\\
d(p_1p_2^2 \mid R) &= (0,2,1,1,2,1,2,2,2,2,2,2,2,2,2),\\
d(p_1^2p_2 \mid R) &= (2,0,1,2,2,2,1,2,2,2,2,2,2,2,2),\\
d(p_1p_3 \mid R) &= (1,1,2,2,1,2,1,2,2,2,2,2,2,2,2),
\end{aligned}$$

$$\begin{aligned}
d(p_1p_3^2 \mid R) &= (1,1,0,2,1,2,2,2,2,2,2,2,2,2,2),\\
d(p_1^2p_3 \mid R) &= (1,2,1,0,2,2,1,2,2,2,2,2,2,2,2),\\
d(p_2p_3 \mid R) &= (1,1,1,1,1,1,2,2,2,2,2,2,2,2,2),\\
d(p_2p_3^2 \mid R) &= (1,1,2,1,1,2,2,2,2,2,2,2,2,2,2),\\
d(p_2^2p_3 \mid R) &= (2,1,1,1,2,1,2,2,2,2,2,2,2,2,2),\\
d(p_1^2p_2^2 \mid R) &= (2,2,1,2,0,2,2,2,2,2,2,2,2,2,2),\\
d(p_1^2p_3 \mid R) &= (1,2,1,2,2,0,2,2,2,2,2,2,2),\\
d(p_2^2p_3 \mid R) &= (2,1,2,1,2,2,0,2,2,2,2,2,2,2),\\
d(p_1p_2p_3 \mid R) &= (2,2,2,2,2,2,2,0,2,2,2,2,2,2),\\
d(p_1^2p_2p_3 \mid R) &= (2,2,2,2,2,2,2,2,0,2,2,2,2,2),\\
d(p_1p_2^2p_3 \mid R) &= (2,2,2,2,2,2,2,2,2,0,2,2,2,2),\\
d(p_1^2p_2^2p_3 \mid R) &= (2,2,2,2,2,2,2,2,2,2,2,0,2,2).
\end{aligned}$$

Since all the distances are distinct, R is a resolving set.

Now we will prove that $R - \{v\} = R'$ is also a resolving set, where v is any vertex that is a member of R.

Let $p_1p_2p_3 \in R$. Then $R - \{p_1p_2p_3\} = R'$. Thus,

$$\begin{aligned}
R' = \Big\{&p_1p_2^2, p_1^2p_2, p_1p_3^2, p_1^2p_3, p_1^2p_2^2, p_1^2p_2p_3, p_2^2p_3, p_1p_2p_3, p_1^2p_2p_3, p_1p_2^2p_3,\\
&p_1p_2p_3^2, p_1^2p_2^2p_3, p_1^2p_2p_3^2, p_1p_2^2p_3^2, p_1^2p_2^2p_3^2\Big\}.
\end{aligned}$$

To show that R' is a resolving set, we must demonstrate that for any two distinct vertices $x, y \in V_l$, there exists at least one vertex $v \in R'$ such that $d(x \mid R') \neq d(y \mid R')$.

Assume x, y are any two vertices from V_l and not equal. Because R is a resolving set, we know there is at least one vertex $v \in R$ such that $d(x \mid R) \neq d(y \mid R)$.

If v is one of the vertices that distinguishes x and y, then:

1. If v is not $p_1p_2p_3$, R' still contains v and we are done.
2. If $v = p_1p_2p_3$, then $d(x \mid R')$ and $d(y \mid R')$ must still differ for at least one vertex in R other than $p_1p_2p_3$.

Thus, $d(x \mid R') \neq d(y \mid R')$ holds, proving that R' is also a resolving set. Therefore, R being a resolving set implies that any subset of R will also be a resolving set.

This completes the proof that if R is a resolving set, then $R - \{v\}$ for any $v \in R$ is also a resolving set.

Case 7. For $\gamma, \eta, \xi > 2$,

$$\begin{aligned}
V_l = \{&p_1,\ p_2,\ p_3,\ \ p_1^2, \dots,\ p_1^\gamma,\ p_2^2, \dots,\ p_2^\eta,\ p_3^2, \dots,\ p_3^\xi,\\
&p_1p_2,\ p_1^2p_2, \dots,\ p_1^\gamma p_2,\ p_1p_3,\ p_1^2p_3, \dots,\ p_1^\gamma p_3,\\
&p_1p_2^2, \dots,\ p_1p_2^\eta,\ p_1p_3^2, \dots,\ p_1p_3^\xi,\ p_2p_3,\\
&p_2^2p_3, \dots,\ p_2^\eta p_3,\ p_2p_3^2, \dots,\ p_2p_3^\xi,\\
&p_1^2p_2^2, \dots,\ p_1^\gamma p_2^\eta,\ p_1^2p_3^2, \dots,\ p_1^\gamma p_3^\xi,\\
&p_2^2p_3^2, \dots,\ p_2^\eta p_3^\xi,\ p_1p_2p_3,\ p_1^2p_2p_3, \dots,\\
&p_1^\gamma p_2p_3,\ p_1p_2^2p_3, \dots,\ p_1p_2^\eta p_3,\\
&p_1p_2p_3^2, \dots,\ p_1p_2p_3^\xi,\ p_1^2p_2^2p_3, \dots,\\
&p_1^\gamma p_2^\eta p_3,\ p_1^2p_2p_3^2, \dots,\ p_1^\gamma p_2p_3^\eta,
\end{aligned}$$

$$p_1p_2^2p_3^2, \dots, \ p_1p_2^{\eta}p_3^{\xi},$$
$$p_1^2p_2^2p_3^2, \dots, \ p_1^{\gamma}p_2^{\eta}p_3^{\xi}\Big\}.$$

Let the set

$$\begin{aligned}
R = \Big\{&p_1^2, \dots, \ p_1^{\gamma}, \ \ p_2^2, \dots, \ p_2^{\eta}, \ p_3^2, \dots, \ p_3^{\xi},\\
&p_1^2p_2, \dots, \ p_1^{\gamma}p_2, \ p_1^2p_3, \dots, \ p_1^{\gamma}p_3,\\
&p_1p_2^2, \dots, \ p_1p_2^{\eta}, \ p_1p_3^2, \dots, \ p_1p_3^{\xi},\\
&p_2^2p_3, \dots, \ p_2^{\eta}p_3, \ p_2p_3^2, \dots, \ p_2p_3^{\xi},\\
&p_1^2p_2^2, \dots, \ p_1^{\gamma}p_2^{\eta}, \ p_1^2p_3^2, \dots, \ p_1^{\gamma}p_3^{\xi},\\
&p_2^2p_3^2, \dots, \ p_2^{\eta}p_3^{\xi}, \ p_1p_2p_3, \ p_1^2p_2p_3, \dots,\\
&p_1^{\gamma}p_2p_3, \ p_1p_2^2p_3, \dots, \ p_1p_2^{\eta}p_3,\\
&p_1p_2p_3^2, \dots, \ p_1p_2p_3^{\xi},\\
&p_1^2p_2^2p_3, \dots, \ p_1^{\gamma}p_2^{\eta}p_3,\\
&p_1^2p_2p_3^2, \dots, \ p_1^{\gamma}p_2p_3^{\eta},\\
&p_1p_2^2p_3^2, \dots, \ p_1p_2^{\eta}p_3^{\xi},\\
&p_1^2p_2^2p_3^2 \ , \dots, \ p_1^{\gamma}p_2^{\eta}p_3^{\xi}\Big\} \subseteq V_l.
\end{aligned}$$

Now, we define the following distributions of p_i given R:

$$\begin{aligned}
d(p_1|R) =&(2, \dots, 2, \dots, 2, \dots, 2, 1, 1, \dots, 1, 1, \dots, 1, \dots, 1, \dots, 1, \dots, 1, 2, \dots, 2, \dots, 2, 1,\\
&\dots, 1, \dots, 1, 2, \dots, 2, 1, 1, \dots),\\
d(p_2|R) =&(2, \dots, 2, \dots, 2, \dots, 1, 1, \dots, 1, 1, 2, \dots, 2, 1, \dots, 1, 2, \dots, 2, 1, 1, \dots, 1, \dots, 1, 1, \dots,\\
&1, \dots, 1, \dots, 1),\\
d(p_3|R) =&(2, \dots, 2, \dots, 2, \dots, 2, 2, 2, \dots, 2, 1, \dots, 1, 2, \dots, 2, 1, \dots, 1, 1, \dots, 1, 2, \dots, 2, 1, \dots,\\
&1, \dots, 1, \dots, 1).
\end{aligned}$$

Next, we present the distributions for p_1^2 and p_1^3:

$$\begin{aligned}
d(p_1^2|R) =&(0, 2, \dots, 2, \dots, 2, \dots, 2, 1, 1, \dots, 1, 1, 1, \dots, 1, 1, \dots, 1, 2, 2, \dots, 2, \dots, 2, 2, \dots,\\
&2, 3, \dots, 3, 1, \dots),\\
d(p_1^3|R) =&(2, 0, 2, \dots, 2, \dots, 2, \dots, 2, 1, 1, \dots, 1, 2, \dots, 2, 1, \dots, 1, 1, 2, 2, \dots, 2, \dots, 2, \dots,\\
&2, 2, 3, \dots, 3, 1, \dots, 1, 3, \dots, 3, 1, \dots, 1).
\end{aligned}$$

Continuing with the distributions for p_2^2 and p_2^3:

$$\begin{aligned}
d(p_2^2|R) =&(2, \dots, 2, 0, 2, \dots, 2, \dots, 2, 1, 1, 2, \dots, 2, \dots, 2, \dots, 2, 1, \dots, 1, 3, \dots, 3, 1, \dots,\\
&1, 3, \dots, 3, 1, \dots, 13, \dots),\\
d(p_2^3|R) =&(2, \dots, 2, 2, 0, 2, \dots, 2, 1, 1, \dots, 1, 2, \dots, 2, 2, \dots, 2, 1, \dots, 1, 3, \dots, 3, 1, \dots, 1, 3,\\
&\dots, 3, 1, \dots, 1, 3, \dots, 3, \dots).
\end{aligned}$$

Next, we have the distributions for p_3^2 and p_3^3:

$$\begin{aligned} d(p_3^2|R) =&(2,\ldots,2,0,2,\ldots,2,2,\ldots,2,1,1,\ldots,1,2,\ldots,2,\ldots,2,\ldots,2,1,1,\ldots,1,3,\ldots,\\ &3,1,\ldots,1,3,\ldots,3),\\ d(p_3^3|R) =&(2,\ldots,2,2,0,2,\ldots,2,2,2,\ldots,2,1,1,\ldots,1,2,\ldots,2,1,1,\ldots,1,2,\ldots,2,\ldots,2,\\ &\ldots,2,1,1,\ldots). \end{aligned}$$

Moving on to the distributions involving products:

$$\begin{aligned} d(p_1p_2|R) =&(1,\ldots,1,\ldots,1,2,\ldots,2,2,\ldots,2,1,\ldots,1,2,\ldots,1,\ldots,1,1,2,\ldots,2,\\ &\ldots,2,\ldots,2,2,\ldots,2),\\ d(p_1^2p_2|R) =&(2,\ldots,2,1,\ldots,1,2,\ldots,2,0,2,\ldots,2,\ldots,2,1,1,\ldots,1,\ldots,1,2,\ldots,2,\\ &\ldots,2,\ldots,2,2,\ldots,2),\\ d(p_1p_3|R) =&(1,\ldots,1,2,\ldots,2,1,\ldots,1,\ldots,1,2,\ldots,2,1,\ldots,1,2,\ldots,2,1,\ldots,1,1,\\ &\ldots,2,\ldots,2,2,2,\ldots),\\ d(p_1^2p_3|R) =&(2,\ldots,2,1,\ldots,1,1,\ldots,1,0,2,\ldots,2,1,\ldots,1,2,\ldots,2,1,\ldots,1,2,\ldots,2,\\ &2,\ldots,2,2,\ldots,2). \end{aligned}$$

Continuing, we have the following distributions:

$$\begin{aligned} d(p_1p_2^2|R) =&(1,\ldots,1,2,\ldots,2,2,\ldots,2,2,\ldots,2,1,\ldots,1,0,2,\ldots,2,1,\ldots,1,2,\\ &\ldots,2,2,\ldots,2,2,\ldots,2),\\ d(p_1p_2^3|R) =&(1,\ldots,1,2,\ldots,2,2,\ldots,2,2,\ldots,2,2,0,2,\ldots,2,1,\ldots,1,2,\ldots,2,2,\\ &\ldots,2,2,\ldots,2),\\ d(p_1p_3^2|R) =&(1,\ldots,1,2,\ldots,2,2,\ldots,2,2,\ldots,2,1,1,1,\ldots,1,2,2,2,2,2,\ldots,2,0,2,\\ &\ldots,2,1,\ldots,1),\\ d(p_1p_3^3|R) =&(1,\ldots,1,3,\ldots,3,2,2,1,1,\ldots,1,2,\ldots,2,2,\ldots,2,2,2,2,2,0,2,\ldots,\\ &2,1,\ldots,1). \end{aligned}$$

Finally, we address the distributions involving $p_2^2p_3^2$ and $p_2^3p_3^2$:

$$\begin{aligned} d(p_2^2p_3^2|R) &= (2,2,2,2,0,2,1,\ldots,1,1,1,1,\ldots,1,3,3,\ldots,3,1,1,\ldots,1,2,\ldots,2,2,\ldots),\\ d(p_2^3p_3^2|R) &= (2,2,2,2,0,2,1,1,\ldots,1,1,1,\ldots,1,3,3,\ldots,3,1,1,\ldots,1,2,\ldots,2,2,\ldots). \end{aligned}$$

Since all the distances are distinct, it follows that R is a resolving set.

Now, we will prove that $R \setminus \{v\} = R'$ is also a resolving set, where v is any vertex that is a member of R.

Let $p_1^2 \in R$. Then, we have $R \setminus \{p_1^2\} = R'$.

$$\begin{aligned} R' = \{&p_1^3,\ldots,p_1^\gamma,p_2^2,\ldots,p_2^\eta,p_3^2,\ldots,p_3^\xi,p_1^2p_2,\ldots,p_1^\gamma p_2,\\ &p_1^2p_3,\ldots,p_1^\gamma p_3,p_1p_2^2,\ldots,p_1p_2^\eta,p_1p_3^2,\ldots,p_1p_3^\xi,p_2^2p_3,\ldots,p_2^\eta p_3,\\ &p_2p_3^2,\ldots,p_2p_3^\xi,p_1^2p_2^2,\ldots,p_1^\gamma p_2^\eta,p_1^2p_3^2,\ldots,p_1^\gamma p_3^\xi,\\ &p_2^2p_3^2,\ldots,p_2^\eta p_3^\xi,p_1^2p_2p_3,\ldots,p_1^\gamma p_2p_3,p_1p_2^2p_3,\ldots,p_1p_2^\eta p_3, \end{aligned}$$

$p_1p_2p_3^2,\dots,p_1p_2p_3^{\xi},p_1^2p_2^2p_3,\dots,p_1^{\gamma}p_2^{\eta}p_3,p_1^2p_2p_3^2,\dots,p_1^{\gamma}p_2p_3^{\eta},$
$p_1p_2^2p_3^2,\dots,p_1p_2^{\eta}p_3^{\xi},p_1^2p_2^2p_3^2,\dots,p_1^{\gamma}p_2^{\eta}p_3^{\xi}\}.$

$$\begin{aligned}
d(p_1\mid R') =&(2,\dots,2,\dots,2,\dots,2,1,1,\dots,1,1,\dots,1,\dots,1,\dots,1,2,\dots,2,\dots,2,1,\dots,\\
&1,\dots,1,2,\dots,2,1,1,\dots),\\
d(p_2\mid R') =&(2,\dots,2,\dots,2,\dots,1,1,\dots,1,1,2,\dots,2,1,\dots,1,2,\dots,2,1,1,\dots,1,\dots,1,2,\dots,\\
&2,1,\dots,1,1,\dots,1,\dots,1,\dots,1,\dots,1,\dots,1,\dots),\\
d(p_3\mid R') =&(2,\dots,2,\dots,2,\dots,2,2,2,\dots,2,1,\dots,1,2,\dots,2,1,\dots,1,1,\dots,1,\dots,1,2,\dots,2,1,\\
&\dots,1,\dots,1,1,\dots,1,\dots),\\
d(p_1^2\mid R') =&(2,\dots,2,\dots,2,\dots,2,1,1,\dots,1,1,1,\dots,1,1,\dots,1,\dots,1,2,2,\dots,2,\dots,2,\dots,2,3,\\
&\dots,3,1,\dots),\\
d(p_1^3\mid R') =&(0,2,\dots,2,\dots,2,\dots,2,1,1,\dots,1,2,\dots,2,1,\dots,1,\dots,1,2,2,\dots,2,\dots,2,\dots,2,\\
&\dots,2,3,\dots,3,1,\dots,1,\dots,1,3,\dots,3,\dots),\\
d(p_2^2\mid R') =&(2,\dots,2,0,2,\dots,2,\dots,2,1,\dots,1,2,\dots,2,\dots,2,\dots,2,2,\dots,2,1,\dots,1,3,\dots,3,1,\\
&\dots,1,\dots,3,\dots,3),\\
d(p_2^3\mid R') =&(2,\dots,2,2,0,2,\dots,2,\dots,2,1,1,\dots,1,2,\dots,2,\dots,2,\dots,2,\dots,1,2,\dots,1,\dots,1,\\
&\dots,1,3,\dots,3,\dots),\\
d(p_3^2\mid R') =&(2,\dots,2,\dots,2,0,2,\dots,2,\dots,2,1,\dots,1,2,\dots,2,\dots,2,\dots,2,\dots,2,1,\dots,1,\dots,\\
&1,2,\dots,2,\dots,2,\dots),\\
d(p_3^3\mid R') =&(2,\dots,2,\dots,2,\dots,2,0,2,\dots,2,\dots,2,1,\dots,1,2,\dots,2,\dots,2,1,\dots,1,\dots,1,2,\dots,\\
&2,\dots,2,\dots),\\
d(p_1p_2\mid R') =&(1,\dots,1,\dots,1,2,\dots,2,2,\dots,2,1,\dots,1,2,\dots,2,\dots,1,\dots,1,\dots,1,\dots,2,\\
&\dots,2,\dots,2),\\
d(p_1^2p_2\mid R') =&(2,\dots,2,1,\dots,1,2,\dots,2,0,2,\dots,2,\dots,2,\dots,2,\dots,2,1,\dots,1,1,\dots,1,\\
&\dots,1,\dots,2,\dots),\\
d(p_1p_3\mid R') =&(1,\dots,1,2,\dots,2,1,\dots,1,\dots,1,2,\dots,2,\dots,1,\dots,1,\dots,1,2,\dots,2,\dots),\\
d(p_1^2p_3\mid R') =&(2,\dots,2,\dots,2,1,\dots,1,1,\dots,1,0,2,\dots,2,1,\dots,1,\dots,2,\dots,2,\dots,2,\dots),\\
d(p_1p_2^2\mid R') =&(1,\dots,1,2,\dots,2,\dots,2,2,\dots,2,1,\dots,1,0,2,\dots,2,\dots,2,\dots,2),\\
d(p_1p_2^3\mid R') =&(1,\dots,1,2,\dots,2,\dots,2,2,\dots,2,1,\dots,1,\dots,2,\dots,2,\dots,2),\\
d(p_1p_3^2\mid R') =&(1,\dots,1,2,\dots,2,\dots,2,1,\dots,1,0,2,\dots,2,\dots,2,\dots,2),\\
d(p_1p_3^3\mid R') =&(1,\dots,1,2,\dots,2,\dots,2,1,\dots,1,\dots,2,\dots,2,\dots,2),\\
d(p_2p_3\mid R') =&(2,\dots,2,1,\dots,1,\dots,1,1,\dots,1,1,\dots,1,\dots,1),\\
d(p_2^2p_3\mid R') =&(2,\dots,2,1,\dots,1,\dots,1,1,\dots,1,1,\dots,1,0,2,\dots),\\
d(p_2p_3^2\mid R') =&(2,\dots,2,1,\dots,1,\dots,1,1,\dots,1,1,\dots,1,1,\dots,1),\\
d(p_2p_3^3\mid R') =&(2,\dots,2,1,\dots,1,\dots,1,1,\dots,1,1,\dots,1,2,\dots,2,\dots),\\
d(p_1p_2p_3\mid R') =&(1,\dots,1,\dots,1,\dots,1,2,\dots,2,2,\dots,2),\\
d(p_1^2p_2p_3\mid R') =&(3,\dots,3,1,\dots,1,2,\dots,2,\dots,2,\dots),\\
d(p_1p_2^2p_3\mid R') =&(1,\dots,1,3,\dots,3,1,\dots,1,2,\dots,2),\\
d(p_1p_2^3p_3\mid R') =&(1,\dots,1,3,\dots,3,1,\dots,1,2,\dots,2,\dots),\\
d(p_1p_2p_3^2\mid R') =&(1,\dots,1,\dots,1,3,\dots,3,2,\dots,2),\\
d(p_1p_2p_3^3\mid R') =&(1,\dots,1,\dots,1,3,\dots,3,2,\dots,2),
\end{aligned}$$

$d(p_1^2p_2^2p_3 \mid R') = (3,\dots,3,1,\dots,1,2,\dots,2,\dots),$
$d(p_2^2p_1p_3 \mid R') = (3,\dots,3,1,\dots,1,2,\dots,2,\dots),$
$d(p_2^3p_1p_3 \mid R') = (3,\dots,3,1,\dots,1,2,\dots,2,\dots),$
$d(p_3^2p_1p_2 \mid R') = (3,\dots,3,1,\dots,1,2,\dots,2,\dots),$
$d(p_3^3p_1p_2 \mid R') = (3,\dots,3,1,\dots,1,2,\dots,2,\dots),$
$d(p_1p_2^2p_3^2 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1p_2^2p_3^3 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1p_2^3p_3^2 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1p_2^3p_3^3 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1^2p_2^2p_3^2 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1^2p_2^2p_3^3 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1^2p_2^3p_3^2 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2),$
$d(p_1^2p_2^3p_3^3 \mid R') = (1,\dots,1,\dots,1,1,\dots,1,2,\dots,2).$

Since all the distances are unique, R' is also a resolving set. Now we will prove the minimality of the fault-tolerant resolving set R. Consider the set $R' - \{v\} = R''$, where v is any vertex that is a member of R'. Let $p_1^3 \in R'$. Then, $R' - \{p_1^3\} = R''$, so

$$\begin{aligned}
R'' = \{&p_1^4,\dots,p_1^\gamma,p_2^2,\dots,p_2^\eta,p_3^2,\dots,p_3^\xi,\\
&p_1p_2,p_1^2p_2,\dots,p_1^\gamma p_2,p_1p_3,p_1^2p_3,\dots,p_1^\gamma p_3,\\
&p_1p_2^2,\dots,p_1p_2^\eta,p_1p_3^2,\dots,p_1p_3^\xi,\\
&p_2p_3,p_2^2p_3,\dots,p_2^\eta p_3,p_2p_3^2,\dots,p_2p_3^\xi,\\
&p_1^2p_2^2,\dots,p_1^\gamma p_2^\eta,p_1^2p_3^2,\dots,p_1^\gamma p_3^\xi,\\
&p_2^2p_3^2,\dots,p_2^\eta p_3^\xi,p_1p_2p_3,\\
&p_1^2p_2p_3,\dots,p_1^\gamma p_2p_3,\\
&p_1p_2^2p_3,\dots,p_1p_2^\eta p_3,\\
&p_1p_2p_3^2,\dots,p_1p_2p_3^\xi,\\
&p_1^2p_2^2p_3,\dots,p_1^\gamma p_2^\eta p_3,\\
&p_1^2p_2p_3^2,\dots,p_1^\gamma p_2p_3^\eta,\\
&p_1p_2^2p_3^2,\dots,p_1p_2^\eta p_3^\xi,\\
&p_1^2p_2^2p_3^2,\dots,p_1^\gamma p_2^\eta p_3^\xi\}.
\end{aligned}$$

Then, we have

$$\begin{aligned}
d(p_1^2|R'') = (&2,\dots,2,\dots,2,\dots,2,1,3,\dots,3,1,3,\dots,3,1,\dots,1,\dots,1,3,3,\dots,3,\dots,3,\dots,3,\\
&\dots,3,1,3,\dots,1,\dots,1,3,\dots,3,\dots,1,\dots,1,3,\dots,3),\\
d(p_1^3|R'') = (&2,\dots,2,\dots,2,\dots,2,1,3,\dots,3,1,3,\dots,3,1,\dots,1,\dots,1,3,3,\dots,3,\dots,3,\dots,3,\\
&\dots,3,1,3,\dots,3,1,\dots,1,1,3,\dots,3,\dots,1,\dots,1,3,\dots,3).
\end{aligned}$$

Since the distances are not distinct, R'' is not a resolving set. Thus, the minimum cardinality of the fault-tolerant resolving set is $(\gamma+1)(\eta+1)(\xi+1) - 8$. □

4. Conclusion

The notion of metric dimension is easily understood. The applications that require the identification of graph nodes are readily apparent owing to their robust correlation with GPS and trilateration. On the contrary, the task of precisely determining the metric dimension of generic graphs is an exceptionally difficult one. In this article, the arithmetic graph A_l, where l is a composite number, is investigated. The fault-tolerant resolving set and the fault-tolerant metric dimension of the arithmetic graph A_l are computed.

Conflict of interest

The authors declare no conflict of interest.

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