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Article

Maxima of the A_α-spectral Radius of Graphs with Given Size and Minimum Degree $\delta \geq 2$

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Abstract: In this paper, we study the A_α-spectral radius of graphs in terms of given size m and minimum degree $\delta \geq 2$, and characterize corresponding extremal graphs completely. Furthermore, we characterize extremal graphs having maximum A_α-spectral radius among (minimally)2-edge-connected graphs with given size m.

Keywords: Size, A_α-spectral radius, Minimum degree, Extremal graph

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1. Introduction

All graphs considered here are simple and undirected. For a graph G, $A(G)$ denotes its adjacency matrix and $D(G)$ denotes the diagonal matrix of its degrees. The matrix $Q(G) = D(G) + A(G)$ is called the signless Laplacian matrix of G. The largest eigenvalue of $A(G)$ is called the spectral radius of G, and the largest eigenvalue of $Q(G)$ is called the signless Laplacian spectral radius of G. For any real number $\alpha \in [0, 1]$, Nikiforov [17] defined the A_α-matrix of G as $A_\alpha(G) = \alpha D(G) + (1-\alpha)A(G)$, which can be regarded as a common generalization of $A(G)$ and $Q(G)$. The largest eigenvalue of $A_\alpha(G)$ is called the A_α-spectral radius of G, denoted by $\rho_\alpha(G)$. For a connected graph G, by the Perron-Frobenius theory of non-negative matrices [17], $\rho_\alpha(G)$ has multiplicity one and there exists a unique positive unit eigenvector corresponding to $\rho_\alpha(G)$. We shall refor to such an eigenvector as the Perron vector of $A_\alpha(G)$.

The investigation on the extremal problems of the spectral radius and the signless Laplacian spectral radius of graphs is an important topic in the theory of graph spectra. For related results, one may refer to [4, 16, 21, 22] and the references therein. Specially, the problem of characterizing the graph with maximal spectral radius for given size is initiated by Brualdi and Hoffman [2], and completely solved by Rowlinson [19]. For further investigation, one may refer to [1, 7, 13, 14, 20, 24] and the references therein. Just recently, one of the hot topics in the study of the Q-spectrum is to characterize the spectral extreme under the conditions of given size and graph parameters. Zhai et al. [26] determined the graph with maximal $Q-$spectral radius among all graphs with given size, and characterized the graph with maximal $Q-$spectral radius among all graphs with given size and clique number (resp., chromatic number). Lou et al. [15] determined the maximal signless Laplacian spectral radius (Laplacian spectral radius) of connected graphs with fixed size and diameter. For more results, one may refer to [8, 10, 25, 27].

The A_α-spectral radius of a graph has been widely concerned. However, the results on the A_α-spectral radius under edge-condition are still relatively little known. Li and Qin [12] generalized the conclusion in [26] to A_α-spectral radius for $1/2 \le \alpha \le 1$. Feng et al. [6] and Huang et al. [9] determined independently the graph having the maximum A_α-spectral radius for $1/2 \le \alpha \le 1$ among all connected graphs of size m and diameter (at least) d.

A friendship graph is one in which every pair of vertices has exactly one common neighbour, denoted by $F_{\frac{m}{3}}$ for given $m \equiv 0(mod3)$. The join of graphs G and H, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by joining each vertex of G with every vertex of H. In this paper, we completely characterize the graphs attaining the maximal A_α-index among all graphs with given size m and minimum degree $\delta \ge 2$ for $\frac{1}{2} \le \alpha < 1$.

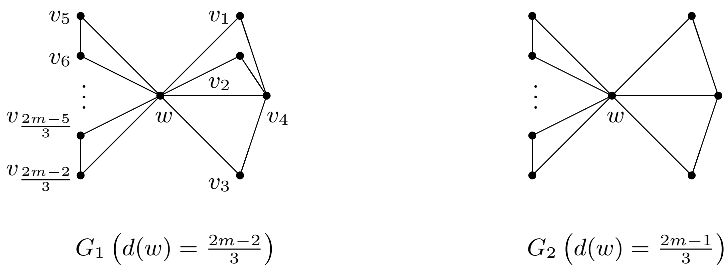


Figure 1. G_1, G_2

Theorem 1. *Let $\frac{1}{2} \le \alpha < 1$ and G be a graph with m edges and minimum degree $\delta \ge 2$, and G_1, G_2 be the graphs shown in Figure 1.*

(i) If $m \ge 24$ and $m \equiv 0(mod3)$, then $\rho_\alpha(G) \le \rho_\alpha(F_{\frac{m}{3}})$, with equality if and only if $G = F_{\frac{m}{3}}$.
(ii) If $m \ge 37$ and $m \equiv 1(mod3)$, then $\rho_\alpha(G) \le \rho_\alpha(G_1)$, with equality if and only if $G = G_1$, where $G_1 = K_1 \vee (\frac{m-7}{3}K_2 \cup K_{1,3})$.
(iii) If $m \ge 29$ and $\equiv 2(mod3)$, then $\rho_\alpha(G) \le \rho_\alpha(G_2)$, with equality if and only if $G = G_2$, where $G_2 = K_1 \vee (\frac{m-5}{3}K_2 \cup P_3)$.

A graph is 2-edge-connected if removing fewer than 2 edges always leaves the remaining graph connected, and is minimally 2-edge-connected if it is 2-edge-connected and deleting any arbitrary chosen edge always leaves a graph which is not 2-edge-connected. For graphs of order n, Chen and Guo [3] showed that $K_{2,n-2}$ attained the maximal spectral radius among all the minimally 2-(edge)-connected graphs. Fan et al. [5] proved that $K_{3,n-3}$ has the largest spectral radius over all minimally 3-connected graphs. For graphs of size m, Guo and Zhang [8,27] gave sharp upper bounds on the $Q(L)$-index of (minimally) 2-connected graphs with given size and characterized the corresponding extremal graphs completely. Noting that a connected graph having no cut edges is 2-edge-connected, we have following corollary.

Corollary 1. *Let $\frac{1}{2} \le \alpha < 1$ and G be a 2-edge-connected graph with m edges.*

(i) If $m \ge 24$ and $m \equiv 0(mod3)$, then $\rho_\alpha(G) \le \rho_\alpha(F_{\frac{m}{3}})$, with equality if and only if $G = F_{\frac{m}{3}}$.
(ii) If $m \ge 37$ and $m \equiv 1(mod3)$, then $\rho_\alpha(G) \le \rho_\alpha(G_1)$, with equality if and only if $G = G_1$, where $G_1 = K_1 \vee (\frac{m-7}{3}K_2 \cup K_{1,3})$.
(iii) If $m \ge 29$ and $\equiv 2(mod3)$, then $\rho_\alpha(G) \le \rho_\alpha(G_2)$, with equality if and only if $G = G_2$, where $G_2 = K_1 \vee (\frac{m-5}{3}K_2 \cup P_3)$.

In this paper, we further study the problem of characterizing graphs among minimally 2-edge-connected graph with maximal A_α- spectral radius. For $m \equiv 1(mod3)$, let G_3 (shown in Figure 2) be the graph obtained from the friendship graph $F_{\frac{m-1}{3}}$ by subdividing an edge

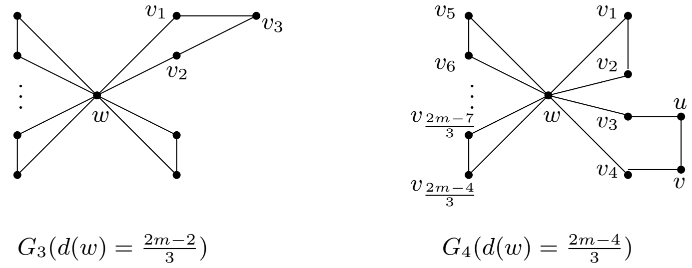


Figure 2. G_3, G_4

once. For $m \equiv 2(mod 3)$, let G_4 (shown in Figure 2) be the graph obtained from the friendship graph $F_{\frac{m-2}{3}}$ by subdividing an edge twice. Employing Theorem 1, we can prove the following theorem.

Theorem 2. *Let $\frac{1}{2} \leq \alpha < 1$ and G be a minimally 2-edge-connected graph with m edges.*

(i) If $m \geq 24$ and $m \equiv 0(mod3)$, then $\rho_\alpha(G) \leq \rho_\alpha(F_{\frac{m}{3}})$, with equality if and only if $G = F_{\frac{m}{3}}$.
(ii) If $m \geq 37$ and $m \equiv 1(mod3)$, then $\rho_\alpha(G) \leq \rho_\alpha(G_3)$, with equality if and only if $G = G_3$.
(iii) If $m \geq 50$ and $m \equiv 2(mod3)$, then $\rho_\alpha(G) \leq \rho_\alpha(G_4)$, with equality if and only if $G = G_4$.

The remainder of the paper is organized as follows. In Section 2, we recall some useful notions and lemmas that will be used later. In Section 3, we give proofs of Theorems 1 and 2 respectively.

2. Preliminaries

For a graph G, $V(G)$ and $E(G)$ denote the vertex set and edge set of G respectively, and $e(G) = |E(G)|$ denotes the number of edges in G. For $v \in V(G)$, $d_G(v)$ or $d(v)$ denotes the degree of v, $N_G(v)$ or $N(v)$ denotes the set of all neighbors of v in G, and $N[v] = N(v) \cup \{v\}$. For a subset S of $V(G)$, $G[S]$ denotes the subgraph of G induced by S, $e(S)$ denotes the number of edges in $G[S]$, and $N_S(v)$ denotes the set of all neighbors of v in S. For two disjoint subsets S and T of $V(G)$, $e(S, T)$ denotes the number of edges with one endpoint in S and the other in T. Let $G - uv$ denote the graph obtained from G by deleting the edge $uv \in E(G)$. Similarly, $G + uv$ is the graph obtained from G by adding an edge $uv \notin E(G)$, where $u, v \in V(G)$. The average degree of the neighbors of a vertex v_i of G is $m(v_i) = \frac{1}{d(v_i)} \sum\limits_{v_iv_j \in E(G)} d(v_j)$. The degree sequence of G is the non-increasing sequence of its vertex degrees. Whenever necessary, the vertices of G can be renumbered so that $d_i \geq d_{i+1}$ for $1 \leq i \leq n$. In that case, we say that G has degree sequence $(d_1, d_2, \cdots, d_n)$, denoted by $\mathbb{D}(G) = (d_1, d_2, \cdots, d_n)$.

Let G be a connected graph on n vertices and $X = (x_1, x_2, \cdots, x_n)^T \in R^n$. Then X can be considered as a function defined on $V(G)$, that is, each vertex x_i is mapped to $x_i = x(v_i)$. One can find in [17] that

$$X^T A_\alpha(G)X = (2\alpha - 1) \sum_{u \in V(G)} x_u^2 d(u) + (1 - \alpha) \sum_{uv \in E(G)} (x_u + x_v)^2,$$

and for arbitrary unit vector $X \in R^n$,

$$\rho_\alpha(G) \geq X^T A_\alpha(G)X, \tag{1}$$

with the equality if and only if X is the Perron vector of $A_\alpha(G)$.

In order to prove our main results, we need the following lemmas.

Lemma 1. ([17]) *If G is a graph with no isolated vertices, then*

$$\rho_\alpha(G) \le \max\{\, \alpha d(u) + (1-\alpha)m(u) \mid u \in V(G)\,\}. \tag{2}$$

Lemma 2. ([17]) *Let G be a graph with n vertices and $\Delta(G) = \Delta$. If $\alpha \in [\frac{1}{2}, 1)$, then*

$$\rho_\alpha(G) \ge \alpha\Delta + \frac{(1-\alpha)^2}{\alpha}. \tag{3}$$

The equality holds if and only if $\alpha = \frac{1}{2}$ and G is the star $K_{1,n-1}$.

Lemma 3. ([17]) *Let G be a connected graph with $\alpha \in [0, 1)$ and H be a proper subgraph of G, then $\rho_\alpha(H) < \rho_\alpha(G)$.*

Lemma 4. ([18]) *Let G be a connected graph, u and v be two vertices of G. Suppose that $v_i \in N_G(v) \setminus N_G(u)$ $(1 \le i \le s)$ and $x = (x_1, x_2, \ldots, x_n)^T$ is the Perron vector of $Q(G)$, where x_i corresponds to the vertex v_i $(1 \le i \le n)$. Let G^* be the graph obtained from G by deleting the edges vv_i and adding the edges uv_i $(1 \le i \le s)$. If $x_u \ge x_v$, then $\rho_\alpha(G) < \rho_\alpha(G^*)$.*

An internal path in some graph is a path $v_0v_1 \ldots v_s$ ($s \ge 1$, or $s \ge 3$ whenever $v_s = v_0$) such that $d(v_0) > 2$, $d(v_s) > 2$, and $d(v_i) = 2$ for $0 < i < s$. Li, Chen and Meng [11] proved the following subdivision theorem.

Lemma 5. ([11]) *Let G be a connected graph with $\alpha \in [0, 1)$ and uv be some edge on an internal path of G. Let G_{uv} denote the graph obtained from G by subdivision of edge uv into edges uw and wv. Then $\rho_\alpha(G_{uv}) < \rho_\alpha(G)$.*

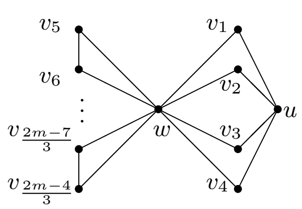
A cycle C of a graph G is said to have a chord if there is an edge of G that joins a pair of non-adjacent vertices of C.

Lemma 6. ([14]) *If G is a minimally 2-edge-connected graph, then no cycle of G has a chord.*

For a connected graph, Yu, Wu and Shu [23] gave a sharp upper bound on Q-index in terms of its degree sequence. The authors of the current paper [28] generalized their result to A_α-index of a connected graph. The following Lemma is a corollary of our result.

Lemma 7. ([28]) *Let G be a simple connected graph with n vertices and degree sequence $d_1 \ge d_2 \ge \cdots \ge d_n$. If $d_1 \ge s \ge d_2$, then $\rho_\alpha(G) \le A(d_1, s)$, where*

$$A(d_1, s) = \frac{1}{2}(\alpha d_1 + s + \alpha - 1 + \sqrt{(s - \alpha d_1 + 1 - \alpha)^2 + 4(1-\alpha)(d_1 - s)}\,).$$



G_5 $\left(d(w) = \frac{2m-4}{3}\right)$

Figure 3. G_5

Lemma 8. *Let $\frac{1}{2} \le \alpha < 1, m \ge 50$, and G_5 be the graph shown in Figure 3. Then $\rho_\alpha(G_5) < \rho_\alpha(G_4)$.*

Proof. Label the vertices is as shown in Figure 1. Let $X = (x_w, x_1, x_2, \cdots, x_{\frac{2m-2}{3}}, x_u)^T$ be a unit eigenvector corresponding to $\rho =: \rho_\alpha(G_5)$ where x_w corresponds to w and x_i corresponds to $v_i(1 \le i \le \frac{2m-4}{3})$ and x_u corresponds to u. By the eigenvalue equation $\rho X = A_\alpha(G_5)X$, we have $x_1 = x_2 = x_3 = x_4$ and $\rho x_u = 4\alpha x_u + 4(1-\alpha)x_1$. It follows that $x_1 = \frac{\rho-4\alpha}{4(1-\alpha)}x_u$. Define $Y = (y_w, y_1, y_2, \cdots, y_{\frac{2m-4}{3}}, y_u, y_v)^T$ such that $y_w = x_w, y_i = x_i$ for $1 \le i \le \frac{2m-4}{3}$, and $y_u = y_v = \frac{\sqrt{2}}{2}x_v$. Clearly,

$$\sum_{i=1}^{\frac{2m-4}{3}} y_i^2 + y_w^2 + y_u^2 + y_v^2 = \sum_{i=1}^{\frac{2m-4}{3}} x_i^2 + x_w^2 + x_u^2 = 1.$$

Noting that $m \ge 50$ and $d_1(G_5) = d(w) = \frac{2m-4}{3}$, by Lemma 2, we have $\rho = \rho_\alpha(G_5) > \frac{2m-4}{3}\alpha \ge 16$. By (1), we have

$$\begin{aligned}
\rho_\alpha(G_4) - \rho \ge & Y^T A_\alpha(G_7)Y - X^T A_\alpha(G_6)X \\
= & (2\alpha-1)(-2x_u^2) + (1-\alpha)((x_1+x_2)^2 + (x_3 + \frac{x_u}{\sqrt{2}})^2 \\
& + (x_4 + \frac{x_u}{\sqrt{2}})^2 + (\frac{x_u}{\sqrt{2}} + \frac{x_u}{\sqrt{2}})^2 - 4(x_1+x_u)^2) \\
= & (2\alpha-1)(-2x_u^2) + (1-\alpha)((2\frac{\rho-4\alpha}{4(1-\alpha)}x_u)^2 + 2(\frac{\rho-4\alpha}{4(1-\alpha)}x_u + \frac{x_u}{\sqrt{2}})^2 \\
& + 2x_u^2 - 4(\frac{\rho-4\alpha}{4(1-\alpha)}x_u + x_u)^2) \\
= & \frac{\rho^2 - (4\sqrt{2}\alpha - 8\alpha - 4\sqrt{2} + 16)\rho + 16\sqrt{2}\alpha^2 - 24\alpha^2 + 32\alpha - 16\sqrt{2}\alpha + 8}{8(1-\alpha)}x_u^2 \\
> & \frac{\rho^2 - (4\sqrt{2}\alpha - 8\alpha - 4\sqrt{2} + 16)\rho}{8(1-\alpha)}x_u^2 \\
> & 0.
\end{aligned}$$

Therefore $\rho_\alpha(G_5) < \rho_\alpha(G_4)$. This completes the proof. □

3. Proofs of Theorems 1 and 2

Proof of Theorem 1. We may assume that G is connected. Otherwise, suppose that $H_i(i = 1, 2, \cdots, k)$ are k connected components of G, where $k \ge 2$. Since $\delta(G) \ge 2$, then $\delta(H_i) \ge 2(1 \le i \le k)$. For $i = 1, 2, \cdots, k$, let v_i be a vertex of H_i, and G^* be the graph obtained from H_i by identifying vertices v_i. Clearly, $\rho_\alpha(G) < \rho_\alpha(G^*)$, and G^* is a connected graph with m edges and minimum degree $\delta \ge 2$. So, in order to complete the proof of Theorem 1, we may assume that G is connected.

Furthermore, we may assume that G is 2-edge-connected. Otherwise, suppose that $u_1v_1 \in E(G)$ is a cut edge of G. Since $\delta(G) \ge 2$, then there exist a path $P = u_k u_{k-1} \cdots u_2 u_1 v_1 v_2 \cdots v_{l-1} v_l$, where $k, l \ge 1$, such that $d(u_k) \ge 3(d(v_l) \ge 3)$ and $u_k(v_l)$ belongs to a cycle $C_1 = u_k u_{k+1} \cdots u_{k+p} u_k (C_2 = v_l v_{l+1} \cdots v_{l+q} v_1)$. Suppose that $|V(G)| = n$. Let $X = (x_{u_1}, x_{u_2}, \cdots, x_{u_{k+p}}, x_{v_1}, x_{v_2}, \cdots, x_{v_{l+q}}, \cdots, x_n)^T$ be a unit eigenvector corresponding to $\rho_\alpha(G)$ where x_{u_i} corresponds to $u_i(1 \le i \le k+p)$, x_{v_j} corresponds to $v_j(1 \le j \le l+q)$. If $x_{u_k} \ge x_{v_l}$, let $G^* = G - v_l v_{l+1} + u_k v_{l+1}$; Otherwise, let $G^* = G - u_k u_{k+1} + v_l u_{k+1}$. In the both cases, G^* is a connected graph with m edges and minimum degree $\delta \ge 2$ and uv is not a cut edge of G^* any more. By Lemma 4, we have $\rho_\alpha(G) < \rho_\alpha(G^*)$. So we may assume that G is 2-edge-connected.

Let $\mathcal{G}_m^2$ denote the set of all 2-edge-connected graphs with m edges. For $G \in \mathcal{G}_m^2$ and $v \in V(G)$, it is easy to see that $d(v) \geq 2$ and $G - v$ has no isolated vertices. Noting that $|E(G-v)| = m - d(v)$, we have

$$d(v) \leq |V(G-v)| \leq 2(m - d(v)).$$

It follows that $d(v) \leq \frac{2m}{3}$ with equality if and only if $G = F_{\frac{m}{3}}$.

For $G \in \mathcal{G}_m^2$, let w be a vertex of G such that

$$\max_{u \in V(G)} \{\alpha d(u) + (1-\alpha)m(u)\} = \alpha d(w) + (1-\alpha)m(w) = \alpha d(w) + \frac{1-\alpha}{d(w)} \sum_{wv \in E(G)} d(v).$$

Noting that $e(N(w)) \leq m - e(N(w), V(G) \setminus N(w))$ and $e(N(w), V(G) \setminus N(w)) \geq d(w)$, we have

$$\sum_{wv \in E(G)} d(v) = 2e(n(w)) + e(N(w), V(G) \setminus N(w)) \leq 2m - d(w).$$

By Lemma 1, we have

$$\rho_\alpha(G) \leq \alpha d(w) + \frac{2m}{d(w)}(1-\alpha) - 1 + \alpha. \tag{4}$$

(i) Let $m \geq 24$ and $m \equiv 0(mod 3)$. It is easy to see that $F_{\frac{m}{3}} \in \mathcal{G}_m^2$. By Lemma 2, we have $\rho_\alpha(F_{\frac{m}{3}}) > \frac{2m\alpha}{3} + \frac{(1-\alpha)^2}{\alpha}$.

If $d(w) = 2$, noting that $e(N(w)) \leq 1$, we have

$$\sum_{wv \in E(G)} d(v) = 2e(N(w)) + e(N(w), V(G) \setminus N(w)) \leq 2 + m - 1 = m + 1.$$

By (2), we have

$$\rho_\alpha(G) \leq 2\alpha + \frac{m+1}{2}(1-\alpha) \leq \frac{2m\alpha}{3} + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(F_{\frac{m}{3}})$$

for $m \geq 9$ and $\frac{1}{2} \leq \alpha < 1$.

If $d(w) = 3$, noting that $e(N(w)) \leq 3$, we have

$$\sum_{wv \in E(G)} d(v) = 2e(N(w)) + e(N(w), V(G) \setminus N(w)) \leq 6 + m - 3 = m + 3.$$

By (2), we have

$$q(G) \leq 3\alpha + \frac{m+3}{3}(1-\alpha) \leq \frac{2m\alpha}{3} + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(F_{\frac{m}{3}})$$

for $m \geq 9$ and $\frac{1}{2} \leq \alpha < 1$.

If $4 \leq d(w) \leq \frac{2m-6}{3}$, let $f(x) = \alpha x + \frac{2m}{x}(1-\alpha)$. It is easy to see that the function $f(x)$ is convex for $x > 0$ and its maximum in any closed interval is attained at one of the ends of this interval. Combining this and (4), we have

$$\begin{aligned} \rho_\alpha(G) &\leq \max\left\{4\alpha + \frac{2m}{4}(1-\alpha),\ \frac{2m-6}{3}\alpha + \frac{3m}{m-3}(1-\alpha)\right\} - 1 + \alpha \\ &\leq \frac{2m\alpha}{3} + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(F_{\frac{m}{3}}). \end{aligned}$$

for $m \geq 12$ and $\frac{1}{2} \leq \alpha < 1$.

$G_6\left(d(w)=\frac{2m-2}{3}\right)$ $G_7\left(d(w)=\frac{2m-2}{3}\right)$

Figure 4. G_6, G_7

If $d(w)=\frac{2m-3}{3}$, then $d_1=d_1(G)=\frac{2m-3}{3}$. Let $|V(G-w)|=\frac{2m-3}{3}+s$, then $2m\geq\frac{2m-3}{3}+2(\frac{2m-3}{3}+s)$. It follows that $0\leq s\leq 1$. This implies that $d_2=d_2(G)\leq 2+3=5$. By Lemma 7, we have

$$\rho_\alpha(G)\leq A(\frac{2m-3}{3},5)<\frac{2m\alpha}{3}+\frac{(1-\alpha)^2}{\alpha}<\rho_\alpha(F_{\frac{m}{3}})$$

for $m\geq 24$ and $\frac{1}{2}\leq\alpha<1$.
If $d(w)=\frac{2m}{3}$, then $G=F_{\frac{m}{3}}$, completing the proof of (i).

(ii) Let $m\geq 37$ and $m\equiv 1(mod 3)$. It is easy to see that $G_1=K_1\vee(\frac{m-2}{3}K_2\cup K_{1,3})\in\mathcal{G}_m^2$. By Lemma 2, we have $\rho_\alpha(G_1)>\frac{2m-2}{3}\alpha+\frac{(1-\alpha)^2}{\alpha}$.
For $2\leq d(w)\leq\frac{2m-8}{3}$, by similar reasoning as in the proof of (i), we can derived that

$$\rho_\alpha(G)\leq\frac{2m-2}{3}\alpha+\frac{(1-\alpha)^2}{\alpha}<\rho_\alpha(G_1)$$

for $m\geq 16$ and $\frac{1}{2}\leq\alpha<1$.
If $d(w)=\frac{2m-5}{3}$, then $d_1=d_1(G)=\frac{2m-5}{3}$. Let $|V(G-w)|=\frac{2m-5}{3}+s$, then $2m\geq\frac{2m-5}{3}+2\left(\frac{2m-5}{3}+s\right)$. It follows that $0\leq s\leq 2$. This implies that $d_2=d_2(G)\leq 2+5=7$. By Lemma 7, we have

$$\rho_\alpha(G)\leq A\left(\frac{2m-5}{3},7\right)<\frac{2m-2}{3}\alpha+\frac{(1-\alpha)^2}{\alpha}<\rho_\alpha(G_1)$$

for $m\geq 37$ and $\frac{1}{2}\leq\alpha<1$.
If $d(w)=\frac{2m-2}{3}$, then $d_1=d_1(G)=\frac{2m-2}{3}$. Let $|V(G-w)|=\frac{2m-2}{3}+s$, then $2m\geq\frac{2m-2}{3}+2\left(\frac{2m-2}{3}+s\right)$. It follows that $0\leq s\leq 1$.

Case 1. $s=0$, it follows that $|V(G-w)|=\frac{2m-2}{3}$. Noting that $|E(G)|=m$, it is well known that $\sum_{i=1}^{|V(G)|}d_i=2m$. Since $\delta\geq 2$, then we known that $\mathbb{D}(G)$ might be

$$\left(\frac{2m-2}{3},4,2,2,2,2,\ldots,2\right)\quad\text{or}\quad\left(\frac{2m-2}{3},3,3,2,2,2,\ldots,2\right).$$

If $\mathbb{D}(G)=(\frac{2m-2}{3},4,2,2,2,2,\ldots,2)$, then $G=G_1$.
If $\mathbb{D}(G)=(\frac{2m-2}{3},3,3,2,2,2,\ldots,2)$, then $G=G_6=K_1\vee(\frac{m-7}{3}K_2\cup p_4)$ or $G=G_7=K_1\vee(\frac{m-10}{3}K_2\cup 2P_3)$, shown in Figure 4. Let $X=(x_w,x_1,x_2,x_3,x_4,\cdots,x_{\frac{2m-2}{3}})^T$ be a unit eigenvector corresponding to $\rho_\alpha(G_6)$ where x_w corresponds to w and x_i corresponds to $v_i(1\leq i\leq\frac{2m-2}{3})$. If $x_2\geq x_3$, let $G^*=G_6-v_4v_3+v_4v_2$; Otherwise, let $G^*=G_6-v_1v_2+v_1v_3$. In the both cases, $G^*=G_1$. By Lemma 4, we have $\rho_\alpha(G_6)<\rho_\alpha(G_1)$. Applying Lemma 4 to the vertices v_2 and v_3 of G_7, we can similarly derive that $\rho_\alpha(G_7)<\rho_\alpha(G_1)$.

Case 2. $s = 1$, then $|V(G-w)| = \frac{2m+1}{3}$. Noting that $|E(G)| = m$, then G has degree sequence $(\frac{2m-2}{3}, 2, 2, 2, 2, 2, 2, \ldots, 2)$. It follows that $G = G_3$. Noting that $wv_2v_3v_1$ is an internal path of G_3, by Lemma 5, we have $\rho_\alpha(G_3) < \rho_\alpha(F_{\frac{m-1}{3}})$. Furthermore, noting that $F_{\frac{m-1}{3}}$ is a proper subgraph of G_6, by Lemma 3, we have $\rho_\alpha(F_{\frac{m-1}{3}}) < \rho_\alpha(G_6)$. Therefore, we have $\rho_\alpha(G_3) < \rho_\alpha(G_6) < \rho_\alpha(G_1)$.

(iii) Let $m \geq 29$ and $m \equiv 2(mod3)$. It is easy to see that $G_2 \in \mathcal{G}_m^2$. By Lemma 2, we have $\rho_\alpha(G_2) > \frac{2m-1}{3}\alpha + \frac{(1-\alpha)^2}{\alpha}$.
For $2 \leq d(w) \leq \frac{2m-7}{3}$, by similar reasoning as in the proof of (i), we can similarly derived that

$$\rho_\alpha(G) < \frac{2m-1}{3}\alpha + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(G_2)$$

for $m \geq 14$ and $\frac{1}{2} \leq \alpha < 1$.
If $d(w) = \frac{2m-4}{3}$, let $|V(G-w)| = \frac{2m-4}{3} + s$, then

$$2m \geq \frac{2m-4}{3} + 2\left(\frac{2m-4}{3} + s\right).$$

It follows that $s \leq 2$. This implies that $d_2 = d_2(G) \leq 2 + 4 = 6$. By Lemma 7, we have

$$\rho_\alpha(G) \leq A\left(\frac{2m-4}{3}, 6\right) \leq \frac{2m-1}{3}\alpha + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(G_2)$$

for $m \geq 29$ and $\frac{1}{2} \leq \alpha < 1$.
If $d(w) = \frac{2m-1}{3}$, let $|V(G-w)| = \frac{2m-1}{3} + s$, then

$$2m \geq \frac{2m-1}{3} + 2\left(\frac{2m-1}{3} + s\right).$$

It follows that $s = 0$. Noting that $|E(G)| = m$, we known that G has degree sequence $(\frac{2m-1}{3}, 3, 2, 2, 2, 2, \ldots, 2)$. It follows that $G = G_2$, completing the proof of (iii).

□

Proof of Theorem 2. Let $\mathcal{H}_m^2$ denote the set of all minimally 2-edge-connected graphs with m edges.

(i) Let $m \geq 24$ and $m \equiv 0(mod3)$. It is easy to see that $F_{\frac{m}{3}} \in \mathcal{H}_m^2 \subseteq \mathcal{G}_m^2$. By Theorem 1(i), we have $\rho_\alpha(G) \leq \rho_\alpha(F_{\frac{m}{3}})$ for $G \in \mathcal{H}_m^2$ and the equality holds if and only if $G = F(\frac{m}{3})$.

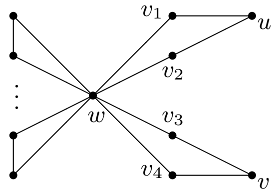
(ii) Let $m \geq 37$ and $m \equiv 1(mod3)$. It is easy to see that $G_4 \in \mathcal{H}_m^2 \subseteq \mathcal{G}_m^2$. By Lemma 2, we have $\rho_\alpha(G_3) > \frac{2m-2}{3}\alpha + \frac{(1-\alpha)^2}{\alpha}$. From the proof of Theorem 1(ii), we know that $\rho_\alpha(G) \leq \frac{2m-2}{3}\alpha + \frac{(1-\alpha)^2}{\alpha}$ for $G \in \mathcal{G}_m^2 \setminus \{G_1, G_3, G_6, G_7\}$. This implies that $\rho_\alpha(G) \leq \rho_\alpha(G_3)$ for $G \in \mathcal{H}_m^2$, and the equality holds if and only if $G = G_3$.

(iii) Let $m \geq 50$ and $m \equiv 2(mod3)$. It is easy to see that $G_4 \in \mathcal{H}_m^2 \subseteq \mathcal{G}_m^2$. By Lemma 2, we have $\rho_\alpha(G_4) > \frac{2m-4}{3}\alpha + \frac{(1-\alpha)^2}{\alpha}$. For $G \in \mathcal{H}_m^2$, let w be a vertex of G such that

$$\max_{u \in V(G)} \{\alpha d(u) + (1-\alpha)m(u)\} = \alpha d(w) + (1-\alpha)m(w) = \alpha d(w) + \frac{1-\alpha}{d(w)} \sum_{wv \in E(G)} d(v),$$

where $2 \leq d(w) \leq \frac{2m-4}{3}$.
For $2 \leq d(w) \leq \frac{2m-10}{3}$, by similar reasoning as in the proof of Theorem 1(i), we can prove that

$$\rho_\alpha(G) \leq \alpha d(w) + (1-\alpha)m(w) \leq \frac{2m-4}{3}\alpha + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(G_4)$$



$G_8\left(d(w)=\frac{2m-4}{3}\right)$

Figure 5. G_8

for $m \geq 20$ and $\frac{1}{2} \leq \alpha < 1$.
If $d(w) = \frac{2m-7}{3}$, let $|V(G-w)| = \frac{2m-7}{3} + s$, then

$$2m \geq \frac{2m-7}{3} + 2\left(\frac{2m-7}{3} + s\right).$$

It follows that $0 \leq s \leq 3$. This implies that $d_2 = d_2(G) \leq 2 + 7 = 9$. By Lemma 7, we have

$$\rho_\alpha(G) \leq A\left(\frac{2m-7}{3}, 9\right) \leq \frac{2m-4}{3}\alpha + \frac{(1-\alpha)^2}{\alpha} < \rho_\alpha(G_4)$$

for $m \geq 50$ and $\frac{1}{2} \leq \alpha < 1$.
If $d(w) = \frac{2m-4}{3}$. let $|V(G-w)| = \frac{2m-4}{3} + s$, then

$$2m \geq \frac{2m-4}{3} + 2\left(\frac{2m-4}{3} + s\right).$$

It follows that $0 \leq s \leq 2$. We consider the following three cases.
Case 1. $s = 0$, then $|V(G-w)| = \frac{2m-4}{3}$ and $|E(G-w)| = \frac{m+4}{3}$. Since G is minimally 2-edge-connected, it follows that $G - w = pK_2 \cup qK_1$, where p nd q are nonnegative integers with $2p + q = \frac{2m-4}{3}$. This implies that $|E(G-w)| \leq \frac{m-2}{3}$, a contradiction.
Case 2. $s = 1$. Let $V(G) \setminus N[w] = \{u\}$. Then $|V(G-w)| = \frac{2m-1}{3}$, and $\mathbb{D}(G) = (\frac{2m-4}{3}, 4, 2, 2, 2, 2, \ldots, 2)$ or $(\frac{2m-4}{3}, 3, 3, 2, 2, 2, \ldots, 2)$. If $\mathbb{D}(G) = (\frac{2m-4}{3}, 4, 2, 2, 2, 2, \ldots, 2)$ and there exist a vertex $v_i \in N(w)$ such that $d(v_i) = 4$, then there exist at least two vertices $v_j, v_k \in N(w)$ such that $v_iv_j, v_iv_k \in E(G)$. Obviously, we obtain a cycle $wv_kv_iv_jw$ with a chord wv_i, a contradiction to Lemma 6.
If $\mathbb{D}(G) = (\frac{2m-4}{3}, 4, 2, 2, 2, 2, \ldots, 2)$ and $d(u) = 4$, then $G = G_5$, shown in Figure 3. By Lemma 8, we have $\rho_\alpha(G_5) < \rho_\alpha(G_4)$.
If $\mathbb{D}(G) = (\frac{2m-4}{3}, 3, 3, 2, 2, 2, \ldots, 2)$, then exists a vertex $v_i \in N(w)$ such that $d(v_i) = 3$. By Lemma 6, we know that $G[N(w)] = pK_2 \cup qK_1$. It follows that $u \in N(v_i)$. Suppose $N(v_i) = \{w, u, v_j\}$. Noting that $d(u) \geq 2$, we deduce that there exists another vertex $v_k \in N(w)$ such that $uv_k \in E(G)$. If $v_k = v_j$, we obtain a cycle wv_iuv_jw with a chord v_iv_j; if $v_k \neq v_j$, we obtain a cycle $wv_jv_iuv_kw$ with a chord wv_i. This contracts Lemma 6.

Case 3. $s = 2$. Let $V(G) \setminus N[w] = \{u, v\}$. Then $|V(G-w)| = \frac{2m+2}{3}$ and the degree sequence of G must be $(\frac{2m-4}{3}, 2, 2, 2, 2, 2, \ldots, 2)$. Noting that $E(G) = m$ and $d(w) = \frac{2m-4}{3}$, it follows that $G = G_4$ or $G = G_8$, shown in Figure 5. Applying Lemma 4 to vertices u and v of G_8, we can derive $\rho_\alpha(G_8) < \rho_\alpha(G_5)$. By Lemma 8, we have $\rho_\alpha(G_5) < \rho_\alpha(G_4)$. Therefore $\rho_\alpha(G_8) < \rho_\alpha(G_4)$.
Combining the above arguments, we complete the proof.

□

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Conflict of Interest

The author declares no conflict of interest.

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