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Article

Neighbor Locating Coloring on Graphs: Three Products

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Abstract: Let G be a simple finite graph. A k-coloring of G is a partition $\pi = \{S_1, \cdots, S_k\}$ of $V(G)$ so that each S_i is an independent set and any vertex in S_i takes color i. A k-coloring $\pi = \{S_1, \cdots, S_k\}$ of $V(G)$ is a neighbor locating coloring if any two vertices $u, v \in S_i$, there is a color class S_j for which, one of them has a neighbor in S_j and the other not. The minimum k with this property, is said to be neighbor locating chromatic number of G, denoted by $\chi_{NL}(G)$ of G. We initiate to study the neighbor locating coloring of graphs resulted from three types of product of two graphs. We investigate the neighbor locating chromatic number of Cartesian, lexicographic and corona product of two graphs. Finally, we untangle the neighbor locating chromatic number of any aforementioned three products of cycles, paths and complete graphs.

Keywords: Coloring, Neighbor locating coloring, (Cartesian, lexicographic and corona) product, Graphs

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1. Introduction

Let $G = (V, E)$ be a simple, finite, connected and undirected graph with the vertex set $V = V(G)$ and the edge set $E = E(G)$. The open neighborhood of v, $N(v)$ is the set of vertices adjacent to v and closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. The minimum and the maximum degree of G is the smallest and largest number of neighbors of a vertex in G and denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$ respectively. For the terminologies and notation not herein, see [1].

For two standard products (Cartesian and lexicographic product) of graphs G and H, the vertex set is $V(G) \times V(H)$ and the adjacency of two vertices are defined as follows. In the *Cartesian product* $G \square H$, two vertices (g, h) and (g', h') are adjacent if g is adjacent to g' in G and $h = h'$ in H, or if $g = g'$ in G and h is adjacent to h' in H. In *lexicographic product* $G[H]$, two vertices (g, h) and (g', h') are adjacent if either $gg' \in E(G)$ or $g = g'$ and $hh' \in E(H)$. The *corona product* $G \circ H$ of two graphs G and H is obtained by tacking one copy of G and $|V(G)|$ copies of H and by joining each vertex of the i-th copy of H to the i-th vertex of G, where $1 \leq i \leq |V(G)|$. We use P_n, C_n and K_n to display the path, cycle and complete graph of order n, respectively. The Cartesian product of $C_m \square C_n$, $P_m \square P_n$, $K_m \square C_n$, and $K_m \square P_n$, are said to be tori graph, grid graph, perfect tori graph and perfect grid graph respectively.

A proper k-coloring of G, ($k \in \mathbb{N}$), is a function f defined from $V(G)$ to a set of colors $[K] = \{1, 2, \cdots, k\}$ in which every two adjacent vertices have different colors. Minimum k

for coloring of a graph G is called the chromatic number of G denoted by $\chi(G) = k$, and the color classes is denoted by $\pi = \{S_1, \cdots, S_k\}$. The color-degree of a vertex v is defined to be the number of different colors of π comprising some vertex of $N(v)$. For a connected graph G, a vertex $v \in V(G)$ and a set of vertices $S \subseteq V(G)$, the distance between v and S is $d(v, S) = \min\{d(v, w) : w \in S\}$.

A k coloring $\pi = \{S_1, S_2, \cdots, S_k\}$ is said to be a metric locating (ML) coloring if for every $i \in \{1, 2, \cdots, k\}$ and for every pair of distinct vertices $u, v \in S_i$, there exists $j \in \{1, 2, \cdots, k\}$ such that $d(u, S_j) \neq d(v, S_j)$. Minimum k for ML-coloring of a graph G, is called (metric-)locating number $\chi_L(G)$ of G, [2–4]. Recently, some authors worked on edge-locating coloring of a graph, which is not without pleasure to see [5].

A k-neighbor locating coloring of a graph G is a partition of $V(G)$ to $\pi = \{S_1, S_2, \cdots, S_k\}$ so that for two vertices $u_1, u_2 \in S_i$, the set of colors of the neighborhood of u_1 is different from the set of colors of the neighborhood of u_2. Minimum k for a neighbor locating coloring (an NL-coloring) of a graph G is called the neighbor locating chromatic number (NL-chromatic number) of G denoted by $\chi_{NL}(G) = k$. This concept has been defined under the name of adjacency locating coloring (L_2-coloring) of G in [6] and briefly worked on. Also, its chromatic number was named L_2-chromatic number of G ($\chi_{L_2(G)}$ of G). For more information on this area see [2, 6–11].

In this paper, we observe some preliminaries results on NL-coloring in Section 2. In Section 3, the NL-chromatic number of Cartesian product of two graphs are studied, in particular the NL-chromatic number of some tori graphs, grid graphs, perfect tori graphs, and perfect grid graphs are determined. In Section 4, the NL-chromatic number of lexicographic product of two graphs are investigated and finally, NL-chromatic number of corona product of two graphs are experimented in Section 5.

2. Preliminary Results

In this section, we explore some preparatory results related to NL-coloring of graphs.

Remark 1. ([8] Remark 1) *Let G be a graph of order n and maximum degree Δ. Let $\Pi = \{S_1, \cdots, S_k\}$ be a k-NL-coloring of G. There exist at most $\binom{k-1}{j}$ vertices in S_i of color-degree j, for every $1 \leq i \leq k$, where $1 \leq j \leq k-1$ and consequently, $|S_i| \leq \sum_{j=1}^{\Delta} \binom{k-1}{j}$. For $\chi_{NL}(G) = k \geq 3$ and $1 \leq j \leq \Delta$, we denote by $a_j(k)$ the maximum number of vertices of color-degree j, and we denote by $\ell(k)$ the maximum number of vertices of color-degree 1 or 2 that is ($\ell(k) = a_1(k) + a_2(k)$). Therefore, we have.*

$$a_1(k) = k(k-1), \qquad a_2(k) = \frac{k(k-1)(k-2)}{2}, \qquad \ell(k) = k\binom{k}{2} = \frac{k^3 - k^2}{2}.$$

Theorem 1. ([7] Theorem 1) *Let G be a non-trivial graph of order n and maximum degree Δ. Let $\chi_{NL}(G) = k$. If G has no isolated vertices and $\Delta \leq k-1$, then*

$$n \leq k \sum_{j=1}^{\Delta} \binom{k-1}{j}.$$

From Theorem 1 and this fact, for positive integers $l \geq k$, $\binom{l}{t} \geq \binom{k}{t}$ ($t \leq k$), then we have.

Corollary 1. *Let G be a non-trivial graph of order n and maximum degree Δ. Let $k = \chi_{NL}(G) \leq l$. If G has no isolated vertices, then*

$$n \leq l \sum_{j=1}^{\Delta} \binom{l-1}{j}.$$

Let G be a graph of order n and maximum degree Δ. From Corollary 1, if m is a positive integer in which $\Delta \leq m-1$ and $n > m\sum_{j=1}^{\Delta}\binom{m-1}{j}$, then $\chi_{NL}(G) > m$. Therefore we have the following.

Corollary 2. *Let G be a graph of order n and maximum degree Δ. Then*

$$\chi_{NL}(G) \geq \min\left\{ l : n \leq l\sum_{j=1}^{\Delta}\binom{l-1}{j} \right\}.$$

This bound is sharp for nontrivial path P_n from Theorem 3.6 of [6] *or* Theorem 17 of [8].

There needs the following.

Theorem 2. ([6] Theorem 3.6). *For a positive integer $n \geq 2$, $\chi_{L_2(P_n)} = m$, where $m = min\{k : k \in \mathbb{N}, n \leq \frac{1}{2}(k^3 - k^2)\}$. More precisely, there exist an adjacency locating m-coloring f_n of the path $P_n = v_1v_2\cdots v_n$ with the color set $\{1, 2, \cdots, m\}$, and two specified colors (say "1" and "2") such that f_n satisfies the following properties.*

(a) $f_n(v_{n-1}) = 2$ and $f_n(v_n) = 1$.
(b) If $n \geq 9$, then $f_n(v_{n-2}) = m$.
(c) If $n \geq 9$ and $n \neq \frac{1}{2}(m^3 - m^2) - 1$, then $f_n(v_1) = 2$ and $f_n(v_2) = 1$.

Theorem 3. ([8] Theorem 17). *Let k, n be integers such that $k \geq 4$ and $\ell(k-1) < n \leq \ell(k)$. Then,*

(1) $\chi_{NL}(P_n) = k$.
(2) $\chi_{NL}(Cn) = k$, if $n \neq \ell(k) - 1$.
(3) $\chi_{NL}(Cn) = k + 1$, if $n = \ell(k) - 1$.

3. NL-Coloring of Cartesian Product of Graphs

In this section we discuss on the neighbor locating coloring of Cartesian product of two graphs and obtain neighbor locating chromatic number of some tori graphs, grid graphs, complete tori and complete grid graphs.

Theorem 4. *Let G and H be two graphs. Then $\chi_{NL}(G\square H) \leq \chi_{NL}(G)\chi_{NL}(H)$. This Bound is sharp.*

Proof. Let $V(G) = \{v_1, v_2, \cdots, v_m\}$ and $V(H) = \{u_1, u_2, \cdots, u_n\}$. Let $\chi_{NL}(G) = k'$ and $\chi_{NL}(H) = k''$ with $C_1 = CNL(G) = \{1, 2, \cdots, k'\}$ as an NL-coloring of G and $C_2 = CNL(H) = \{1, 2, \cdots, k''\}$ as an NL-coloring of H. Now we give rise an NL-coloring of $G\square H$. Let $\{w_{ij} = (v_i, u_j) : 1 \leq i \leq m, 1 \leq j \leq n\}$ be the set of vertices $G\square H$. We make an NL-coloring of $G\square H$ with $k'k''$ colors. Let

$$C = \{rs : 1 \leq r \leq k' \; and \; 1 \leq s \leq k''\} = \{11, 21, \cdots, k'1,\ 12, 22, \cdots, k'2,\ \cdots,\ 1k'', 2k'', \cdots, k'k''\}$$

be a set of $k'k''$ labels in which the vertex w_{ij} is assigned by the label rs whenever the vertex v_i has been assigned with label r and u_j with label s. It is sufficient to show that for both vertices with the same color, their color neighbors are different.
For this, we bring up three positions.

1. If we have two or more similar colors in a row, for instance w_{ik} and w_{il} are assigned with same color rs in a ith row. Then vertex v_i is assigned by r and u_k, u_l are assigned by s. Since u_k, u_l has same color in H, $N(u_k) \cap C_2 \neq N(u_l) \cap C_2$. Assume that the label t is in $N(u_k) \cap C_2 \setminus N(u_l) \cap C_2$, then rt is in $N(w_{ik})$ and is not in $N(w_{il})$. This shows that, the color neighbors of w_{ik} and w_{il} are different.

2. If we have two or more similar colors in a column, that is the vertices w_{kj} and w_{lj} have similar color, then we can observe the proof as part 1.

3. Let w_{ij} and w_{lk} have same color where $i \neq l$ and $j \neq k$. Let rs be the color of w_{ij} and w_{lk}. Then the vertices v_i, v_k are assigned with color r and the vertices u_j, u_l are assigned with color s. Assume that t is the color of one of the neighbors of v_i say v_p while no neighbors of v_l are assigned by t. Thus w_{pj} has color ts. On the other hand, no vertex like w_{qz} with color ts cannot be appeared in $N(w_{lk})$ since otherwise, v_q is assigned with color t in $N_H(v_l)$. This denotes, the color neighbors of w_{ij} and w_{lk} are different. Therefore C is a neighbor locating coloring of $G\square H$.

For observing the sharpness, consider $G = H = P_2$, then $\chi_{NL}(P_2) = 2$, $P_2\square P_2 = C_4$ and $\chi_{NL}(C_4) = 4$. □

3.1. Tori graphs

The tori graph $C_m\square C_n$ has maximum degree $\Delta = 4$ and is a free isolated vertices graph. Therefore we have.

Theorem 5. *Let $G = C_m\square C_n$ be a tori graph. Then,*

$$\min\{k : \frac{\ell(k) + k\binom{k}{4}}{mn} \geq 1\} \leq \chi_{NL}(C_m\square C_n) \leq \chi_{NL}(C_m)\chi_{NL}(C_n).$$

Proof. The upper bound is resulted from Theorem 4.

The lower bound. By definition of Cartesian product, the tori graph $C_m\square C_n$ is a regular graph with degree $\Delta(G) = \delta(G) = 4$, and

$$\begin{aligned} k\sum_{j=1}^{4}\binom{k-1}{j} &= k\left(\binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3} + \binom{k-1}{4}\right) \\ &= k\binom{k}{2} + k\binom{k}{4} = \ell(k) + k\binom{k}{4}. \end{aligned}$$

Therefore the Corollary 2 prove the lower bound. □

Now we investigate the neighbor locating chromatic number of some special tori graphs.

Proposition 1. *The following holds.*

(i) $\chi_{NL}(C_3\square C_3) = 5$,
(ii) $\chi_{NL}(C_3\square C_4) = 4$,
(iii) $\chi_{NL}(C_3\square C_5) = 5$,
(iv) $\chi_{NL}(C_4\square C_4) = 5$,
(v) $5 \leq \chi_{NL}(C_4\square C_5) \leq 6$,
(vi) $5 \leq \chi_{NL}(C_5\square C_5) \leq 6$,
(vii) $5 \leq \chi_{NL}(C_6\square C_6) \leq 6$,
(viii) $6 \leq \chi_{NL}(C_9\square C_9) \leq 7$.

Proof. (i) On the contrary, suppose that $\chi_{NL}(C_3\square C_3) < 5$. It is clear, $\chi_{NL}(C_3\square C_3) \geq 4$. Let $\{u_{ij} : 1 \leq i \leq 3, 1 \leq j \leq 3\}$ be the set of vertices of $C_3\square C_3$. If $\pi = \{S_1, S_2, S_3, S_4\}$ be the set of colors, then it is well known that u_{22} cannot have color-degree 1 or 2. So there are at least three colors in its neighbors. Let $\{u_{22}\} \subseteq S_1$, $\{u_{12}\} \subseteq S_2$, $\{u_{23}, u_{32}\} \subseteq S_3$ and $\{u_{21}\} \subseteq S_4$. Then it is easy to check that u_{11} does not accept color 1 or 3 and it must be labeled with fifth color, a contradiction.

(ii) Let $\{u_{ij} : 1 \leq i \leq 3, 1 \leq j \leq 4\}$ be the set of vertices $C_3 \square C_4$. It is clear $\chi_{NL}(C_3 \square C_4) \geq 4$. On the other hand, if $\pi = \{S_1, S_2, S_3, S_4\}$ is the set of colors, and consider $S_1 = \{u_{24}, u_{31}, u_{33}\}$, $S_2 = \{u_{21}, u_{32}, u_{34}\}$, $S_3 = \{u_{11}, u_{13}, u_{22}\}$, $S_4 = \{u_{12}, u_{14}, u_{23}\}$. then it can be an NL-coloring of $C_3 \square C_4$. Therefore $\chi_{NL}(C_3 \square C_4) = 4$.

(iii) It is well known that $\chi_{NL}(C_3 \square C_5) \geq 4$. On the contrary, assume that, $\chi_{NL}(C_3 \square C_5) = 4$. Let $\pi = \{S_1, S_2, S_3, S_4\}$. Then any vertex of $C_3 \square C_5$ has color degree 2 or 3 and from Observation 1 for $m = 3$ and $n = 5$, there exist at most 10 vertices of color degree 2 in $C_3 \square C_5$. On the other hand, there exist at most $4\binom{3}{3} = 4$ vertices of color degree 3, thus there must exist at least 11 vertices of color degree 2 in $C_3 \square C_5$, a contradiction. Therefore $\chi_{NL}(C_3 \square C_5) \geq 5$. Now we give an NL-coloring of $C_3 \square C_5$ with 5 colors. Let $\pi = \{S_1, S_2, S_3, S_4, S_5\}$ with $S_1 = \{u_{22}, u_{25}, u_{31}, u_{33}\}$, $S_2 = \{u_{23}, u_{34}\}$, $S_3 = \{u_{12}, u_{15}, u_{24}\}$, $S_4 = \{u_{11}, u_{14}, u_{32}, u_{35}\}$ and $S_5 = \{u_{13}, u_{21}\}$ is an NL coloring of $C_3 \square C_5$. Therefore $\chi_{NL}(C_3 \square C_5) = 5$.

(iv) On the contrary, assume that $\chi_{NL}(C_4 \square C_4) = 4$. At first, we can easy to see that, there does not exist any vertex of color degree 1 in $C_4 \square C_4$. Also by a routine investigation, there exist at most two vertices of color degree 2 in each column of $C_4 \square C_4$. On the other hand, since $k = 4$, there must exist at least $4\binom{3}{2} = 12$ vertices of color degree 2, a contradiction. Therefore $\chi_{NL}(C_4 \square C_4) \geq 5$. Now we give an NL-coloring with 5 colors as follows. Let $\pi = \{S_1, S_2, S_3, S_4, S_5\}$ with $S_1 = \{u_{22}, u_{31}, u_{33}, u_{44}\}$, $S_2 = \{u_{14}, u_{34}\}$, $S_3 = \{u_{12}, u_{24}, u_{41}, u_{43}\}$, $S_4 = \{u_{11}, u_{23}, u_{32}\}$ and $S_5 = \{u_{13}, u_{21}, u_{42}\}$. Therefore $\chi_{NL}(C_4 \square C_4) = 5$.

(v) At first we assert $\chi_{NL}(C_4 \square C_5) \geq 5$. Assume on the contrary that, $\chi_{NL}(C_4 \square C_5) = 4$. Since $C_4 \square C_5$ has only the vertices of color degree 2 or 3 with 4 colors, from Theorem 5 $\chi_{NL}(C_4 \square C_5) > 4$. The following denotes $\chi_{NL}(C_4 \square C_5) \leq 6$. Let $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ with $S_1 = \{u_{22}, u_{25}, u_{31}, u_{33}\}$, $S_2 = \{u_{23}, u_{34}, u_{41}\}$, $S_3 = \{u_{12}, u_{15}, u_{24}\}$, $S_4 = \{u_{14}, u_{32}, u_{35}, u_{43}\}$, $S_5 = \{u_{13}, u_{21}, u_{45}\}$ and $S_6 = \{u_{11}, u_{42}, u_{44}\}$. Therefore $5 \leq \chi_{NL}(C_4 \square C_5) \leq 6$.

(vi) It is clear that, $\chi_{NL}(C_5 \square C_5) \geq 5$. Now, consider $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ with $S_1 = \{u_{22}, u_{25}, u_{31}, u_{33}, u_{45}\}$, $S_2 = \{u_{23}, u_{34}, u_{51}\}$, $S_3 = \{u_{12}, u_{15}, u_{24}, u_{42}, u_{44}\}$, $S_4 = \{u_{14}, u_{32}, u_{35}, u_{53}\}$, $S_5 = \{u_{13}, u_{21}, u_{43}, u_{55}\}$ and $S_6 = \{u_{11}, u_{41}, u_{52}, u_{54}\}$. Then π is an NL-coloring of $C_5 \square C_5$. Therefore $5 \leq \chi_{NL}(C_5 \square C_5) \leq 6$.

(vii) It is easy to see that, $\chi_{NL}(C_6 \square C_6) \geq 5$. We show an NL-coloring of $C_6 \square C_6$ with 6 colors. Consider the coloring $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ with $S_1 = \{u_{25}, u_{33}, u_{42}, u_{51}, u_{53}, u_{66}\}$, $S_2 = \{u_{21}, u_{23}, u_{32}, u_{34}, u_{41}, u_{55}, u_{62}\}$, $S_3 = \{u_{11}, u_{15}, u_{22}, u_{43}, u_{45}, u_{54}\}$, $S_4 = \{u_{13}, u_{24}, u_{26}, u_{35}, u_{44},$
$u_{52}, u_{61}, u_{64}\}$, $S_5 = \{u_{14}, u_{16}, u_{31}, u_{46}, u_{63}, u_{65}\}$ and $S_6 = \{u_{12}, u_{36}, u_{56}\}$. Therefore $5 \leq \chi_{NL}(C_6 \square C_6) \leq 6$.

(viii) And finally we investigate $\chi_{NL}(C_9 \square C_9)$. From Theorem 5, $\chi_{NL}(C_9 \square C_9) \geq 6$. Consider $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ with

$$S_1 = \{u_{11}, u_{14}, u_{34}, u_{47}, u_{53}, u_{55}, u_{62}, u_{73}, u_{75}, u_{82}, u_{88}\},$$
$$S_2 = \{u_{16}, u_{24}, u_{36}, u_{45}, u_{54}, u_{56}, u_{59}, u_{71}, u_{83}, u_{95}, u_{97}\},$$
$$S_3 = \{u_{17}, u_{21}, u_{38}, u_{42}, u_{44}, u_{49}, u_{61}, u_{65}, u_{67}, u_{76}, u_{87}, u_{99}\},$$
$$S_4 = \{u_{13}, u_{19}, u_{26}, u_{28}, u_{33}, u_{39}, u_{47}, u_{52}, u_{57}, u_{66}, u_{68}, u_{74}, u_{79}, u_{94}, u_{96}\},$$
$$S_5 = \{u_{32}, u_{35}, u_{37}, u_{41}, u_{43}, u_{58}, u_{63}, u_{78}, u_{84}, u_{86}, u_{92}\},$$
$$S_6 = \{u_{22}, u_{25}, u_{27}, u_{48}, u_{64}, u_{69}, u_{81}, u_{93}, u_{98}\} \text{ and}$$
$$S_7 = \{u_{12}, u_{15}, u_{18}, u_{23}, u_{29}, u_{31}, u_{51}, u_{72}, u_{77}, u_{85}, u_{89}, u_{91}\}.$$

This coloring is an NL-coloring of $C_9 \square C_9$ with 7 colors. Therefore $6 \leq \chi_{NL}(C_9 \square C_9) \leq 7$. □

Regarding to the Theorem 5, the following problem can be explored.

Problem 1. *Let $n \geq 3$ be a positive integer. For $k \geq 5$, if*

$$\ell(k-1) + (k-1)\binom{k-1}{4} < n^2 \leq \ell(k) + (k)\binom{k}{4},$$

then $\chi_{NL}(C_n \square C_n) \in \{k, k+1\}$.

3.2. Grid graphs

In this subsection we discuss on NL-coloring of grid graphs. The grid graph $P_m \square P_n$ has maximum degree $\Delta = 4$ for $n, m \geq 3$, and has maximum degree $\Delta = 3$ for $m = 2$ and $n \geq 3$, furthermore, any grid graph is free isolated vertices graph. Therefore we have.

Proposition 2. *Let $G = P_2 \square P_n$ be grid graph (lader graph) where $n \geq 3$. Then*

$$\min\{k : \frac{k^2 - k + k\binom{k}{3}}{2n} \geq 1\} \leq \chi_{NL}(P_2 \square P_n) \leq 2\chi_{NL}(P_n).$$

The lower bound is sharp for $P_2 \square P_n$ ($n \in \{3, 4, 5\}$) and the upper bound is sharp for $P_2 \square P_2$.

Proof. The upper bound and its sharpness are resulted from Theorem 4.

For lower bound, since $\Delta(P_2 \square P_n) = 3$,

$$\begin{aligned} k\sum_{j=1}^{3}\binom{k-1}{j} &= k\left(\binom{k-1}{1} + \binom{k-1}{2} + \binom{k-1}{3}\right) \\ &= k^2 - k + k\binom{k}{3}. \end{aligned}$$

Now using Corollary 2, we observe the proof. □

For grid graphs $P_m \square P_n$ for $m, n \geq 3$, we have $\Delta(P_m \square P_n) = 4$. Therefore using similar proof of Theorem 5 we have the following.

Theorem 6. *Let $G = P_m \square P_n$ be the grid graph obtained from Cartesian product of P_m and P_n. Then*

$$\min\{k : \frac{\ell(k) + k\binom{k}{4}}{mn} \geq 1\} \leq \chi_{NL}(P_m \square P_n) \leq \chi_{NL}(P_m)\chi_{NL}(P_n).$$

The lower bound is sharp for $P_m \square P_n$ where $m = n \in \{3, 4, 5, 6\}$, see Proposition 3.

Proposition 3. *For grid graphs $P_n \square P_n$, $n \in \{3, 4, 6, 9\}$, we have.*

(i) $\chi_{NL}(P_3 \square P_3) = 4$,
(ii) $\chi_{NL}(P_4 \square P_4) = 4$,
(iii) $\chi_{NL}(P_6 \square P_6) = 5$,
(iv) $\chi_{NL}(P_9 \square P_9) = 6$.

Proof. At first, assume that $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $v_{ij} = (v_i, v_j)$ for $1 \leq i \leq n$ and $1 \leq j \leq n$ be the vertex in a ith row and jth column of $P_n \square P_n$.

(i) Suppose to the contrary that $\chi_{NL}(P_3 \square P_3) \leq 3$. It is easy to see that, v_{22} has color-degree 1 or 2. If the color degree of v_{22} is 1, then $v_{12}, v_{21}, v_{23}, v_{32}$ have same color. According to the definition of neighbor locating coloring $v_{11}, v_{13}, v_{31}, v_{33}$ should be assigned with different colors, a contradiction.
If the color degree of v_{22} is 2 and without loss of generality, the color of v_{22} is 1 and v_{12}, v_{21}, v_{23} and v_{32} accept two colors 2 or 3. So, at least one of the vertices of v_{11}, v_{13}, v_{31} and v_{33} for example v_{31}, should be assigned with color 1, and so v_{22} and v_{31} with color 1 have same color neighbor, which is a contradiction. On the other hand, it is easy to see that $\chi_{NL}(P_3 \square P_3) \leq 4$. Therefore $\chi_{NL}(P_3 \square P_3) = 4$.

(ii) For 4-NL coloring of $P_4 \square P_4$, from Theorem 6, $\chi_{NL}(P_4 \square P_4) \geq 4$. On the other hand, the tables T_1 and T_2 show that $\chi_{NL}(P_4 \square P_4) = 4$.

(iii) Theorem 6 denotes $\chi_{NL}(P_6 \square P_6) \geq 5$. On the other hand, the tables T_2 and T_3 show that $\chi_{NL}(P_6 \square P_6) = 5$.

(iv) From Theorem 6 we have $\chi_{NL}(P_9 \square P_9) \geq 6$. Now tables T_3 and T_4 show that $\chi_{NL}(P_9 \square P_9) = 6$.

1	2
3	4

T_1

3	2	4	1
2	1	2	4
1	3	4	3
4	1	3	2

T_2

4	2	5	2	5	3
5	3	2	4	1	4
1	2	1	2	4	5
5	1	3	4	3	4
1	4	1	3	2	5
2	5	3	5	3	1

T_3

1	6	2	4	6	5	6	3	1
6	3	5	3	2	4	1	4	3
3	2	4	2	5	2	5	3	5
6	3	5	3	2	4	1	4	6
2	4	1	2	1	2	4	5	4
3	1	5	1	3	4	3	4	3
2	6	1	4	1	3	2	5	6
3	1	2	5	3	5	3	1	2
6	4	1	6	1	3	6	5	1

T_3

Therefore the results hold. □

According to Theorem 6 and Proposition 3, we expect to have the following problem.

Problem 2. *Let $n \geq 3$ be a positive integer. For $k \geq 5$, if*

$$\ell(k-1) + (k-1)\binom{k-1}{4} < n^2 \leq \ell(k) + (k)\binom{k}{4},$$

then $\chi_{NL}(P_n \square P_n) = k$.

3.3. Perfect tori graphs and perfect grid graphs

For neighbor locating coloring of perfect tori graph and perfect grid graph, we have.

Lemma 1. *Let $\chi_{NL}(K_m \square C_n) = k$. If $m+1 \leq k \leq 2m-1$, then at least one vertex of any column have color degree at least m.*

Proof. Let u_{1i} be the i-th vertex of the first column with color in S_i. Suppose on the contrary, the first column has the property that, every vertex has color degree $m-1$, then the second and end columns must be colored with colors S_j, $1 \leq j \leq m$ which have already used for the 1st column. Now, each vertex in the second column with color in S_i, must have a neighbor in the third column with a color in S_t where $t \geq m+1$. Since S_t can be nominated for at most $m-1$ colors, one vertex in the third column can be in S_j, $1 \leq j \leq m$. Let vertex u_{3l} be in S_1. Then two vertices u_{2l} and u_{1r} with same color in S_r $(1 \leq r \leq m)$ find same neighbor colors $\{S_1, S_2, \cdots, S_{r-1}, S_{r+1}, \cdots S_m\}$, that is a contradiction. □

The following give us an upper sharp bound for neighbor locating chromatic number for $K_m \square P_n$ and $K_m \square C_n$.

Proposition 4. *Let $G = K_m \square P_n$ be the perfect grid graph obtained from Cartesian product of K_m and P_n with $m \geq 3$, $n \geq 1$. Then $\chi_{NL}(K_m \square P_n) \leq m+n-1$. This bound is sharp for $K_3 \square P_3$.*

Proof. Each vertex in $K_m \square P_n$ has $m-1$ neighbors in its column and at most two neighbors in its row. There is also n copy of K_m in $K_m \square P_n$. We color the graph by the columns.
The first column must accept m colors. By assigning a new color to every other column, we will clearly have a neighbor locating coloring for $K_m \square P_n$. We consequently, will have $\chi_{NL}(K_m \square P_n) \leq m+n-1$. □

Using similar proof of Proposition 4, we have.

Proposition 5. *Let $G = K_m \square C_n$ be the perfect tori graph obtained from Cartesian product of K_m and C_n with $m, n \geq 3$. Then $\chi_{NL}(K_m \square C_n) \leq m+n-1$ if $3 \leq m \leq 6$. This bound is sharp for $K_3 \square C_3$.*

In the fallow, we display some Cartesian product of $K_m \square C_3$ which $m+1 \leq \chi_{NL}(K_m \square C_n) \leq m+2$. For $m=3$, with an easy calculation is verifiable that $\chi_{NL}(K_3 \square C_n) = 5 = 3+2$, and for the $m \geq 7$, we have $\chi_{NL}(K_m \square C_n) = m+1$ according to the table shown below.

2	$m+1$	4
3	1	5
4	5	6
5	6	7
.	.	.
.	.	.
.	.	.
$m-2$	$m-1$	m
$m-1$	m	$m+1$
m	3	1
$m+1$	4	2

T_5

$\chi_{NL}(K_m \square C_n) = m+1$

Regarding to the table T_5 and χ_{NL}-coloring of $K_3\Box C_3$ we have.

Corollary 3. *For $m \geq 3$,*

1. $\chi_{NL}(K_m\Box C_3) \leq m+2$ *if* $3 \leq m \leq 6$. *This bound is sharp.*
2. $\chi_{NL}(K_m\Box C_3) = m+1$ *if* $m \geq 7$.

According to the Corollary 3, one can have.

Problem 3. *For* $4 \leq m \leq 6$, $\chi_{NL}(K_m\Box C_3) = m+2$.

4. Lexicographic Product of Graphs

In this section we discuss on the neighbor locating coloring of lexicographic product of two graphs with emphasis on paths, cycles and complete graphs. We start with the following.

Taking the proof of Theorem 4 as a plan, we can have the following theorem.

Theorem 7. *Let G and H be two graphs. Then $\chi_{NL}(G[H]) \leq \chi_{NL}(G)\chi_{NL}(H)$. This bound is sharp.*

Proof. The proof is quite similar to the proof of Theorem 4. For sharpness consider $G = P_2$ and $H = P_n$ for any n or $H = C_n$ for $n \geq 3$, and see Proposition 6. □

For $n = 1$ and $n = 2$, $P_2[P_1] = K_2$ and $P_2[P_2] = K_4$, and next $\chi_{NL}(K_2) = 2$, $\chi_{NL}(K_4) = 4$. We hence deduce $\chi_{NL}(P_2[P_1]) = 2\chi_{NL}(P_1)$ and $\chi_{NL}(P_2[P_2]) = 2\chi_{NL}(P_2)$. Now we extend the mentioned results for any P_n and C_n.

Proposition 6. *Let $n \geq 3$. Then,*

(i) $\chi_{NL}(P_2[P_n]) = 2\chi_{NL}(P_n)$.
(ii) $\chi_{NL}(P_2[C_n]) = 2\chi_{NL}(C_n)$.

Proof. (i) Let $v_1, v_2, \cdots, v_n$ be the vertices of P_n and u_1, u_2 be the vertices of P_2. Then $V(P_2[P_n]) = \{(u_1, v_i), (u_2, v_i) : 1 \leq i \leq n\}$. Since any (u_1, v_i) is adjacent to any (u_2, v_j), hence they receive distinct colors. Therefore, we must use $\chi_{NL}(P_n) = k$ colors for first row and $\chi_{NL}(P_n) = k$ colors for the second row and then $\chi_{NL}(P_2[P_n]) = 2k = 2\chi_{NL}(P_n)$.

(ii) It is proved such as the proof of part (i). □

It is expected we nurture the Proposition 6 as.

Theorem 8. *If $k \geq 3$ and $n \geq 4$ are integers and $\chi_{NL}(P_n) = k$, then*

(i) $\chi_{NL}(P_n[P_n]) = 2\chi_{NL}(P_n) = 6$, if $n \in \{3, 4\}$,
(ii) $\chi_{NL}(P_n[P_n]) = 2\chi_{NL}(P_n) + \lfloor\frac{n-2}{2}\rfloor$, if $n \in \{5, 6\}$,
(iii) $\chi_{NL}(P_n[P_n]) \leq 2\chi_{NL}(P_n) + \lfloor\frac{n-2}{2}\rfloor$, if $n \geq 7$.

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $V(P_n[P_n]) = \{u_{ij} : 1 \leq i, j \leq n\}$. From the definition of the Lexicographic product, each vertex of the first row is adjacent to the vertices of the second row, each vertex of the last row is adjacent to the vertices of the $n-1$-th row and each vertex of the i-th row is adjacent to the vertices of the $i-1$-th and $i+1$-th rows for $2 \leq i \leq n-1$.

(i) For $n=3$, it is easy to see that, $\chi_{NL}(P_3[P_3]) = 2\chi_{NL}(P_3) = 6$.
Let $n=4$. From the relation between vertices mentioned as above, we need at least 6 colors for NL-coloring of $P_4[P_4]$. Let $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6\}$ where $S_1 = \{u_{22}, u_{24}, u_{44}\}$, $S_2 = \{u_{12}, u_{14}, u_{34}\}$, $S_3 = \{u_{21}, u_{41}, u_{43}\}$, $S_4 = \{u_{11}, u_{31}, u_{33}\}$, $S_5 = \{u_{23}, u_{42}\}$ and $S_6 = \{u_{13}, u_{32}\}$. This gives an NL-coloring of $P_4[P_4]$ with 6 colors. Therefore $\chi_{NL}(P_4[P_4]) = 6$.

(ii) Let $n \in \{5,6\}$. We show that $\chi_{NL}(P_n[P_n]) = 2\chi_{NL}(P_n) + \lfloor \frac{n-2}{2} \rfloor$.

$\chi_{NL}(P_5[P_5])$. It is obvious that, the first and second rows needs at least 6 colors for NL-coloring. On the other hand, for avoiding to the same color neighbors for vertices of the first and third rows or for vertices of the second and fourth rows, we need at lest one color that has not been used for first and second rows. Thus $\chi_{NL}(P_5[P_5]) \geq 7$. Let $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7\}$ where $S_1 = \{u_{11}, u_{14}, u_{31}, u_{34}\}$, $S_2 = \{u_{12}, u_{32}, u_{52}\}$, $S_3 = \{u_{13}, u_{15}, u_{33}, u_{34}, u_{51}, u_{54}\}$, $S_4 = \{u_{21}, u_{24}, u_{53}, u_{55}\}$, $S_5 = \{u_{21}, u_{42}, u_{44}\}$, $S_6 = \{u_{23}, u_{25}, u_{41}\}$ and $S_7 = \{u_{43}, u_{45}\}$. Then π is an NL-coloring of $P_5[P_5]$ with 7 colors. This shows that, $\chi_{NL}(P_5[P_5]) \leq 7$. Therefore $\chi_{NL}(P_5[P_5]) = 7 = 2\chi_{NL}(P_5) + \lfloor \frac{5-2}{2} \rfloor$.

$\chi_{NL}(P_6[P_6])$. The first and second rows need 6 colors for NL-coloring of $P_6[P_6]$. If the third row take the colors of the first row, then the fourth row must take at least one color that has not been used for the second row. As well, if the fifth row take the color of the third row, then the sixth row must take at least one color that has not been used for the second and fourth rows. Thus for NL-coloring of $P_6[P_6]$ we need at least 8 distinct colors. That is $\chi_{NL}(P_6[P_6]) \geq 8$. Now we give an NL-coloring of $P_6[P_6]$ with 8 colors. Let $\pi = \{S_1, S_2, S_3, S_4, S_5, S_6, S_7, S_8\}$ where $S_1 = \{u_{11}, u_{16}, u_{31}, u_{36}, u_{64}, u_{66}\}$, $S_2 = \{u_{12}, u_{14}, u_{32}, u_{34}, u_{51}, u_{53}\}$, $S_3 = \{u_{13}, u_{15}, u_{33}, u_{35}\}$, $S_4 = \{u_{21}, u_{26}, u_{54}, u_{56}\}$, $S_5 = \{u_{22}, u_{24}, u_{44}, u_{46}\}$, $S_6 = \{u_{23}, u_{25}, u_{41}, u_{43}, u_{62}, u_{65}\}$, $S_7 = \{u_{52}, u_{55}\}$ and $S_8 = \{u_{42}, u_{45}, u_{61}, u_{63}\}$. Then π is an NL-coloring of $P_6[P_6]$ with 8 colors. This shows that, $\chi_{NL}(P_6[P_6]) \leq 8$. Therefore,$\chi_{NL}(P_6[P_6]) = 8 = 2\chi_{NL}(P_6) + \lfloor \frac{6-2}{2} \rfloor$.

(iii) We want to show that for $n \geq 7$, $\chi_{NL}(P_n[P_n]) \leq 2\chi_{NL}(P_n) + \lfloor \frac{n-2}{2} \rfloor$. If $n \geq 7$, then we use $\chi_{NL}(P_n) = k$ colors $\{1, 2, \cdots, k\}$ for odd rows and we assign k colors $\{k+1, k+2, \cdots, 2k\}$ for second row, and for any other even row, we use a new color beside the colors have been used for the second row. Since we have $\lfloor \frac{n}{2} \rfloor$ even rows, we need at most $\lfloor \frac{n-2}{2} \rfloor$ new colors. This coloring is an NL-coloring for $P_n[P_n]$ with $2k + \lfloor \frac{n-2}{2} \rfloor$. Therefore $\chi_{NL}(P_n[P_n]) \leq 2\chi_{NL}(P_n) + \lfloor \frac{n-2}{2} \rfloor$. □

Regarding to the Theorem 8, maybe have the following problem.

Problem 4. *For which integers $n \geq 7$, $\chi_{NL}(P_n[P_n]) = 2\chi_{NL}(P_n) + \lfloor \frac{n-2}{2} \rfloor$?*

In $G[H]$, if $G = C_3$ and $H \in \{C_n, P_n\}$, then from definition of lexicographic product of two graphs, any two vertices of $C_3[C_n]$ or of $C_3[P_n]$ are adjacent. Let G_{in} represent the i-th row of $C_3[C_n]$ or of $C_3[P_n]$. Then, there needs $\chi_{NL}(C_n)$ or $\chi_{NL}(P_n)$ colors for NL-coloring of G_{in}. Therefore $\chi_{NL}(C_3[C_n]) \geq 3\chi_{NL}(C_n)$ or $\chi_{NL}(C_3[P_n]) \geq 3\chi_{NL}(P_n)$. Now from Theorem 7 we have.

Observation 9. *Let $n \geq 3$. Then,*

(i) $\chi_{NL}(C_3[C_n]) = 3\chi_{NL}(C_n)$.
(ii) $\chi_{NL}(C_3[P_n]) = 3\chi_{NL}(P_n)$.

Remark 2. *In the graph $G[H]$, if $G = P_3$ and $H \in \{C_n, P_n\}$, then from definition of lexicographic product of two graphs, any vertex of second row of $P_3[C_n]$ or of $P_3[P_n]$ is adjacent to any vertex of the first and third rows of $P_3[C_n]$ or $P_3[P_n]$ respectively. Thus there need at least $2\chi_{NL}(C_n)$ or $2\chi_{NL}(P_n)$ for NL-coloring of $P_3[C_n]$ or of $P_3[P_n]$ respectively. On the other hand,* *each vertex of first and third rows have all colors of the second row in their color neighbors. Therefore if u_{1j} and u_{3l} of the first and third rows are in a same color class, then it must* $\{u_{1(j+1)}, u_{1(j-1)}\} \neq \{u_{3(l+1)}, u_{3(l-1)}\}$ (mod n). *Therefore we can adapt the NL-coloring of P_{1_n}, and P_{3_n}, from the NL-coloring of P_{2n} and also adapt the NL-coloring of C_{1_n}, and C_{3_n}, from the NL-coloring of C_{2n}.*

Now from the Remark 2, we can have.

Theorem 10. *Let $n \geq 3$. If $\ell(k-1) < n \leq \ell(k)$ and $\ell(k+t-1) < 2n \leq \ell(k+t)$, then*

(i) $\chi_{NL}(P_3[P_n]) \leq 2\chi_{NL}(P_n) + t + 2$.
(ii) $\chi_{NL}(P_3[C_n]) \leq 2\chi_{NL}(C_n) + t + 2$.

Proof. (i) Let $P_{i_n} = u_{i1}u_{i2}\cdots u_{in}$ be the i-th row of $P_3[P_n]$. Let $\chi_{NL}(P_n) = k$. Then $\ell(k-1) < n \leq \ell(k)$, and $\ell(k+t-1) < 2n \leq \ell(k+t)$. Hence $\chi_{NL}(P_{2n}) = k+t = m$ for $t \geq 0$. We partition the path P_{2n} with the set of vertices $\{v_1, v_2, \cdots, v_{n-1}, v_n, v_{n+1}, v_{n+2}, \cdots, v_{2n-1}, v_{2n}\}$ with χ_{NL}-colors, to two paths $P_{1_n} = u_{11}u_{12}\cdots u_{1n}$ and $P_{3_n} = u_{31}u_{32}\cdots u_{3n}$ where $u_{1j} = v_j$ and $u_{3j} = v_{n+j}$.
Suppose that, $c(x)$ denote the color of x. From the NL-coloring of P_{2n}, $(c(u_{11}), c(u_{12})) \neq (c(u_{3n}), c(u_{3(n-1)}))$. Now we bring up a few situations.

(a) If $(c(u_{11}),\ c(u_{12})) \notin \{(c(u_{1n}), c(u_{1(n-1)})),\ (c(u_{31}), c(u_{32}))\}$ *and* $(c(u_{3n}), c(u_{3(n-1)})) \notin \{(c(u_{1n}),\ c(u_{1(n-1)})),\ (c(u_{31}), c(u_{32}))\}$, then the colors assigned to P_{1_n} and P_{3_n} are NL-coloring.
(b) If $(c(u_{11}), c(u_{12})) = (c(u_{1n}), c(u_{1(n-1)}))$, then $(c(u_{3n}), c(u_{3(n-1)})) \neq (c(u_{1n}), c(u_{1(n-1)}))$. Thus in this situation, we assign a new color to u_{11} and the rest of the colors remain unchanged. Therefore P_{1_n} and P_{3_n} are NL-colored with at most $m+1$ colors.
(c) If $(c(u_{11}), c(u_{12})) = (c(u_{1n}), c(u_{1(n-1)}))$ and $(c(u_{3n}), c(u_{3(n-1)})) = (c(u_{1n}), c(u_{1(n-1)}))$, then we assign two new colors, one to u_{11}, and one other to u_{31}. These coloring is an NL-coloring for P_{1_n} and P_{3_n} with at most $m+2$ colors.

(ii) Now we discuss on $\chi_{NL}(P_3[C_n])$. Let $C_{i_n} = u_{i1}u_{i2}\cdots u_{in}$ be the i-th row of $P_3[C_n]$. Let $\chi_{NL}(C_n) \in \{k, k+1\}$. Then $\ell(k-1) < n \leq \ell(k)$, and $\ell(k+t-1) < 2n \leq \ell(k+t)$. Hence $\chi_{NL}(C_{2n}) = k+t = m$ for $t \geq 0$. We partition the cycle C_{2n} with the set of vertices $\{v_1, v_2, \cdots, v_{n-1}, v_n, v_{n+1}, v_{n+2}, \cdots, v_{2n-1}, v_{2n}\}$ with NL-colors, to two cycles $C_{1_n} = u_{11}u_{12}\cdots u_{1n}$ and $C_{3_n} = u_{31}u_{32}\cdots u_{3n}$ where $u_{1j} = v_j$ and $u_{3j} = v_{n+j}$. From the NL-coloring of C_{2n}, we bring up a few situations.

1. If $c(u_{1n}) = c(u_{3n})$, then C_{1_n} has no same color vertices with same color neighbors. In this situation $(c(u_{11}), c(u_{12})) \neq (c(u_{31}), c(u_{32}))$, and it observes that C_{3_n} also has no same color vertices with same color neighbors. Therefore this NL-coloring of C_{2n} can be used for C_{1_n} and C_{3_n}.
2. Let $c(u_{1n}) \neq c(u_{3n})$. If $(c(u_{11}), c(u_{12})) = (c(u_{31}), c(u_{32}))$, then C_{1_n} has no same color vertices with same color neighbors. But maybe, C_{3_n} has same color vertices with same color neighbors, for this problem, we assign a new color to u_{31}.
3. Let $c(u_{1n}) \neq c(u_{3n})$ and $(c(u_{11}), c(u_{12})) \neq (c(u_{31}), c(u_{32}))$, to avoid finding vertices with the same color that have same color neighbors, we use two new colors for u_{11} and u_{31}. These coloring is an NL-coloring for C_{1_n} and C_{3_n} with at most $m+2$ colors.

□

Here we give some examples $P_3[P_n]$ and $P_3[C_n]$ with $\chi_{NL}(P_3[P_n]) = 2\chi_{NL}(P_n) + 1$, $\chi_{NL}(P_3[P_n]) = 2\chi_{NL}(P_n) + 2$, $\chi_{NL}(P_3[C_n]) = 2\chi_{NL}(C_n)$ and $\chi_{NL}(P_3[C_n]) = 2\chi_{NL}(C_n) + 1$.

Example 1.

• For $n = 4$, $\chi_{NL}(P_3[P_4]) = 6 = 2\chi_{NL}(P_4)$ with $c(u_{11}, u_{12}, u_{13}, u_{14}) = (3, 1, 2, 1)$, $c(u_{21}, u_{22}, u_{23}, u_{24}) = (4, 5, 6, 5)$ and $c(u_{31}, u_{32}, u_{33}, u_{34}) = (3, 2, 3, 1)$.

• For $n = 4$, $\chi_{NL}(P_3[C_4]) = 8 = 2\chi_{NL}(C_4)$ with $c(u_{11},\ u_{12},\ u_{13},\ u_{14}) = (1, 2, 3, 4)$, $c(u_{21},\ u_{22},\ u_{23},\ u_{24}) = (5, 6, 7, 8)$ and $c(u_{31}, u_{32}, u_{33}, u_{35}) = (1, 3, 2, 4)$.

• For $n = 5$, $\chi_{NL}(P_3[P_5]) = 7 = 2\chi_{NL}(P_5) + 1$ with $c(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}) = (1, 2, 3, 1, 2)$, $c(u_{21}, u_{22}, u_{23}, u_{24}, u_{25}) = (4, 5, 6, 4, 5)$ and $c(u_{31}, u_{32}, u_{33}, u_{34}, u_{35}) = (7, 2, 3, 7, 2)$.

• For $n = 5$, $\chi_{NL}(P_3[C_5]) = 7 = 2\chi_{NL}(C_5) + 1$ with $c(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}) = (1, 2, 1, 2, 3)$, $c(u_{21}, u_{22}, u_{23}, u_{24}, u_{25}) = (4, 5, 4, 5, 6)$ and $c(u_{31}, u_{32}, u_{33}, u_{34}, u_{35}) = (7, 2, 7, 1, 2)$.

• For $n = 6$, $\chi_{NL}(P_3[P_6]) = 7 = 2\chi_{NL}(P_6) + 1$ with $c(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}) = (1, 2, 3, 2, 3, 1)$, $c(u_{21}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}) = (4, 5, 6, 5, 6, 4)$ and $c(u_{31}, u_{32}, u_{33}, u_{34}, u_{35}, u_{36}) = (2, 7, 2, 3, 7, 3)$.

• For $n = 6$, $\chi_{NL}(P_3[C_6]) = 9 = 2\ \chi_{NL}(C_6) + 1$ with $c(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}, u_{16}) = (1, 3, 4, 2, 3, 2)$, $c(u_{21}, u_{22}, u_{23}, u_{24}, u_{25}, u_{26}) = (5, 6, 7, 8, 6, 8)$ and $c(u_{31}, u_{32}, u_{33}, u_{34}, u_{35}, u_{36}) = (1, 9, 2, 3, 9, 3)$.

In the same way we can obtain $\chi_{NL}(P_3[P_7]) = \chi_{NL}(P_3[P_8]) = \chi_{NL}(P_3[P_9]) = 7$, $\chi_{NL}(P_3[C_7]) = \chi_{NL}(P_3[C_9]) = 7$ and $\chi_{NL}(P_3[C_8]) = 9$.

In the next example, it is displayed a $\chi_{NL}(P_3[C_{24}]) = 10 = 2\chi_{NL}(C_{24}) + 2$.

Example 2. $\chi_{NL}(P_3[P_{49}]) = 12 = 2\chi_{NL}(P_{49}) + 2$ and $\chi_{NL}(P_3[C_{49}]) = 13 = 2\chi_{NL}(C_{49}) + 1$. Since $\ell(4) < 49 \leq \ell(5)$ and $49 = \ell(5) - 1$, $\chi_{NL}(P_{49}) = 5$ and $\chi_{NL}(P_{98}) = 7$ and $\chi_{NL}(C_{49}) = 6$ and $\chi_{NL}(C_{98}) = 7$, the vertices of $P_{1_{49}}$, $P_{3_{49}}$, $C_{1_{49}}$ and $C_{3_{49}}$ cannot be NL-colored with 6 colors, otherwise $\chi_{NL}(P_{98})$, $\chi_{NL}(C_{98}) \leq 6$ which is impossible. The following is an χ_{NL}-coloring of $P_3[P_{49}]$ with 12 colors and $P_3[C_{49}]$ with 13 colors.

$P_{2_{49}}$ is NL colored with colors $8, 9, 10, 11, 12$ as usual.
The NL-colors of the vertices of $P_{1_{49}}$ is
$7, 1, 5,\ 6, 1, 6,\ 1, 5, 4,\ 1, 3, 5,\ 1, 3, 2,\ 6, 7, 2,\ 6, 3, 2,\ 4, 6, 4,\ 6, 2, 4,\ 5, 2, 5,\ 2, 6, 5,\ 6, 5, 2,\ 4, 7, 2,\ 4, 3, 6,\ 3, 6, 4,\ 3, 5, 3,\ 7$ for the vertices $u_{11}, \cdots, u_{1(49)}$ respectively.

The NL-colors of the vertices of $P_{3_{49}}$ is
$7, 5, 6,\ 3, 5, 4,\ 5, 4, 6,\ 5, 4, 3,\ 4, 3, 2,\ 5, 3, 2,\ 7, 3, 2,\ 1, 5, 2,\ 1, 2, 1,\ 4, 2, 1,\ 6, 2, 1,\ 7, 3, 1,\ 3, 6, 1,\ 3, 4, 1,\ 4, 1, 6,\ 4, 1, 7,\ 5$ for the vertices $u_{31}, \cdots, u_{4(49)}$ respectively.

$C_{2_{49}}$ is NL colored with colors $8, 9, 10, 11, 12, 13$ as usual. By changing the color u_{11} from 7 to 4 in χ_{NL}-coloring of $P_3[P_{49}]$, we attain a χ_{NL}-coloring of $P_3[C_{49}]$ with 13 colors.

If we use the Remark 2, and Theorem 10, for $C_4[P_n]$ and $C_4[C_n]$ we can adopt.

Theorem 11. *Let $n \geq 4$. If $\ell(k-1) < n \leq \ell(k)$ and $\ell(k+t-1) < 2n \leq \ell(k+t)$, then*

(i) $\chi_{NL}(C_4[P_n]) \leq 2\chi_{NL}(P_n) + 2t + 4$.
(ii) $\chi_{NL}(C_4[C_n]) \leq 2\chi_{NL}(C_n) + 2t + 4$.

Proof. Let Let P_{i_n} and C_{i_n} respectively be the i-th row of $C_4[P_n]$ and $C_4[C_n]$ for $1 \leq i \leq 4$.

(i) By definition all vertices P_{2_n} and P_{4_n} are adjacent to all vertices of P_{1_n} and P_{3_n} and vice versa. Therefore, if two vertices u_{1j} and u_{3t} respectively in P_{1_n} and P_{3_n} have same colors under any NL-coloring, then $\{c(u_{1(j-1)}), c(u_{1(j+1)})\} \neq \{c(u_{3(t-1)}), c(u_{1(t+1)})\}$ $(mod\, n)$, and in the same way, if two vertices u_{2j} and u_{4t} respectively in P_{2_n} and P_{4_n} have same colors under any NL-coloring, then $\{c(u_{2(j-1)}), c(u_{2(j+1)})\} \neq \{c(u_{4(t-1)}), c(u_{4(t+1)})\}$ $(mod\, n)$. Now using the Remark 1 we need at most $\chi_{NL}(P_n) + t + 2$ colors for NL-coloring of P_{1_n} and P_{3_n} and we need at most $\chi_{NL}(P_n) + t + 2$ colors for NL-coloring of P_{2_n} and P_{4_n}.

(ii) There is a same way for part 2 and we left the proof.

□

From the Examples 1, 2, maybe the Theorems 10 and 11 regulated as.

Problem 5. *For path P_n and cycle C_n, we have.*

1. $\chi_{NL}(P_3[P_n]) \leq 2\chi_{NL}(P_n) + t + 1$.
2. $\chi_{NL}(P_3[C_n]) \leq 2\chi_{NL}(C_n) + t + 1$.
3. $\chi_{NL}(C_4[P_n]) \leq 2\chi_{NL}(P_n) + 2t + 2$.
4. $\chi_{NL}(C_4[C_n]) \leq 2\chi_{NL}(C_n) + 2t + 2$.

5. Corona Product of Graphs

In this section we investigate neighbor locating coloring of the corona product of two graphs in terms of neighbor locating coloring of each of them. We start with a general result.

Theorem 12. *Let G and H be two graphs. Then $\chi_{NL}(G \circ H) \leq |V(G)| + \chi_{NL}(H)$. This bound is sharp.*

Proof. For this, if we assign distinct colors to the vertices of G and from definition of corona of two graphs the color of the i-th vertex of G must be different to the colors of the vertices of i-th copy of H. On the other hand, if any two vertices in a copy of H have same color, then it is clear, they have different color neighbors in this copy. If two vertices of two copies of H have same color, then one of them is in i-th copy and the other is in j-th copy where the color of i-th vertex in G and the color of j-th vertex in G are different and then two vertices in two copies of H with same color find different color neighbors. This bound is sharp for $K_2 \circ K_n$, see Theorem 13. □

Theorem 13. *Let $n \geq 3$ and $m \geq 1$ be integers. Then* $\chi_{NL}(K_n \circ K_m) = \begin{cases} n & m \leq n-2 \\ m+2 & m \geq n-1 \end{cases}$.

Proof. Suppose that $m \leq n-2$. Since $\chi_{NL}(K_n) = n$, let $\pi = \{S_1, S_2, \cdots, S_n\}$ be the set of colors assigned to K_n. Let the i-th vertex of K_n is assigned by color i and K_m^i be the i-th copy of K_m. Then we consider $\pi(K_m^i) = \{S_{i+1}, \cdots, S_{i+m}\}$ $(mod\, n)$ for $1 \leq i \leq n$. We assert that these assignments give us an NL-coloring of $K_n \circ K_m$. For this, every vertex of K_m^i has at most m color neighbors and any vertex in K_n has $n-1 \geq m+1$ color neighbors.

As well we say $\pi_i = \pi(K_m^i) = \{S_{i+1}, S_{i+2}, \cdots, S_{i+m}\}$ $(mod\, n)$. Then π_i can be have common colors at most with $\pi_{i+1}, \pi_{i+2}, \cdots, \pi_{i+m}, \pi_{i-1}, \pi_{i-2}, \cdots, \pi_{i-m}$ and π_i does not have common color with π_j for $j \geq i+m+1$ or $j \leq i-m-1$ if any. On the other hand, for any $1 \leq k \leq m$, every vertex of K_m^{i+k} does not have the color $i+k-1$ $(mod\, n)$ as a color neighbor but any vertex of K_m^i has this property, also for any $1 \leq k \leq m$, every vertex of K_m^{i-k} does not have the color $i+m-k+1$ $(mod\, n)$ as a color neighbor but any vertex of K_m^i has this property.

Therefore, π is a minimum-NL-coloring of $K_n \circ K_m$ for $m \leq n-2$, and then $\chi_{NL}(K_n \circ K_m) = n$.

Suppose that $m \geq n-1$, then $\omega(K_n \circ K_m) = m+1$. Same as part 1, we need n distinct color for K_n and since $m \geq n-1$ at least $m+1$ distinct color must be used for the i-th vertex of K_n and the vertices of K_m^i. Therefore, we need at least $m+2$ colors for NL-coloring of $K_n \circ K_m$. Now we give an NL coloring of $K_n \circ K_m$ with $m+2$ colors. For this, we bring up three situations.

1. Suppose that $m = n-1$ and $\pi = \{S_1, S_2, \cdots, S_n, S_{n+1}\}$. Then we assign the colors as follows.
   $\pi(K_n) = \{S_1, S_2, \cdots, S_n\}$, where i-th color has been used for i-th vertex of K_n. Then

   $$\pi(K_m^i) = \{S_1, S_2, \cdots, S_{i-2}, S_{i+1}, \cdots, S_n, S_{n+1}\} \textit{ for } 1 \leq i \leq n.$$

   We show that π is a minimum NL-coloring of G with $m+2 = n+1$ colors. We straightforward understand, any vertex of K_m^i does not have color $i-1$ in its color neighbors and any vertex of K_n with j-th color $(j \neq i-1)$ has $i-1$ as a color neighbor. On the other hand, every vertex of K_m^i has color $j-1$ in the color neighbor and every vertex of K_m^j has color $i-1$ in the color neighbor for $i \neq j$. Therefore the mentioned coloring is a minimum NL-coloring.

2. Suppose that $m = n$ and $\pi = \{S_1, S_2, \cdots, S_n, S_{n+1}, S_{n+2}\}$. We show that π is a minimum NL-coloring for $K_n \circ K_m$ with $m+2 = n+2$ colors. Then we consider the assigned colors as follows.
   $\pi(K_n) = \{S_1, S_2, \cdots, S_n\}$, where i-th color has been used for i-th vertex of K_n, and

   $$\pi(K_m^i) = \{S_1, S_2, \cdots, S_{i-2}, S_{i+1}, \cdots, S_{n+1}, S_{n+2}\} \textit{ for } 1 \leq i \leq n.$$

   It is straightforward to understand, any vertex of K_m^i $(1 \leq i \leq n)$ has color $n+1$ or $n+2$ in its color neighbors and any vertex of K_n does not have one as a color neighbor. On the other hand, every vertex of K_m^i has color $j-1$ in the color neighbor and every vertex of K_m^j has color $i-1$ in the color neighbor for $i \neq j$. Therefore the mentioned coloring is a minimum NL-coloring.

3. Suppose that $m \geq n+1$ and $m = n+r$ and $\pi = \{S_1, S_2, S_3 \cdots, S_{n+r+1}, S_{n+r+2}\}$. Then, from part 2, consider $\pi(K_n) = \{S_1, S_2, \cdots, S_n\}$, and

   $$\pi(K_m^i) = \{S_1, S_2, \cdots, S_{i-2}, S_{i+1}, \cdots, S_{m+1}, S_{m+2}\} \textit{ for } 1 \leq i \leq n.$$

   Now similar proof of part 2, shows that π is a minimum NL-coloring of $K_n \circ K_m$ for $m \geq n+1$.

□

Theorem 14. *Let $G = P_m \circ P_n$ be a graph. Then,*

$$\min\{k+m : \frac{(k+m)\left(\binom{k+m}{3} + \binom{k+m}{k+2}\right)}{m+mn} \geq 1\} \leq \chi_{NL}(P_m \circ P_n) \leq m + \chi_{NL}(P_n).$$

The upper bounds is sharp.

Proof. For upper bound, use Theorem 12 and for the sharpness, consider $P_3 \circ P_n$ for $n \geq 1$.

Lower bound; in graph $P_m \circ P_n$, the vertices of a copy of P_n have color-degree 2 or 3 and the vertices of P_m have color-degree $k+2$ or $k+3$, where $k = \chi_{NL}(P_n)$. On the other hand $\chi_{NL}(P_m \circ P_n) \leq m+k$. From Corollary 1, we have

$$m + mn \leq (k+m)\left(\binom{k+m-1}{2} + \binom{k+m-1}{3} + \binom{k+m-1}{k+1} + \binom{k+m-1}{k+2}\right) =$$

$$(k+m)\left(\binom{k+m}{3} + \binom{k+m}{k+2}\right).$$

Now using Corollary 2 the lower bound is observed. □

In the following, for any integer $k \geq 3$, we construct a graph G in which $\chi_{NL}(G \circ P_2) = k$.

Proposition 7. *For every $k \geq 3$, $\chi_{NL}(P_{\frac{k(k-1)(k-2)}{6}} \circ P_2) = k$.*

Proof. For any positive integer m, there are m K_3 in $P_m \circ P_2$ in which $2m$ vertices have exactly color-degree 2. Therefore, from Corollary 1

$$n(P_m \circ P_2) = m + 2m \leq l\binom{l-1}{2} + m \leq \frac{3}{2}\binom{l-1}{2}.$$

Since $\frac{l(l-1)(l-2)}{6} \leq l\binom{l-1}{2}$, from Corollary 2, $\chi_{NL}(P_{\frac{k(k-1)(k-2)}{6}}) \geq k$. Now we give an NL-coloring for $P_{\frac{k(k-1)(k-2)}{6}} \circ P_2$ with k-colors. It is well known that, $6 \mid k(k-1)(k-2)$. Thus there exist two situations as follows.

1. Let $6 \mid (k-1)(k-2)$ and $t = \frac{(k-1)(k-2)}{6}$. Then we have a path with tk vertices and tk paths P_2 which each of them is adjacent to a vertex of P_{tk} and the vertices of P_{tk} with $V(P_{tk}) = \{v_{jk+i} : 1 \leq i \leq k \; and \; 0 \leq j \leq t-1\}$ and the vertices of $V(mP_2) = \{u_{jk+i}, w_{jk+i} : 1 \leq i \leq k \; and \; 0 \leq j \leq t-1\}$.
   For coloring of $P_{tk} \circ P_2$, we assign color i to the vertex v_{jk+i} for $1 \leq i \leq k$ and $0 \leq j \leq t-1$ of P_{tk}, and assign color $i+1$, $j+i+2$ to the vertices u_{jk+i}, w_{jk+i} for $1 \leq i \leq k$ and $1 \leq j \leq t-2$ and assign color $i+2, i+3$ to the vertices $u_{(t-1)k+i}, w_{(t-1)k+i}$ $(mod\ k)$ respectively. This coloring is an NL-coloring with k colors. Therefore, in this position $\chi_{NL}(P_{\frac{k(k-1)(k-2)}{6}} \circ P_2) = k$

2. Let $6 \nmid (k-1)(k-2)$ and $\frac{k(k-1)(k-2)}{6} = sk+r$ where $1 \leq r \leq k-1$. Then we have a path with $sk+r$ vertices and $sk+r$ paths P_2 which each of them is adjacent to a vertex of P_{sk+r} and the vertices of P_{sk+r} with $V(P_{sk+r}) = \{v_{jk+i} : 1 \leq i \leq k \; and \; 0 \leq j \leq s-1\} \cup \{v_{sk+i} : 1 \leq i \leq r\}$ and the vertices of $V(mP_2) = \{u_{jk+i}, w_{jk+i} : 1 \leq i \leq k \; and \; 0 \leq j \leq s-1\} \cup \{u_{sk+i}, w_{sk+i} : 1 \leq i \leq r\}$.

For coloring of $P_{sk+r} \circ P_2$, we assign color i to the vertex v_{jk+i} for $1 \leq i \leq k$ and $0 \leq j \leq s$ of P_{sk+r} and assign color $i+1, j+i+2$ to the vertices u_{jk+i}, w_{jk+i} for $1 \leq i \leq k$ and $1 \leq j \leq s-1$ and assign color $i+2, i+3$ to the vertices u_{sk+i}, w_{sk+i} $(1 \leq i \leq r)$, $(mod\ k)$ respectively. This coloring is an NL-coloring with k colors. Therefore, in this position $\chi_{NL}(P_{\frac{k(k-1)(k-2)}{6}} \circ P_2) = k$. Consequently, the result holds. □

One of the relation in NL coloring of corona product, attaining $\chi_{NL}(K_m \circ P_n)$. For this, at first we state a proposition.

Proposition 8.

$$\chi_{NL}(K_m \circ P_n) = \begin{cases} m+2 & m = n = 3 \\ n+1 & 4 \leq m = n \leq 5, \text{ or } m = 4 \text{ and } n \leq 3, \text{ or } m = 3 \text{ and } n \leq 2. \end{cases}$$

Proof. 1. If $m = n = 3$, then it is clear $\chi_{NL}(K_3 \circ P_3) \geq 4$. Assume contradictorily, $\chi_{NL}(K_3 \circ P_3) = 4$ and vertices of K_3 are assigned with colors $1, 2, 3$. Then the path P_3 adjacent to the vertex with color 1, are $2, 3, 4$ and two other P_3, take colors $1, 2, 4$ and $1, 3, 4$ respectively. With this coloring, either two vertices with color 4 attain a same color neighbor or two vertices with one of colors $1, 2$ or 3 attain a same color neighbor, that is a contradiction. Therefore $\chi_{NL}(K_3 \circ P_3) \geq 5$. Now we assign colors $1, 4, 5$; $2, 4, 5$ and $3, 4, 5$ to the vertices P_3 adjacent to the vertices with color 2, 3 and 1 respectively. The former coloring is an NL-coloring of $K_3 \circ P_3$.

2. Let $m = n = 4$. Then $\chi_{NL}(K_4 \circ P_4) \geq 5$. Now we give an NL coloring with 5 colors. Let v_i $(1 \leq i \leq 4)$ be the vertices of K_4 and u_{ji}, $(1 \leq j \leq 4$ be the vertices of P_{4_i}, the path P_4 adjacent to v_i. The assignment color i to v_i; color $i - 1$ $(mod\ 4)$ to u_{1i}; color $i + 1$ $(mod\ 4)$ to u_{2i}; color 5 to u_{3i}; and color $i + 2$ $(mod\ 4)$ to u_{4i} give an NL-coloring for $1 \leq i \leq 4$. Therefore $\chi_{NL}(K_4 \circ P_4) = 5$.

Let $m = n = 5$. Then $\chi_{NL}(K_5 \circ P_5) \geq 6$. We give an NL coloring with 6 colors. Let v_i $(1 \leq i \leq 5)$ be the vertices of K_5 and u_{ji}, $(1 \leq j \leq 5)$ be the vertices of P_{5_i}, the path P_5 adjacent to v_i. The assignment color i to v_i; color $i - 1$ $(mod\ 5)$ to u_{1i}; color $i + 1$ $(mod\ 5)$ to u_{2i}; color 6 to u_{3i}; color $i + 2$ $(mod\ 5)$ to u_{4i}; and color $i + 3$ $(mod\ 5)$ to u_{5i} give an NL-coloring for $1 \leq i \leq 4$. Therefore $\chi_{NL}(K_5 \circ P_5) = 6$.

For $m = 4$ and $n \leq 3$; or $m = 3$ and $n \leq 2$, there exist similar reason, and we left the proof. □

Theorem 15. *Let m and n be positive integers, $\ell(k-1) + 1 \leq n \leq \ell(k)$, and $n = tk + r$ $(0 \leq r \leq k - 1)$.*

$$\text{If } m > k \text{ and } \binom{m-1}{3} \geq 3(r\lceil \frac{n}{k} \rceil + (k - r)\lfloor \frac{n}{k} \rfloor), \text{ then } \chi_{NL}(K_m \circ P_n) = m$$

Proof. Since $\ell(k-1)+1 \leq n \leq \ell(k)$, $\chi_{NL}(P_n) = k$. Let the vertex v_i of K_m be assigned by color i for $1 \leq i \leq m$. Let P_{n_i} be the path P_n adjacent to v_i, and $V(P_{n_i}) = \{u_{ji} : 1 \leq j \leq n\}$. From the data, we assign colors $i + 1, i + 2, \cdots, i + k$ $(mod\ n)$ to the vertices of $V(P_{n_i})$ $(1 \leq i \leq m)$. There has been used mn colors for all P_{ni}s so that every color is appeared in k paths P_{ni}s, and is iterated n times. In addition to that, each vertex is iterated $\lceil \frac{n}{k} \rceil$ times in r paths P_{ni}s and is iterated $\lfloor \frac{n}{k} \rfloor$ times in $k - r$ paths P_{ni}s. On the other hand, each vertex has color degree at most 3. Hence, each vertex with color i has at most $(r\lceil \frac{n}{k} \rceil + (k - r)\lfloor \frac{n}{k} \rfloor)$ clusters of three colors. In the other words, there exist $(r\lceil \frac{n}{k} \rceil + (k - r)\lfloor \frac{n}{k} \rfloor)$ clusters of four colors in which one of colors is i. Also, if we consider the m colors of K_m, for each vertex with color i, there exist $\binom{m-1}{3}$ clusters of three colors so that i forms clusters of four colors that one of colors is i. Since by the data, $\binom{m-1}{3} \geq 3(r\lceil \frac{n}{k} \rceil + (k - r)\lfloor \frac{n}{k} \rfloor)$, there does not exist two same color vertices with same color neighbors. Therefore $\chi_{NL}(K_m \circ P_n) = m$. □

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