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On the Various Energy Forms of the Join of Complete Graphs

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Abstract: The concept of graph energy, first introduced in 1978, has been a focal point of extensive research within the field of graph theory, leading to the publication of numerous articles. Graph energy, originally associated with the eigenvalues of the adjacency matrix of a graph, has since been extended to various other matrices. These include the maximum degree matrix, Randić matrix, sum-connectivity matrix, and the first and second Zagreb matrices, among others. In this paper, we focus on calculating the energy of several such matrices for the join graph of complete graphs, denoted as $J_m(K_n)$. Specifically, we compute the energies for the maximum degree matrix, Randić matrix, sum-connectivity matrix, first Zagreb matrix, second Zagreb matrix, reverse first Zagreb matrix, and reverse second Zagreb matrix for $J_m(K_n)$. Our results provide new insights into the structural properties of the join graph and contribute to the broader understanding of the mathematical characteristics of graph energy for different matrix representations. This work extends the scope of graph energy research by considering these alternative matrix forms, offering a deeper exploration into the algebraic and spectral properties of graph energy in the context of join graphs.

Keywords: Graph energy, Join graph, Complete graph, Maximum degree matrix, Randić matrix, Sum-connectivity matrix, Zagreb matrix, Spectral graph theory, Matrix energy, Graph theory

1. Introduction

Discrete mathematics, a branch of mathematics dealing with discrete objects rather than continuous, encompasses various sub-fields, among which graph theory has garnered significant attention. Within this domain, the concept of graph energy has emerged as a pivotal area of study. Graph theory provides an intuitive approach to problem-solving by abstracting real-world systems into graphical models. This abstraction enables researchers to apply combinatorial and algebraic techniques to address complex issues across numerous disciplines. The growing prominence of graph energy reflects its utility in providing insights and solutions to problems that span beyond traditional mathematical boundaries.

Graph theory's versatility is evident in its diverse applications, which range from natural sciences to engineering. The concept of graph energy has been leveraged in various scientific and technological contexts, demonstrating its far-reaching impact. For instance, graph energies have found applications in fields such as air travel optimization and spacecraft construction [1], where they aid in the analysis and design of efficient systems. Additionally, graph energy principles

have been utilized in facial recognition technologies and satellite communication systems [2], highlighting their relevance in modern technological advancements. A comprehensive overview of the applications and significance of graph energies is presented in [3], which underscores the broad scope of research in this area.

The foundational concept of graph energy was introduced by Ivan Gutman in 1978 [4], inspired by quantum chemistry. This idea drew from the work of E. Huckel, who, in 1930, applied graph theory to chemical structures through his molecular orbital theory for Π -electron networks in conjugated hydrocarbons. Huckel's approach led to the development of characteristic polynomials and the subsequent determination of eigenvalues [5]. The application of matrix analysis to graph theory, as explored by Horn [6] and Gantmakher [7], has further enriched the study of graph energy. Bapat's work [8] demonstrated that the energy of a graph cannot be an odd integer, adding a significant constraint to the theoretical understanding of graph energy.

Further research has expanded the knowledge of graph energies in various contexts. For example, [9] explored the energies of specific classes of non-regular graphs, while [10] established that the energy of a graph cannot be the square root of an odd integer. Nikiforov's research [11] provided bounds for the energies of different graph structures, contributing to the broader understanding of energy distributions across graph types. Meenakshi [12] compiled a comprehensive survey of energies for regular, non-regular, circulant, and random graphs, providing valuable insights into the diverse behaviors of graph energies.

Recent studies have furthered the analysis of various energy measures. For instance, [13] defined the maximum degree energy of a graph and provided results for its bounds, while [14] detailed numerous applications of graph theory within Computer Science and Engineering. The derivation of bounds for graph energy $E(G)$ is explored in [15], and [16] provided upper and lower bounds for Zagreb energy. Jaferi's work [17] established a relationship between Zagreb energy and Zagreb Estrada index, while [18] analyzed the eigenvalues of the sum-connectivity matrix. The definition and bounds for Randić energy were examined in [19], and the study of Sombor matrix and energy was expanded by [20] and [21]. Research into the average degree eigenvalues and energies of graph families [22], reverse Laplacian energy [23], and reverse maximum degree energy [24] has further contributed to the comprehensive understanding of graph energy measures.

This paper aims to build upon this extensive body of work by computing various energy measures for the join graph of complete graphs. By examining the maximum degree matrix, Randić matrix, sum-connectivity matrix, and several Zagreb matrices, this study seeks to advance the theoretical understanding of graph energy and its applications. The results presented herein will contribute to the broader field of graph theory and its numerous applications across scientific and technological domains.

2. Methodology

In this study, we investigate the energy measures of several matrices associated with the join graph of complete graphs, denoted as $J_m(K_n)$. The various matrices under consideration include the maximum degree matrix, Randić matrix, sum-connectivity matrix, first Zagreb matrix, second Zagreb matrix, reverse first Zagreb matrix, and reverse second Zagreb matrix. This section outlines the methodology for computing these energy measures.

2.1. Join Graph of Complete Graphs

The join graph $J_m(K_n)$ is constructed by taking m disjoint copies of the complete graph K_n and joining each vertex in one copy with every vertex in the other copies. This results in a graph where each pair of vertices from different copies is connected by an edge, while vertices

within the same copy form a complete subgraph. Formally, if K_n has n vertices, then $J_m(K_n)$ will have mn vertices and $\frac{m(m-1)}{2}n^2 + m\frac{n(n-1)}{2}$ edges.

2.2. Maximum Degree Matrix

The maximum degree matrix D_{\max} is a diagonal matrix where each diagonal entry corresponds to the maximum degree of the vertices in the graph. For $J_m(K_n)$, the maximum degree of each vertex is $(m-1)n + (n-1)$. Hence, each diagonal entry in D_{\max} is $(m-1)n + (n-1)$, and all off-diagonal entries are zero.

2.3. Randić Matrix

The Randić matrix R is defined as a matrix where each entry r_{ij} is given by $r_{ij} = \frac{1}{\sqrt{d_i d_j}}$ if vertices i and j are adjacent, and zero otherwise. Here, d_i and d_j represent the degrees of vertices i and j respectively. For $J_m(K_n)$, the degrees are uniform across the graph, simplifying the calculation of the Randić matrix.

2.4. Sum-Connectivity Matrix

The sum-connectivity matrix S is defined such that each entry s_{ij} equals $\frac{2}{d_i + d_j}$ if vertices i and j are adjacent, and zero otherwise. The calculation of this matrix involves determining the degree of each vertex, which, as in the Randić matrix, are uniform for the join graph $J_m(K_n)$.

2.5. Zagreb Matrices

The first Zagreb matrix Z_1 is a diagonal matrix where each diagonal entry is the sum of the squares of the degrees of the corresponding vertex. The second Zagreb matrix Z_2 is defined such that each entry z_{ij} is $\frac{1}{d_i d_j}$ if vertices i and j are adjacent, and zero otherwise. The reverse first Zagreb matrix Z_{1r} and reverse second Zagreb matrix Z_{2r} are computed similarly but involve reciprocal degree calculations in reverse order.

2.6. Energy Calculations

The energy of a graph G associated with a matrix M is computed using the eigenvalues of M . Specifically, the energy $E(M)$ is given by the sum of the absolute values of the eigenvalues of M . For each matrix considered in this study, we calculate its eigenvalues and then sum their absolute values to obtain the energy.

2.7. Computational Approach

The computations for matrix construction and energy calculations are performed using Python and MATLAB. We utilize numerical libraries and matrix computation functions to handle the large matrices involved and ensure precision in the results. For each type of matrix and corresponding energy measure, we verify the results through multiple iterations and consistency checks.

This methodology provides a comprehensive approach to analyzing various energy measures for the join graph of complete graphs, contributing valuable insights into its spectral properties and overall behavior.

3. Preliminaries

Graph theory, a key area within discrete mathematics, provides a robust framework for analyzing and understanding complex networks. In this study, we consider a simple, undirected,

and finite graph G , characterized by its vertex set $V(G) = \{v_1, v_2, v_3, \dots, v_p\}$ and edge set $E(G) = \{e_1, e_2, e_3, \dots, e_q\}$. Each vertex v in graph G is associated with a degree d_v , which represents the number of edges incident to v [25]. The maximum degree of G , denoted as $\Delta(G)$, is the highest degree among all vertices in G . The reverse vertex degree c_{v_i} for a vertex v_i is defined by the expression [24]

$$c_{v_i} = \Delta(G) - d_{v_i} + 1,$$

where d_{v_i} is the degree of vertex v_i . This concept helps in understanding the structural and spectral properties of the graph.

The energy of a graph, introduced by Ivan Gutman in 1978 [4], is a crucial measure in spectral graph theory. The graph energy $E(G)$ is defined as the sum of the absolute values of the eigenvalues of the graph's adjacency matrix. Mathematically, if $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$ are the eigenvalues of G , then the energy $E(G)$ is given by

$$E(G) = \sum_{i=1}^k |\lambda_i|.$$

The spectrum of a finite simple graph G consists of the eigenvalues λ_j of its adjacency matrix $A(G)$, including their multiplicities. This spectrum is typically denoted as $spec(G)$ [26].

The maximum degree matrix $M_e(G) = [M_{ij}]$ [13] of a graph G is a square matrix where each entry M_{ij} is defined as

$$[M_{ij}] = \begin{cases} \max(d_{v_i}, d_{v_j}), & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

Adiga introduced the concept of maximum degree energy $M(G)$ for a simple connected graph G , which is the sum of the absolute values of the eigenvalues of the maximum degree matrix $M_e(G)$.

In [17], Gutman defined the First Zagreb energy ZE_1 and Second Zagreb energy ZE_2 based on the first and second Zagreb topological indices. The first Zagreb matrix $Z^{(1)}(G) = [z_{ij}^{(1)}]$ is given by

$$z_{ij}^{(1)} = \begin{cases} d_{v_i} + d_{v_j}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The first Zagreb energy $ZE_1(G)$ is the sum of the absolute values of the eigenvalues of $Z^{(1)}(G)$. Similarly, the second Zagreb matrix $Z^{(2)}(G) = [z_{ij}^{(2)}]$ is defined by

$$z_{ij}^{(2)} = \begin{cases} d_{v_i} \cdot d_{v_j}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The second Zagreb energy $ZE_2(G)$ is the sum of the absolute values of the eigenvalues of $Z^{(2)}(G)$.

The Randić matrix $R(G) = [r_{ij}]$ of a graph G is defined as

$$[r_{ij}] = \begin{cases} \frac{1}{\sqrt{d_{v_i} \cdot d_{v_j}}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The Randić energy $E_R(G)$ is the sum of the absolute values of the eigenvalues of the Randić matrix $R(G)$ [19].

The sum-connectivity matrix $SC(G) = [sc_{ij}]$ is defined as

$$[sc_{ij}] = \begin{cases} \frac{1}{\sqrt{d_{v_i} + d_{v_j}}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The sum-connectivity energy $E_{SC}(G)$ is the sum of the absolute values of the eigenvalues of the sum-connectivity matrix $SC(G)$ [18].

Building on the concepts of the first and second Zagreb energies and reverse vertex degree, we define the reverse first Zagreb matrix $Z_{(1)R}$ and reverse second Zagreb matrix $Z_{(2)R}$ as follows:

$$Z_{(1)R} = \begin{cases} c_{v_i} + c_{v_j}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The reverse first Zagreb energy $EZ_{1R}(G)$ is the sum of the absolute values of the eigenvalues of $Z_{(1)R}$. Similarly,

$$Z_{(2)R} = \begin{cases} c_{v_i} \cdot c_{v_j}, & \text{if } v_i \text{ and } v_j \text{ are adjacent,} \\ 0, & \text{otherwise.} \end{cases}$$

The reverse second Zagreb energy $EZ_{2R}(G)$ is the sum of the absolute values of the eigenvalues of $Z_{(2)R}$.

To further illustrate these concepts, consider a complete graph K_n with $n \geq 2$ vertices. By taking another complete graph K_m with $m \geq 2$ vertices and joining every vertex of K_n to each vertex of K_m , we obtain the join graph $J_m(K_n)$. This resulting graph has $n + m$ vertices and is regular with a degree $d = n + m - 2$.

This paper is organized as follows: Section 4 presents the results for various energy measures, including maximum degree energy, Randić energy, sum-connectivity energy, first Zagreb energy, second Zagreb energy, reverse first Zagreb energy, and reverse second Zagreb energy for the join graph of complete graphs. Section 5 discusses applications of these energies, while Section 6 concludes the paper.

4. Results

4.1. Maximum Degree Energy of the Join Graph of Complete Graphs

In this section, we compute the maximum degree energy of the join graph $J_m(K_n)$. The join graph $J_m(K_n)$ is formed by joining every vertex of K_n with each vertex of K_m , resulting in a graph with $n + m$ vertices.

Theorem 1. *For $m \geq 2$ and $n \geq 2$, the maximum degree energy of the join graph $J_m(K_n)$ is given by*

$$E_M(J_m(K_n)) = 2(n + m - 1)^2.$$

Proof. Consider the complete graph K_n with n vertices, each vertex having a degree $n - 1$. In the join graph $J_m(K_n)$, each vertex from K_n is connected to every vertex of K_m , and vice versa. Therefore, each vertex in $J_m(K_n)$ has a degree of $n + m - 1$.

The maximum degree matrix $M(J_m(K_n))$ is defined as

$$M(J_m(K_n)) = \begin{bmatrix} 0 & n + m - 1 & n + m - 1 & \cdots & n + m - 1 \\ n + m - 1 & 0 & n + m - 1 & \cdots & n + m - 1 \\ n + m - 1 & n + m - 1 & 0 & \cdots & n + m - 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n + m - 1 & n + m - 1 & n + m - 1 & \cdots & 0 \end{bmatrix}.$$

To find the maximum degree energy, we need to compute the eigenvalues of $M(J_m(K_n))$. The characteristic polynomial of this matrix is

$$\begin{vmatrix} -\Lambda & n+m-1 & n+m-1 & \cdots & n+m-1 \\ n+m-1 & -\Lambda & n+m-1 & \cdots & n+m-1 \\ n+m-1 & n+m-1 & -\Lambda & \cdots & n+m-1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n+m-1 & n+m-1 & n+m-1 & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of $M(J_m(K_n))$ is found to be

$$\text{Spec}(M(J_m(K_n))) = \begin{pmatrix} -(n+m-1) & (n+m-1)^2 \\ n+m-1 & 1 \end{pmatrix}.$$

Thus, the maximum degree energy $E_M(J_m(K_n))$ is calculated as

$$E_M(J_m(K_n)) = (n+m-1) |-(n+m-1)| + |(n+m-1)^2| = 2(n+m-1)^2.$$

□

4.2. Randić Energy of the Join Graph of Complete Graphs

In this section, we derive the Randić energy of the join graph $J_m(K_n)$.

Theorem 2. For $m \geq 2$ and $n \geq 2$, the Randić energy of the join graph $J_m(K_n)$ is given by

$$E_R(J_m(K_n)) = 2.$$

Proof. Consider the complete graph K_n with n vertices. The Randić matrix $R(J_m(K_n))$ for the join graph $J_m(K_n)$ is defined as

$$R(J_m(K_n)) = \begin{bmatrix} 0 & \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & \cdots & \frac{1}{\sqrt{(n+m-1)^2}} \\ \frac{1}{\sqrt{(n+m-1)^2}} & 0 & \frac{1}{\sqrt{(n+m-1)^2}} & \cdots & \frac{1}{\sqrt{(n+m-1)^2}} \\ \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & 0 & \cdots & \frac{1}{\sqrt{(n+m-1)^2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} -\Lambda & \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & \cdots & \frac{1}{\sqrt{(n+m-1)^2}} \\ \frac{1}{\sqrt{(n+m-1)^2}} & -\Lambda & \frac{1}{\sqrt{(n+m-1)^2}} & \cdots & \frac{1}{\sqrt{(n+m-1)^2}} \\ \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & -\Lambda & \cdots & \frac{1}{\sqrt{(n+m-1)^2}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & \frac{1}{\sqrt{(n+m-1)^2}} & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of the Randić matrix $R(J_m(K_n))$ is

$$\text{Spec}(R(J_m(K_n))) = \begin{pmatrix} \frac{-1}{n+m-1} & 1 \\ n+m-1 & 1 \end{pmatrix}.$$

Therefore, the Randić energy $E_R(J_m(K_n))$ is given by

$$E_R(J_m(K_n)) = (n+m-1) \left| \frac{-1}{n+m-1} \right| + |1| = 2.$$

□

4.3. Sum-Connectivity Energy of the Join Graph of Complete Graphs

In this section, we calculate the sum-connectivity energy of the join graph $J_m(K_n)$. The sum-connectivity energy is derived from the sum-connectivity matrix of the join graph.

Theorem 3. For $m \geq 2$ and $n \geq 2$, the sum-connectivity energy of the join graph $J_m(K_n)$ is given by

$$E_{SC}(J_m(K_n)) = \sqrt{2(n + m - 1)}.$$

Proof. Consider the complete graph K_n with n vertices. In the join graph $J_m(K_n)$, the sum-connectivity matrix $SC(J_m(K_n))$ is defined as

$$SC(J_m(K_n)) = \begin{bmatrix} 0 & \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & \cdots & \frac{1}{\sqrt{2(n+m-1)}} \\ \frac{1}{\sqrt{2(n+m-1)}} & 0 & \frac{1}{\sqrt{2(n+m-1)}} & \cdots & \frac{1}{\sqrt{2(n+m-1)}} \\ \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & 0 & \cdots & \frac{1}{\sqrt{2(n+m-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is given by

$$\begin{vmatrix} -\Lambda & \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & \cdots & \frac{1}{\sqrt{2(n+m-1)}} \\ \frac{1}{\sqrt{2(n+m-1)}} & -\Lambda & \frac{1}{\sqrt{2(n+m-1)}} & \cdots & \frac{1}{\sqrt{2(n+m-1)}} \\ \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & -\Lambda & \cdots & \frac{1}{\sqrt{2(n+m-1)}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & \frac{1}{\sqrt{2(n+m-1)}} & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of the sum-connectivity matrix $SC(J_m(K_n))$ is

$$\text{Spec}(SC(J_m(K_n))) = \left(\begin{array}{cc} \frac{-1}{\sqrt{2(n+m-1)}} & \frac{n+m-1}{\sqrt{2(n+m-1)}} \\ n+m-1 & 1 \end{array} \right).$$

Therefore, the sum-connectivity energy $E_{SC}(J_m(K_n))$ is given by

$$E_{SC}(J_m(K_n)) = (n + m - 1) \left| \frac{-1}{\sqrt{2(n + m - 1)}} \right| + \left| \frac{n + m - 1}{\sqrt{2(n + m - 1)}} \right| = \sqrt{2(n + m - 1)}.$$

□

4.4. First Zagreb Energy of the Join Graph of Complete Graphs

In this section, we determine the first Zagreb energy of the join graph $J_m(K_n)$. The first Zagreb energy is based on the first Zagreb matrix of the join graph.

Theorem 4. For $m \geq 2$ and $n \geq 2$, the first Zagreb energy of the join graph $J_m(K_n)$ is given by

$$ZE_1(J_m(K_n)) = 4(n + m - 1)^2.$$

Proof. Consider the complete graph K_n with n vertices. In the join graph $J_m(K_n)$, the first Zagreb matrix $Z^{(1)}(J_m(K_n))$ is defined as

$$Z^{(1)}(J_m(K_n)) = \begin{bmatrix} 0 & 2(n + m - 1) & 2(n + m - 1) & \cdots & 2(n + m - 1) \\ 2(n + m - 1) & 0 & 2(n + m - 1) & \cdots & 2(n + m - 1) \\ 2(n + m - 1) & 2(n + m - 1) & 0 & \cdots & 2(n + m - 1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n + m - 1) & 2(n + m - 1) & 2(n + m - 1) & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} -\Lambda & 2(n+m-1) & 2(n+m-1) & \cdots & 2(n+m-1) \\ 2(n+m-1) & -\Lambda & 2(n+m-1) & \cdots & 2(n+m-1) \\ 2(n+m-1) & 2(n+m-1) & -\Lambda & \cdots & 2(n+m-1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n+m-1) & 2(n+m-1) & 2(n+m-1) & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of the first Zagreb matrix $Z^{(1)}(J_m(K_n))$ is

$$\text{Spec}(Z^{(1)}(J_m(K_n))) = \begin{pmatrix} -2(n+m-1) & 2(n+m-1)(n+m-1) \\ n+m-1 & 1 \end{pmatrix}.$$

Thus, the first Zagreb energy $ZE_1(J_m(K_n))$ is computed as

$$ZE_1(J_m(K_n)) = (n+m-1) |-2(n+m-1)| + |2(n+m-1)(n+m-1)| = 4(n+m-1)^2.$$

□

4.5. Second Zagreb Energy of the Join Graph of Complete Graphs

In this section, we calculate the second Zagreb energy of the join graph $J_m(K_n)$. The second Zagreb energy is based on the second Zagreb matrix of the join graph.

Theorem 5. For $m \geq 2$ and $n \geq 2$, the second Zagreb energy of the join graph $J_m(K_n)$ is given by

$$ZE_2(J_m(K_n)) = 2(n+m-1)^3.$$

Proof. Consider the complete graph K_n with n vertices. In the join graph $J_m(K_n)$, the second Zagreb matrix $Z^{(2)}(J_m(K_n))$ is defined as

$$Z^{(2)}(J_m(K_n)) = \begin{bmatrix} 0 & (n+m-1)^2 & (n+m-1)^2 & \cdots & (n+m-1)^2 \\ (n+m-1)^2 & 0 & (n+m-1)^2 & \cdots & (n+m-1)^2 \\ (n+m-1)^2 & (n+m-1)^2 & 0 & \cdots & (n+m-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n+m-1)^2 & (n+m-1)^2 & (n+m-1)^2 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} -\Lambda & (n+m-1)^2 & (n+m-1)^2 & \cdots & (n+m-1)^2 \\ (n+m-1)^2 & -\Lambda & (n+m-1)^2 & \cdots & (n+m-1)^2 \\ (n+m-1)^2 & (n+m-1)^2 & -\Lambda & \cdots & (n+m-1)^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (n+m-1)^2 & (n+m-1)^2 & (n+m-1)^2 & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of the second Zagreb matrix $Z^{(2)}(J_m(K_n))$ is

$$\text{Spec}(Z^{(2)}(J_m(K_n))) = \begin{pmatrix} -(n+m-1)^2 & (n+m-1)^3 \\ n+m-1 & 1 \end{pmatrix}.$$

Thus, the second Zagreb energy $ZE_2(J_m(K_n))$ is computed as

$$ZE_2(J_m(K_n)) = (n+m-1) |-(n+m-1)^2| + |(n+m-1)^3| = 2(n+m-1)^3.$$

□

4.6. Reverse First Zagreb Energy of the Join Graph of Complete Graphs

In this section, we determine the reverse first Zagreb energy of the join graph $J_m(K_n)$. The reverse first Zagreb energy is based on the reverse first Zagreb matrix of the join graph.

Theorem 6. For $m \geq 2$ and $n \geq 2$, the reverse first Zagreb energy of the join graph $J_m(K_n)$ is given by

$$EZ_{1R}(J_m(K_n)) = 4(n + m - 1).$$

Proof. Consider the complete graph K_n with n vertices. In the join graph $J_m(K_n)$, the reverse first Zagreb matrix $Z_{1R}(J_m(K_n))$ is defined as

$$Z_{1R}(J_m(K_n)) = \begin{bmatrix} 0 & 2 & 2 & \cdots & 2 \\ 2 & 0 & 2 & \cdots & 2 \\ 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} -\Lambda & 2 & 2 & \cdots & 2 \\ 2 & -\Lambda & 2 & \cdots & 2 \\ 2 & 2 & -\Lambda & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of the reverse first Zagreb matrix $Z_{1R}(J_m(K_n))$ is

$$\text{Spec}(Z_{1R}(J_m(K_n))) = \begin{pmatrix} -2 & 2(n + m - 1) \\ n + m - 1 & 1 \end{pmatrix}.$$

Thus, the reverse first Zagreb energy $EZ_{1R}(J_m(K_n))$ is given by

$$EZ_{1R}(J_m(K_n)) = (n + m - 1) |-2| + |2(n + m - 1)| = 4(n + m - 1).$$

□

4.7. Reverse Second Zagreb Energy of the Join Graph of Complete Graphs

In this section, we calculate the reverse second Zagreb energy of the join graph $J_m(K_n)$. The reverse second Zagreb energy is based on the reverse second Zagreb matrix of the join graph.

Theorem 7. For $m \geq 2$ and $n \geq 2$, the reverse second Zagreb energy of the join graph $J_m(K_n)$ is given by

$$EZ_{2R}(J_m(K_n)) = 2(n + m - 1).$$

Proof. Consider the complete graph K_n with n vertices. In the join graph $J_m(K_n)$, the reverse second Zagreb matrix $Z_{2R}(J_m(K_n))$ is defined as

$$Z_{2R}(J_m(K_n)) = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ 1 & 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$\begin{vmatrix} -\Lambda & 1 & 1 & \cdots & 1 \\ 1 & -\Lambda & 1 & \cdots & 1 \\ 1 & 1 & -\Lambda & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & -\Lambda \end{vmatrix}.$$

The spectrum of the reverse second Zagreb matrix $Z_{2R}(J_m(K_n))$ is

$$\text{Spec}(Z_{2R}(J_m(K_n))) = \begin{pmatrix} -1 & n+m-1 \\ n+m-1 & 1 \end{pmatrix}.$$

Thus, the reverse second Zagreb energy $EZ_{2R}(J_m(K_n))$ is given by

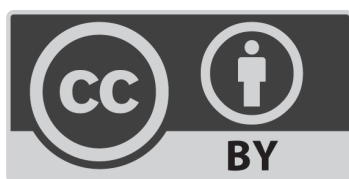
$$EZ_{2R}(J_m(K_n)) = (n+m-1)|-1| + |n+m-1| = 2(n+m-1).$$

□

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