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# Analyzing Pathway Energy And Time Complexity In Flower Families Through Vertex-Disjoint Paths

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ABSTRACT

This research delves into the pathway energy framework for flower families, a class of simple connected graphs, whose path matrix $P$ is constructed such that each entry $P_{ij}$ quantifies the maximum number of vertex-disjoint paths. By analyzing the characteristic values of this matrix, we establish the pathway energy bounds specific to these flower graph families. Additionally, a comprehensive algorithm is developed to evaluate the time complexity across different flower family configurations, utilizing numerous trials to capture their average, maximum, and minimum computational behaviors. This analysis offers a comparative study of the structural intricacies that lead to increased computational complexity, highlighting which graph topologies tend to impose higher algorithmic challenges. The proposed method introduces a refined and adaptable approach, deepening the exploration of characteristic graph properties and their computational impact, thereby expanding the practical applications of these findings in graph theory.

*Keywords:* Characteristics values, Flower families, Energy, Pathway Energy, Time complexity

## 1. Introduction

Consider an undirected graph $G(V, E)$, where $V$ represents the set of vertices and $E$ denotes the set of edges [5]. The concept of graph energy, first introduced by Ivan Gutman [10, 23] in 1978, emerged as a refined extension of the Hückel Molecular Orbital (HMO) theory, originally formulated to quantify the total $\pi$-electron energy in molecular systems.

Following this, the concept of graph energy resurfaced as a significant and highly researched theme within chemical graph theory [23], leading to a surge in academic attention, with more than a

hundred research papers published annually [11, 12]. Numerous properties and well-established bounds associated with graph energy have since been developed and explored extensively [2, 3, 6].A molecular graph $G$ with $n$ vertices. The eigenvalues of the $(0, 1)$-adjacency matrix of $G$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$, constitute the spectrum of $G$. The total sum of the absolute values of the eigenvalues of $A(G)$ defines the energy of the graph $G$ [10, 24].

In this framework, Patekar et al. introduced a novel concept called the pathway matrix [17, 20], which leads to the definition of path energy for a graph, extending the classical notion of graph energy. Define a matrix $P = (p_{ij})$ of dimension $n \times n$, where for $i \neq j$, $p_{ij}$ represents the greatest number of vertex-disjoint paths from vertex $v_i$ to vertex $v_j$. For the case where $i = j$, set $p_{ij} = 0$. In the paper by Sabeen et al. [19], the author explores the $P^{(k)}$ pathway energy across various graph classes, deriving key conclusions about the behavior of $P^{(k)}$ path energy. Similarly, Raza et al. [18] in 2023 dig into the spectrum of graphs. In a related study, Xu and Zhou [22] identify the graphs that attain both upper and lower bounds on the path index for graphs with specified structural parameters. In the book by Cvetkovski et al. [7], the authors examine a variety of inequalities, some of which serves as key instruments in tightening the bounds and enhancing the precision of energy calculations within graph structures. We utilize these tools to derive improved upper and lower bounds on the pathway energy for graphs with suitable parameters. Hutter et al. [13] explored the use of machine learning to predict algorithm runtime based on problem-specific features. A simulation of graphs for reduced order modelling in city-scale district energy grids, leveraging spectral graph theory, was recently proposed by Simonsson et al. [21]. Duraj and Mezic [9, 15] introduced a novel complexity measure for directed graphs, termed spectral complexity. The spectrum of the path matrix, can be optimally resolved in polynomial time, despite the inherent complexity of the graphs.

## 2. Motivation and Methodology

Graph theory is fundamental in optimizing complex networks, yet calculating pathway energy, key to measuring network connectivity through vertex-disjoint paths-poses significant challenges. Efficient computation of pathway energy is critical for designing robust and high-performing networks. To address the complexity of calculating pathway energy for intricate flower families, we propose a novel algorithmic approach.

Our method constructs a matrix $P$ that represents the maximum number of non-overlapping paths between distinct vertex pairs. From the characteristic values of $P$, we derive pathway energy bounds, including upper, lower, improved upper, improved lower, and exact limits. We further develop an algorithm that performs $n$-trial analyses to assess the time complexity of these graphs, by determining average, maximum, and minimum execution times. This approach not only reveals computational challenges across various graph configurations but also provides a streamlined algorithmic framework. By combining advanced mathematical techniques with innovative algorithms, our study offers new insights into network complexity, significantly enhancing both theoretical understanding and practical applications in graph theory.

## 3. Preliminaries

In this section, the ideas and inequalities that provide the foundation for our work were discussed.

**Definition 3.1.** (Wheel) [16] A wheel graph $W_n$ is build by connecting each vertex of a cycle $C_n$ to

a single hub vertex. This hub is referred to as the central vertex, and the cycle $C_n$ is termed the rim of $W_n$.

**Definition 3.2.** (Helm) [16] A helm graph $H_n$ is a graph constructed by linking a pendant edge to every vertex on the outer cycle $C_n$ of a wheel graph $W_n$.

**Definition 3.3.** (Flower) [16] A flower graph is generated by connecting a helm graph $H_n$'s pendent vertices to its center vertex, denoted as $F_n$.

**Definition 3.4.** (Sunflower graph) [8] A sunflower graph $SF_n$ is constructed by transforming each edge of the circumference of a wheel graph $W_n$ into a triangle. Two triangles are connected by a common vertex if and only if their corresponding edges are adjacent.

**Definition 3.5.** (Closed Sunflower graph) [4] When independent vertices of a sun flower graph $SF_n$ that are not contiguous to its center vertex are joined, a closed sun flower graph $CSF_n$ is created, which causes a cycle to occur on $n$ vertices.

**Definition 3.6.** (Blossom graph) [16] When all the vertices of a closed sun flower graph $CSF_n$ are joined to its center vertex, the resulting graph is called a blossom graph $Bl_n$.

**Theorem 3.7.** (Cauchy-Schwarz inequality) [1, 7, 14] *If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are real numbers, we have*

$$\left( \sum_{i=1}^{n} a_i b_i \right)^2 \leq \left( \sum_{i=1}^{n} a_i^2 \right) \left( \sum_{i=1}^{n} b_i^2 \right). \tag{1}$$

**Theorem 3.8.** (Arithmetic and Geometric mean inequality(AM-GM))[1, 7, 14] *Let $n$ be a number of vertex and $\lambda_i$ and $\lambda_j$ be the characteristic values.*

$$\frac{1}{n(n-1)} \sum_{i \neq j} |\lambda_i| |\lambda_j| \geq \left( \prod_{i \neq j} |\lambda_i| |\lambda_j| \right)^{\frac{1}{n(n-1)}}. \tag{2}$$

**Theorem 3.9.** (Chebyshev's Inequality (monotonic Condition)) [7] *If $\{x_i\}$ and $\{y_i\}$ are non-decreasing, then:*

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} x_i y_j \geq \frac{2}{n(n-1)} \left( \sum_{i=1}^{n} x_i \right) \left( \sum_{i=1}^{n} y_i \right), \tag{3}$$

*with the monotonicity condition from Chebyshev's Inequality.*

**Theorem 3.10.** (Weighted AM–GM inequality) [7] *Let $a_i \in (0, \infty)$, $i = 1, 2, \ldots, n$, and $\alpha_i \in [0, 1]$, $i = 1, 2, \ldots, n$, be such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then*

$$a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_n^{\alpha_n} \leq \alpha_1 a_1 + \alpha_2 a_2 + \cdots + \alpha_n a_n. \tag{4}$$

**Theorem 3.11.** (Weighted Cauchy–Schwarz inequality) [7] *Let $a_i, b_i \in \mathbb{R}$ and $m_i \in \mathbb{R}^+$ for $i =$*

$1, 2, \ldots, n$. *Then:*

$$\left(\sum_{i=1}^{n} a_i b_i m_i\right)^2 \leq \left(\sum_{i=1}^{n} a_i^2 m_i\right)\left(\sum_{i=1}^{n} b_i^2 m_i\right), \tag{5}$$

*with equality if and only if* $a_1 b_1 = a_2 b_2 = \cdots = a_n b_n$.

## 4.  Deriving the Pathway Energy Boundaries on Flower Graphs Families

The different pathway energy extremities such as lower, upper, improved upper, and improved lower bounds for the flower graph families are explored in this section.

**Proposition 4.1.** *If* $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$ *are the characteristic values of* $A_p(F_n)$ *then,*

$$\sum_{i=1}^{2n+1} \mu_i^2 = 4\Lambda + 3\Upsilon + 2\top.$$

**Proof.** Let us examine the vertex-disjoint pathway matrix $A_p(F_n)$ associated with a flower graph. The elements of this matrix can be expressed in a structured manner:

$$A_p(F_n) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n+1} \\ v_{n+2} \\ v_{n+3} \\ \vdots \\ v_{2n+1} \end{array} \begin{array}{cccccc|cccc} v_1 & v_2 & v_3 & \cdots & v_{n+1} & v_{n+2} & v_{n+3} & \cdots & v_{2n+1} \\ \left(\begin{array}{ccccc} 0 & 4 & 4 & \cdots & 4 \\ 4 & 0 & 3 & \cdots & 3 \\ 4 & 3 & 0 & \cdots & 3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 3 & 3 & \cdots & 0 \end{array}\right. & \left. \begin{array}{cccc} 2 & 2 & \cdots & 2 \\ 2 & 2 & \cdots & 2 \\ 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & \cdots & 2 \end{array} \right) \end{array}$$

$$A_p(F_n) = \begin{cases} \omega_{ij} = 0, & \text{sum of all null entries on the main diagonal where } i = j, \\ \Lambda_{ij} = 4, & \text{max disjoint paths between inner block,} \\ & \text{if } 2 \leq i \leq n+1, 2 \leq j \leq n+1, \\ \Upsilon_{ij} = 3, & \text{min disjoint paths in the inner block where } i \neq j, \ i, j \leq n+1, \\ \top_{ij} = 2, & \text{disjoint paths between inner and outer blocks} \\ & \text{if } i \geq n+2, \ 1 \leq j \leq 2n+1, 1 \leq i \leq n+1, j \geq n+2 \end{cases}$$

The following equation defines the characteristic polynomial of this $2n+1 \times 2n+1$ square matrix:

$$a_0 \mu^{2n+1} + a_1 \mu^{2n} + a_2 \mu^{2n-1} + \ldots + a_n = 0,$$

where the characteristic values are denoted as $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$. The squared characteristic values added together can be expressed as follows:

$$\sum_{i=1}^{2n+1} \mu_i^2 = \left[\sum_{i=j}^{2n+1} (\omega_{ij})\right] + \left[\sum_{i<j}^{n+1} (\Upsilon_{ij}) + \sum_{i>j}^{n+1} (\Upsilon_{ij})\right] + \left[\sum_{i\geq 2}^{n+1}\sum_{j\geq 2}^{n+1} (\Lambda_{ij})\right]$$

$$+ \left[\left(\sum_{i=1}^{n}\sum_{j=n+1}^{2n+1} (\top_{ij})\right) + \left(\sum_{i=n+1}^{2n+1}\sum_{j=1}^{2n+1} (\top_{ij})\right) + \left(\sum_{i=n+1}^{2n+1}\sum_{j=n+1}^{2n+1} (\top_{ij})\right)\right]$$

$$= 4\Lambda + 3\Upsilon + 2\top.$$

Here $\Lambda, \top$, and $\Upsilon$ denotes the sum of all the elements 4, 3 and 2 vertex-disjoint path matrix respectively. $\qquad\square$

### 4.1. The pathway energy lower bounds for flower graph

**Theorem 4.2.** *For a flower graph $F_n$, $n \geq 4$, with $l = 2n + 1$ vertices and $4n$ edges.*

*(i) The lower bound of pathway energy is given by:*

$$E_p(F_n) \geq \sqrt{4\Lambda + 3\Upsilon + 2\top + \frac{2}{(l)(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2},$$

*where $w_i$ are the weights, $\sum_{i=1}^{l} w_i = 1$ gives the total sum of the characteristic value weights.*

*(ii) The improved lower bound of pathway energy is:*

$$E_p(F_n) \geq \sqrt{4\Lambda + 3\Upsilon + 2\top + (l)(l-1)D^{\frac{2}{l}}} \quad where \quad D = |\det A_p(F_n)|.$$

**Proof.** Consider the flower graph $F_n$ with an vertex-disjoint pathway matrix of order $l \times l$, where $l = 2n + 1$ represents the vertices. The characteristic values $\mu_1, \mu_2, \ldots, \mu_l$ are obtained by solving the characteristic polynomial $|A_p(F_n) - \mu I| = 0$. The pathway energy $E_p(F_n)$ is the sum of the absolute values of the characteristic values of the vertex-disjoint path matrix. The following condition describes this.

$$[E_p(F_n)]^2 = \left( \sum_{i=1}^{l} |\mu_i| \right)^2.$$

Moreover, this can also be written as the product of the absolute characteristic value summations:

$$[E_p(F_n)]^2 = \left( \sum_{i=1}^{l} |\mu_i| \right) \left( \sum_{j=1}^{l} |\mu_j| \right).$$

$$[E_p(F_n)]^2 = \left( \sum_{i=1}^{l} |\mu_i|^2 \right) + \left( \sum_{i \neq j} |\mu_i||\mu_j| \right).$$

**Case 1:** Proof for the pathway energy lower bounds of flower graph

From the Chebyshev's Inequality (3), we derive that,

$$[E_p(F_n)]^2 \geq \sum_{i=1}^{l} |\mu_i|^2 + \frac{2}{l(l-1)} \left( \sum_{i=1}^{l} |\mu_i| \right) \left( \sum_{i=1}^{l} |\mu_i| \right).$$

From the Weighted AM–GM inequality (4), we conclude,

$$[E_p(F_n)]^2 \geq \sum_{i=1}^{l} |\mu_i|^2 + \frac{2}{l(l-1)} \left( \sum_{i=1}^{l} \mu_i w_i \right) \left( \sum_{i=1}^{l} \mu_i w_i \right) \quad \text{where} \sum_{i=1}^{l} w_i = 1,$$

$$\geq \sum_{i=1}^{l} |\mu_i|^2 + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i}) \prod_{i=1}^{l} (\mu_i^{w_i}),$$

$$\geq 4\Lambda + 3\Upsilon + 2\top + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2 \ldots\ldots\ldots \text{[by Theorem 4.1]}.$$

Thus, the lower bounds of the pathway energy for flower graph is,

$$E_p(F_n) \geq \sqrt{4\Lambda + 3\curlyvee + 2\top + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2}.$$

**Case 2:** Proof for the pathway energy improved lower bounds.

From the AM-GM inequalities in Eq. (2), we derive that

$$\frac{1}{l(l-1)} \sum_{i \neq j} |\mu_i||\mu_j| \geqslant \left( \prod_{i \neq j} |\mu_i||\mu_j| \right)^{\frac{1}{l(l-1)}},$$
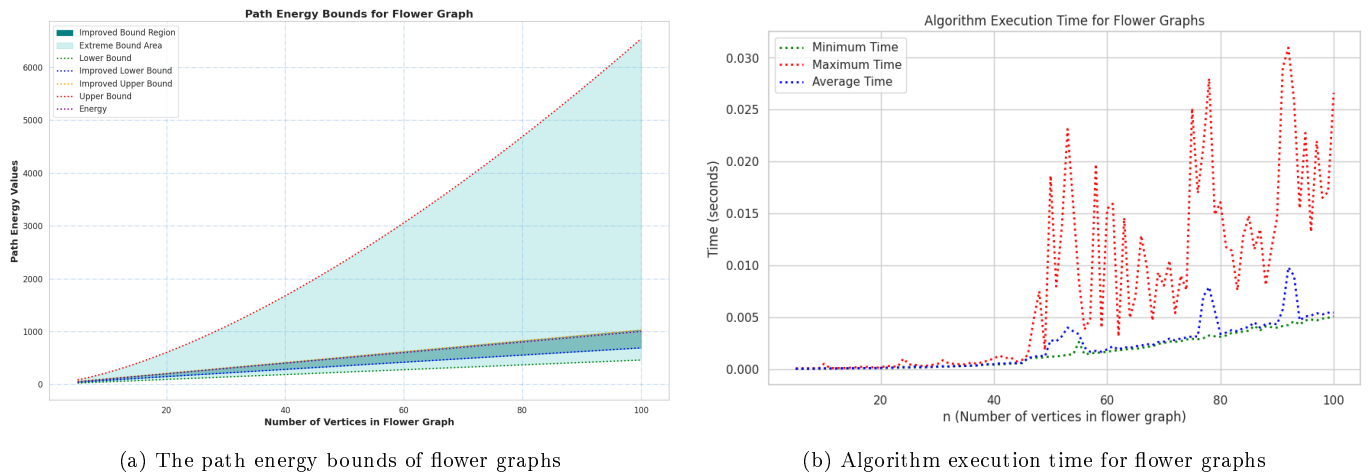
$$[E_p(F_n)]^2 \geq \sum_{i=1}^{l} |\mu_i|^2 + l(l-1) \left( \prod_{i \neq j} |\mu_i||\mu_j| \right)^{\frac{1}{l(l-1)}}$$

$$= \sum_{i=1}^{l} |\mu_i|^2 + l(l-1) \left| \prod_{i=1}^{l} (\mu_i) \right|^{\frac{2}{l}}$$

$$= 4\Lambda + 3\curlyvee + 2\top + l(l-1)D^{\frac{2}{l}} \quad \text{by Theorem 4.1.}$$

Thus, the improved lower bounds of the pathway energy for flower graph is,

$$E_p(F_n) \geq \sqrt{4\Lambda + 3\curlyvee + 2\top + l(l-1)D^{\frac{2}{l}}}.$$

$\square$

**Observation 4.3.** The pathway energy of lower bounds, improved lower bounds and time complexity analysis $O(n^3)$ for flower graphs with $n$ varying from 4 to 100 are computed and plotted in Figure 1: (a) and (b).



(a) The path energy bounds of flower graphs      (b) Algorithm execution time for flower graphs

**Fig. 1.** The comparison of path energy bounds (a) and Time complexity execution (b) for flower graphs

### 4.2.  The pathway energy upper bound for flower graph

**Theorem 4.4.** *For flower graph $F_n$ with $n \geq 4$,*
  *(i)  The upper bound of pathway energy is given by:* $E_p(F_n) \leq \sqrt{l(4\Lambda + 3 \curlyvee + 2\top)}$
  *(ii)  The improved upper bound of pathway energy is:* $E_p(F_n) \leq \sqrt{5(4\Lambda + 3 \curlyvee + 2\top)}.$

**Proof.** The methodology employed to ascertain the characteristic values echoes that utilized in Theorem 4.2. Let $\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots \geq \mu_{n-1} \geq \mu_n \geq \ldots \geq \mu_l$ denote the characteristic values of the $l \times l$ vertex-disjoint path matrix of $F_n$.

**Case 1:** Proof for the pathway energy upper bound of flower graph.
From the Cauchy-Schwarz inequality in the Eq. (1) we derive that,

$$\left(\sum_{i=1}^{l} \alpha_i \beta_i\right)^2 \leq \left(\sum_{i=1}^{l} \alpha_i^2\right)\left(\sum_{i=1}^{l} \beta_i^2\right).$$

To establish the upper bound, consider the expression $[E_p(F_n)]^2 \leq \left(\sum_{i=1}^{l} \alpha_i \beta_i\right)^2$, where the pathway energy is bounded by the right-hand side. Substituting $\alpha_i = 1$ and $\beta_i = |\mu_i|$ into the inequality yields the desired result.

$$[E_p(F_n)]^2 \leq \left(\sum_{i=1}^{l} 1.|\mu_i|\right)^2 \leq \left(\sum_{i=1}^{l} 1\right)^2 . \left(\sum_{i=1}^{l} |\mu_i|^2\right) \ldots\ldots\ldots \text{by } (1),$$
$$= l(4\Lambda + 3 \curlyvee + 2\top)\ldots\ldots\ldots\ldots\ldots \text{by Theorem } 4.1,$$
$$[E_p(F_n)] \leq \sqrt{l(4\Lambda + 3 \curlyvee + 2\top)}.$$

Thus, the upper bound of the pathway energy of the flower graph concluded by,

$$E_p(F_n) \leq \sqrt{l(4\Lambda + 3 \curlyvee + 2\top)}.$$

**Case 2:** Proof for the pathway energy improved upper bound.
The Weighted Cauchy Schwarz inequality in (5) allows us to deduce that,

$$\left(\sum_{i=1}^{l} a_i b_i w_i\right)^2 \leq \left(\sum_{i=1}^{l} a_i^2 w_i\right)\left(\sum_{i=1}^{l} b_i^2 w_i\right).$$

For proving the improved upper bound, consider $[E_p(F_n)]^2 \leq \left(\sum_{i=1}^{l} a_i.b_i.w_i\right)^2$, such that the pathway energy of the graph is less than RHS of the inequality, and substituting $a_i = \frac{1}{(l)^{\frac{3}{2}}} |\mu_i|$ and $b_i = 1$ $w_i = \sqrt{5}$, in the above inequality,

$$[E_p(F_n)]^2 \leq \left(\sum_{i=1}^{l} \frac{\mu_i^2}{(l)^3} . \sqrt{5}\right)\left(\sum_{i=1}^{l} 1.\sqrt{5}\right),$$
$$\leq \frac{\sqrt{5}l}{l^3}\left(\sum_{i=1}^{l} \mu_i^2\right).l^2\sqrt{5},$$
$$= 5(4\Lambda + 3 \curlyvee + 2\top)\ldots\ldots\ldots\ldots\ldots \text{by Theorem } 4.1,$$
$$[E_p(F_n)] \leq \sqrt{5(4\Lambda + 3 \curlyvee + 2\top)}.$$

Thus, the improved upper bound of the pathway energy for flower graph is,

$$[E_p(F_n)] \leq \sqrt{5(4\Lambda + 3 \Upsilon + 2\top)}.$$

$\square$

**Observation 4.5.** The pathway energy of upper bounds, improved upper bounds and time complexity analysis $O(n^3)$ for flower graphs with $n$ varying from 4 to 100 are computed and plotted in Figure 1: (a) and (b).

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**Algorithm 1** Bounds of a flower graph with time complexity

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**Require:** Number of vertices $n \geq 4$

1: **Step 1: Adjacency Matrix Setup**
2: Initialize $A \leftarrow \text{zeros}(2n + 1, 2n + 1)$
3: Define vertices: diagonal $= 0$, inner $= \text{list}(1, \ldots, n + 1)$, outer $= \text{list}(n + 2, \ldots, 2n + 1)$
4: $A[0][i] \leftarrow 4$ for $i \in$ inner, for $2 \leq i \leq n + 1, i + 1 \leq j \leq n + 1$   (inner block)
5: $A[i][j] \leftarrow 3$ for $i, j \leq n + 1, i \neq j$   (inner block)
6: $A[i][j] \leftarrow 2$ for  $i \geq n + 2,\ 1 \leq j \leq 2n + 1, 1 \leq i \leq n + 1, j \geq n + 2$   (outer block)
7: **Step 2: Compute characteristic values and Energy**
8: compute the characteristic values $\mu \leftarrow \text{chval}(A)$
9: Calculate total energy $E \leftarrow \sum |\mu_i|$
10: **Step 3: Upper and Lower Bounds Calculation**
11: Sum of squared characteristic values $R \leftarrow \sum \mu_i^2$
12: Upper Bound $T \leftarrow \sqrt{(2n + 1) \times R}$
13: Lower Bound $K \leftarrow \sqrt{R + (2n + 1) \times 2n \times |\det(A)|^{\frac{2}{2n+1}}}$
14: **Step 4: Improved Upper and Lower Bound Calculation**
15: Improved Upper Bound $T_{improved} \leftarrow \sqrt{5 \times R}$
16: Improved Lower Bound $K_{improved} \leftarrow \sqrt{R + \frac{2}{(2n+1)(2n)} \times \prod_{i=1}^{2n+1} |\mu_i|^2}$
17: **Step 5: Time Complexity Analysis**
18: Time complexity of characteristic value computation is $O((2n + 1)^3)$
19: Overall time complexity:    $O(f(n))$,    where    $f(n) = (2n + 1)^3 +$ computation time
20: **Output**: Energy $E$, Bounds $T, K, T_{improved}, K_{improved}$, Time Complexity $O(f(n))$.

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## 5.   The Pathway Energy Boundaries on Sunflower Graph

The different pathway energy bounds of sunflower graph $SF_n$ like lower, upper, improved lower, and improved upper bounds are investigated in this section.

**Proposition 5.1.** *The characteristic values of $A_p(SF_n)$ are $\sum\limits_{i=1}^{2n+1} \mu_i^2 = 4\delta + 3\varkappa + 2\varpi$, if $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$ are the characteristic values of $A_p(SF_n)$.*

**Proof.** The structure of the vertex-disjoint path matrix $A_p(SF_n)$ for the sunflower graph is depicted as follows,

$$A_p(SF_n) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n+1} \\ v_{n+2} \\ v_{n+3} \\ \vdots \\ v_{2n+1} \end{array} \begin{pmatrix} \begin{array}{ccccc|ccccc} v_1 & v_2 & v_3 & \cdots & v_{n+1} & v_{n+2} & v_{n+3} & \cdots & v_{2n+1} \\ 0 & 3 & 3 & \cdots & 3 & 2 & 2 & \cdots & 2 \\ 3 & 0 & 4 & \cdots & 3 & 2 & 2 & \cdots & 2 \\ 3 & 4 & 0 & \cdots & 4 & 2 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 3 & 3 & 4 & \cdots & 0 & 2 & 2 & \cdots & 2 \\ \hline 2 & 2 & 2 & \cdots & 2 & 0 & 2 & \cdots & 2 \\ 2 & 2 & 2 & \cdots & 2 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2 & 2 & 2 & \cdots & 2 & 2 & 2 & \cdots & 0 \end{array} \end{pmatrix}$$

$$A_p(SF_n) = \begin{cases} F_{ij} = 0, & \text{Total count of zeros positioned along the main diagonal } i = j, \\ \delta_{ij} = 4, & \text{Largest set of disjoint paths inside the central block, where} \\ & |i - j| = 1 \text{ or } |i - j| = n - 1, \ 1 \leq i, j \leq n + 1, \\ \varkappa_{ij} = 3, & \text{the inner block's minimum disjoint path if } i \neq j, \ i, j \leq n + 1, \\ \varpi_{ij} = 2, & \text{vertex-disjoint paths spanning the inner and outer regions} \\ & \text{if } i \geq n + 2, \ 1 \leq j \leq 2n + 1, 1 \leq i \leq n + 1, j \geq n + 2 \end{cases}$$

The characteristic equation of this $2n + 1 \times 2n + 1$ square matrix is given by:

$$a_0 \mu^{2n+1} + a_1 \mu^{2n} + a_2 \mu^{2n-1} + \ldots + a_n = 0,$$

where the characteristic values are denoted as $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$. The sum of the squares of these characteristic values is defined as:

$$\sum_{i=1}^{2n+1} \mu_i^2 = \left[ \sum_{i=j}^{2n+1} (F_{ij}) \right] + \left[ \sum_{i<j}^{n+1} (\varkappa_{ij}) + \sum_{i>j}^{n+1} (\varkappa_{ij}) \right] + \left[ \sum_{\substack{1 \leq i \leq n \\ |i-j|=1}} \sum_{\substack{1 \leq j \leq n \\ |i-j|=n-1}} (\delta_{ij}) \right]$$

$$+ \left[ \left( \sum_{i=1}^{n} \sum_{j=n+1}^{2n+1} (\varpi_{ij}) \right) + \left( \sum_{i=n+1}^{2n+1} \sum_{j=1}^{2n+1} (\varpi_{ij}) \right) + \left( \sum_{i=n+1}^{2n+1} \sum_{j=n+1}^{2n+1} (\varpi_{ij}) \right) \right]$$

$$= 4\delta + 3\varkappa + 2\varpi.$$

In this instance, $\delta, \varkappa,$ and $\varphi$ denote the total of all elements 4, 3, and 2 respectively, present in the vertex-disjoint path matrices. $\square$

## 5.1. The pathway energy lower bounds for sunflower graph

**Theorem 5.2.** *For a sunflower graph $SF_n$ with $n \geq 5$, there are $l = 2n + 1$ vertices and $4n$ edges.*
*(i) The lower bound of pathway energy is given by:*

$$E_p(SF_n) \geq \sqrt{4\delta + 3\varkappa + 2\varpi + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2},$$

*where $w_i$ are the weights, $\sum_{i=1}^{l} w_i = 1$ gives the total sum of the characteristic value weights.*

(ii) *The improved lower bound of pathway energy is:*

$$E_p(SF_n) \geq \sqrt{4\delta + 3\varkappa + 2\varpi + (l)(l-1)D^{\frac{2}{l}}} \quad where \quad D = |\det A_p(SF_n)|.$$

**Proof.** The proof is similar to Theorem 4.2 and is left to the interest of the reader. $\square$

**Observation 5.3.** The pathway energy of lower bounds, improved lower bounds and time complexity analysis $O(n^3)$ for sunflower graphs with order of $n$ varying from 5 to 100 are shown in Figure 2: (a) and (b).



(a) The path energy bounds of sunflower graphs          (b) Algorithm execution time for sunflower graphs
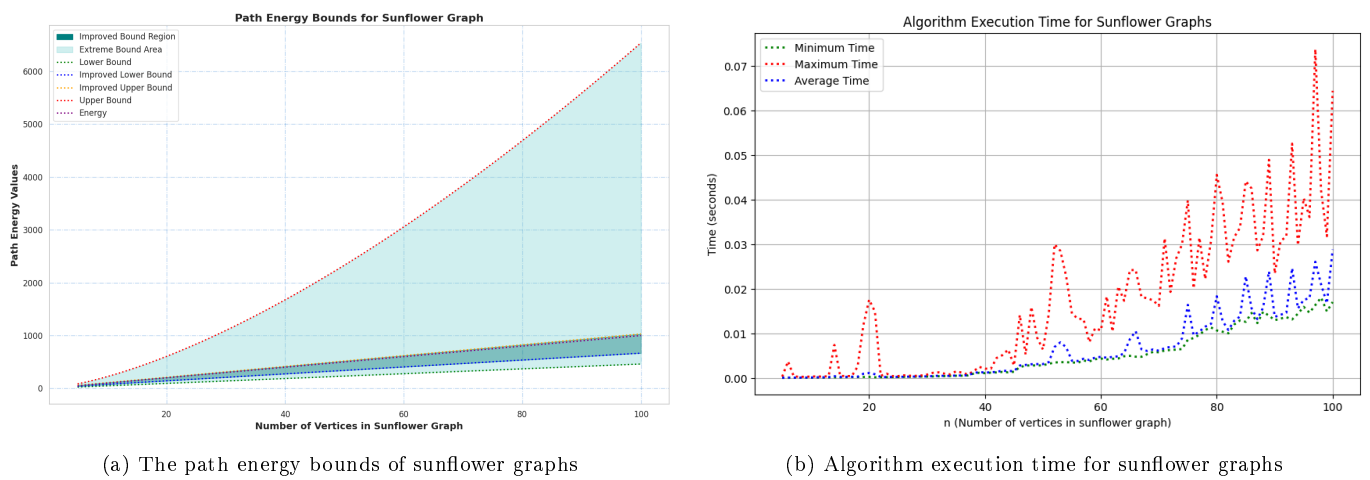
**Fig. 2.** The comparison of path energy bounds (a) and Time complexity execution (b) for sunflower graphs.

## 5.2.    The pathway energy upper bound for sunflower graph

**Theorem 5.4.** *For sunflower graph $SF_n$ with $n \geq 5$,*
  (i) *The upper bound of pathway energy is given by:* $E_p(SF_n) \leq \sqrt{l(4\delta + 3\varkappa + 2\varpi)}$
  (ii) *The improved upper bound of pathway energy is:* $E_p(SF_n) \leq \sqrt{5(4\delta + 3\varkappa + 2\varpi)}$.

**Proof.** The proof is similar to Theorem 4.4 and is left to the interest of the reader. $\square$

**Observation 5.5.** The pathway energy of upper bounds, improved upper bounds and time complexity analysis $O(n^3)$ for sunflower graphs with order of $n$ varying from 5 to 100 are shown in Figure 2: (a) and (b).

**Algorithm 2** Bounds of a sunflower graph with time complexity

**Require:** Number of vertices $n \geq 5$
 1: **Step 1: Adjacency Matrix Setup**
 2: Initialize $A \leftarrow \text{zeros}(2n+1, 2n+1)$
 3: Define vertices: diagonal $= 0$, inner $= \text{list}(1, \ldots, n+1)$, outer $= \text{list}(n+1, \ldots, 2n+1)$
 4: Fill $A$ as follows:
 5: $A[i][j] \leftarrow 4$ for $i \in$ inner, $for\ |i-j| = n-1,\ 1 \leq i,j \leq n+1$  (inner block)
 6: $A[i][j] \leftarrow 3$ for $i,j \leq n+1, i \neq j$  (inner block)
 7: $A[i][j] \leftarrow 2$ for $i \geq n+2,\ 1 \leq j \leq 2n+1, 1 \leq i \leq n+1, j \geq n+2$  (outer block)
 8: **Step 2: characteristic value and Energy Calculation**
 9: Compute characteristic values $\mu \leftarrow \text{chval}(A)$
10: Calculate total energy $E \leftarrow \sum |\mu_i|$
11: **Step 3: Upper, and Lower Bounds Calculation**
12: Sum of squared characteristic values $R \leftarrow \sum \mu_i^2$
13: Upper Bound $T \leftarrow \sqrt{(2n+1) \times R}$
14: Lower Bound $K \leftarrow \sqrt{R + (2n+1) \times 2n \times |\det(A)|^{\frac{2}{2n+1}}}$
15: **Step 4: Improved Upper and Lower Bound Calculation**
16: Improved Upper Bound $T_{improved} \leftarrow \sqrt{5 \times R}$
17: Improved Lower Bound $K_{improved} \leftarrow \sqrt{R + \frac{2}{(2n+1)(2n)} \times \prod_{i=1}^{2n+1} |\mu_i|^2}$
18: **Step 5: Time Complexity Analysis**
19: Time complexity of characteristic value computation is $O((2n+1)^3)$
20: Overall time complexity:   $O(f(n))$,   where   $f(n) = (2n+1)^3 +$ computation time
21: **Output**: Energy $E$, Bounds $P, T, K, T_{improved}, K_{improved}$, Time Complexity $O(f(n))$

## 6.  The Path Energy Bounds on Closed Sunflower Graph

This section explicates the distinct path energy bounds for the closed sunflower graph $CSF_n$, comprising the lower, upper, exact energy, improved lower and upper bound.

**Proposition 6.1.** *If* $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$ *are the characteristic values of* $A_p(CSF_n)$ *then,*

$$\sum_{i=1}^{2n+1} \mu_i^2 = 5\kappa + 4\chi.$$

**Proof.** The arrangement of elements in the vertex-disjoint path matrix $A_p(CSF_n)$ of a closed sunflower graph is shown below:

$$A_p(CSF_n) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_{n+1} \\ v_{n+2} \\ v_{n+3} \\ \vdots \\ v_{2n+1} \end{array} \begin{array}{ccccc} v_1 & v_2 & v_3 & \cdots & v_{n+1} \end{array} \begin{array}{cccc} v_{n+2} & v_{n+3} & \cdots & v_{2n+1} \end{array}$$

$$\left( \begin{array}{ccccc|cccc} 0 & 5 & 5 & \cdots & 5 & 4 & 4 & \cdots & 4 \\ 5 & 0 & 5 & \cdots & 5 & 4 & 4 & \cdots & 4 \\ 5 & 5 & 0 & \cdots & 5 & 4 & 4 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 5 & 5 & 5 & \cdots & 0 & 4 & 4 & \cdots & 4 \\ \hline 4 & 4 & 4 & \cdots & 4 & 0 & 4 & \cdots & 4 \\ 4 & 4 & 4 & \cdots & 4 & 4 & 0 & \cdots & 4 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 4 & 4 & 4 & \cdots & 4 & 4 & 4 & \cdots & 0 \end{array} \right) \qquad (6)$$

$$A_p(SF_n) = \begin{cases} \lambda_{ij} = 0, & \text{Sum of zeros on the matrix's main diagonal where } i = j, \\ \kappa_{ij} = 5, & \text{Optimal disjoint path count within the inner block} \\ & \text{where } i \neq j, \ i, j \leq n+1, \\ \chi_{ij} = 4, & \text{maximum possible disjoint paths between outer block,} \\ & \text{if } i \geq n+2, \ 1 \leq j \leq 2n+1, 1 \leq i \leq n+1, j \geq n+2. \end{cases}$$

The characteristic equation associated with this $2n+1 \times 2n+1$ square matrix is:

$$a_0 \mu^{2n+1} + a_1 \mu^{2n} + a_2 \mu^{2n-1} + \ldots + a_n = 0,$$

where the characteristic values are denoted as $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \ldots, \mu_{2n+1}$.
The squared sum of these characteristic values is expressed as,

$$\sum_{i=1}^{2n+1} \mu_i^2 = \left[ \sum_{i=j}^{2n+1} (\lambda_{ij}) \right] + \left[ \sum_{i<j}^{n+1} (\kappa_{ij}) + \sum_{i>j}^{n+1} (\kappa_{ij}) \right]$$
$$+ \left[ \left( \sum_{i=1}^{n} \sum_{j>n}^{2n+1} (\chi_{ij}) \right) + \left( \sum_{i>n}^{2n+1} \sum_{j=1}^{2n+1} (\chi_{ij}) \right) + \left( \sum_{i>n+1}^{2n+1} \sum_{j>n+1}^{2n+1} (\chi_{ij}) \right) \right]$$
$$= 0 \, (\alpha) + 5 \, (\kappa) + 4 \, (\chi)$$
$$= 5\kappa + 4\chi.$$

In this context, $\kappa$ and $\chi$ represent the sums of all the elements 5 and 4 in the path matrices, respectively. □

## 6.1. The path energy lower bound for the closed sunflower graph

**Theorem 6.2.** *Let $CSF_n$ be a closed sunflower graph comprising $l = 2n+1$ vertices and $5n$ edges.*
  *(i) The lower bound of pathway energy is given by,*

$$E_p(CSF_n) \geq \sqrt{5\kappa + 4\chi + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2},$$

  *where $w_i$ is the weights of the characteristic values and their sum is $\sum_{i=1}^{l} w_i = 1$.*
  *(ii) The improved pathway energy lower bound is given by,*

$$E_p(CSF_n) \geq \sqrt{5\kappa + 4\chi + l(l-1)D^{\frac{2}{l}}} \quad \text{where } D = |\det A_p(CSF_n)|.$$

**Proof.** The vertex-disjoint path matrix of the closed sunflower network $CSF_n$ has order $l \times l$, where the vertices are represented by $l = 2n+1$. Solving the polynomial equation $|A_p(CSF_n) - \mu I| = 0$ yields the characteristic values $\mu_1, \mu_2, \ldots, \mu_l$. Following the method in Theorem 4.2 and using terminology that has already been explained, the pathway energy $E_p(CSF_n)$ is defined as the total of the absolute values of these characteristic values.

$$[E_p(CSF_n)]^2 = \left( \sum_{i=1}^{l} |\mu_i| \right)^2 + \left( \sum_{i \neq j} |\mu_i||\mu_j| \right).$$

**Case 1:** Proof for the pathway energy lower bound of $\mathbf{CSF_n}$

From the Chebyshev's Inequality (3), we derive that,

$$[E_p(CSF_n)]^2 \geq \sum_{i=1}^{l}|\mu_i|^2 + \frac{2}{l(l-1)}\left(\sum_{i=1}^{l}|\mu_i|\right)\left(\sum_{i=1}^{l}|\mu_i|\right).$$

From the Weighted AM–GM inequality (4), we conclude,

$$[E_p(CSF_n)]^2 \geq \sum_{i=1}^{l}|\mu_i|^2 + \frac{2}{l(l-1)}\left(\sum_{i=1}^{l}\mu_i w_i\right)\left(\sum_{i=1}^{l}\mu_i w_i\right) \quad \text{where} \quad \sum_{i=1}^{l}w_i = 1,$$

$$\geq 5\kappa + 4\chi + \frac{2}{l(l-1)}\prod_{i=1}^{l}(\mu_i^{w_i})^2 \ldots\ldots\ldots\text{by (Theorem 6.1)}.$$

Thus, the lower bound is

$$E_p(CSF_n) \geq \sqrt{5\kappa + 4\chi + \frac{2}{l(l-1)}\prod_{i=1}^{l}(\mu_i^{w_i})^2}.$$

**Case 2:** Proof for the pathway energy improved lower bound.

From the AM-GM inequalities in Eq. (2), we derive that

$$[E_p(CSF_n)]^2 = \sum_{i=1}^{l}|\mu_i|^2 + l(l-1)\left|\prod_{i=1}^{l}(\mu_i)\right|^{\frac{2}{l}} = 5\kappa + 4\chi + l(l-1)D^{\frac{2}{l}} \quad \text{[by Theorem 6.1]}.$$

Thus, the improved lower bound of the pathway energy for closed sunflower graph is,

$$E_p(CSF_n) \geq \sqrt{5\kappa + 4\chi + l(l-1)D^{\frac{2}{l}}}.$$

$\square$

**Observation 6.3.** The pathway energy of lower bounds, improved lower bounds and time complexity analysis $O(n^3)$ for closed sunflower graphs with order of $n$ varying from 5 to 100 are depicted in Figure 3: (a) and (b).
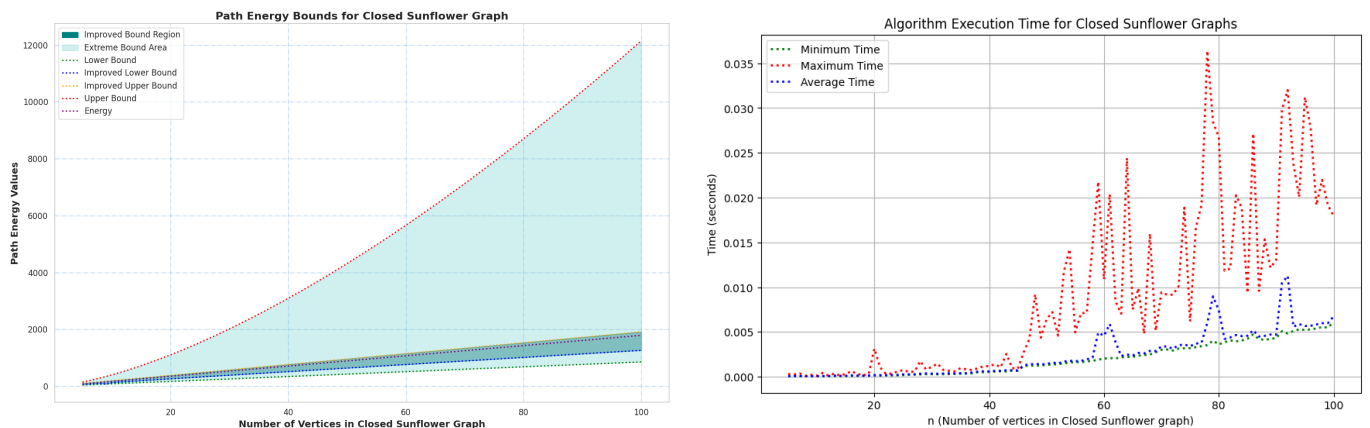


(a) The path energy bounds of closed sunflower graphs

(b) Algorithm execution time for closed sunflower graphs

**Fig. 3.** The comparison of path energy bounds (a) and time complexity execution (b) for closed sunflower graphs.

### 6.2. The exact pathway energy bound for closed sunflower graph

**Theorem 6.4.** *For the closed sunflower graph $CSF_n$, where $n \geq 5$, the exact bounds on the pathway energy are as follows:*

$$E_p[CSF_n] = (9n - 4) + \mid (n - 10)(0.47) + (0.47) \mid + \mid (n - 10)(8.53) + 85.53 \mid$$

*corrected to two decimal places.*

**Proof.** The vertex-disjoint path configuration of the $CSF_n$ graph is elaborated in detail in the path adjacency matrix (6). It has $2n + 1$ roots, implying that it contains $2n + 1$ characteristic values.

The characteristic equation of (6) is

$$(-\mu)^{2n+1} + \mathrm{tr}(-\mu)^{2n} + \cdots + \det(A_p(CSF_n)) = 0. \tag{7}$$

Upon solving the Eq. (7), the $2n + 1$ characteristic values are determined to be,

$$\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_{n-1} = 5, \quad \mu_n = \mu_{n+1} = \mu_{n+2} = \cdots = \mu_{2n-1} = 4,$$

$$\mu_{2n} = (n - 10)(0.47) + 0.47, \text{ and } \mu_{2n+1} = (n - 10)(8.53) + 83.53.$$

The Spectrum is expressed in the form of,

$$\mathrm{Spec}\,(\mathbb{A}_p(CSF_n)) = \begin{bmatrix} -5 & -4 & (n - 10)(0.47) + (0.47) & (n - 10)(8.53) + 83.53 \\ n & n - 1 & 1 & 1 \end{bmatrix}.$$

The exact closed sunflower graph pathway energy is given by,

$$E_p[(CSF_n)] = \sum_{i=1}^{2n+1} |\mu_i| = \mid -5 \mid (n) + \mid -4 \mid (n - 1)$$
$$+ \mid (n - 10)(0.47) - 0.47 \mid + \mid (n - 10)(8.53) + 85.53 \mid$$
$$= 9n - 4 + \mid (n - 10)(0.47) + (0.47) \mid + \mid (n - 10)(8.53) + 85.53 \mid.$$

The value $\mid -5 \mid$ occurs $(n)$ times, $\mid -4 \mid$ repeated $(n - 1)$ times, $[(n - 10)(0.47)] - 0.47$, and $[(n - 10)(8.53)] + 85.53$ arrives once.

Therefore the pathway energy exact bound for closed sunflower graph is denoted by $E_p(CSF_n) = 9n - 4 + \mid (n - 10)(0.47) + (0.47) \mid + \mid (n - 10)(8.53) + 85.53 \mid$.      $\square$

**Observation 6.5.** The pathway exact energy and time complexity analysis $O(n^3)$ for closed sunflower graphs with order of $n$ varying from 5 to 100 are shown in Figure 3: (a) and (b).

### 6.3. The path energy upper bound for closed sunflower graph

**Theorem 6.6.** *For Closed sunflower $CSF_n$ with $n \geq 5$,*
  *(i) The upper bound of pathway energy is given by: $E_p(CSF_n) \leq \sqrt{l(5\kappa + 4\chi)}$*
  *(ii) The improved upper bound of pathway energy is: $E_p(CSF_n) \leq \sqrt{5(5\kappa + 4\chi)}$.*

**Proof.** In this segment to establish the upper bounds, follow the approach of characteristic values determination in Theorem 4.2. For the purpose of the proof, it is a essential to take into account the following consideration,

**Case 1:** Proof for the pathway energy upper bound of $\mathbf{CSF_n}$.

To establish the upper bound, apply the Cauchy-Schwarz inequality in (1).

Consider $[E_p(CSF_n)]^2 \leq \left(\sum_{i=1}^{l} \alpha_i \beta_i\right)^2$, where the pathway energy is bounded above. Substituting $\alpha_i = 1$ and $\beta_i = |\mu_i|$ into this inequality yields the desired result.

$$[E_p(CSF_n)]^2 \leq \left(\sum_{i=1}^{l} 1.|\mu_i|\right)^2 \leq \left(\sum_{i=1}^{l} 1\right)^2 . \left(\sum_{i=1}^{l} |\mu_i|^2\right) \ldots\ldots\ldots [\text{by } (1)]$$

$$= l(5\kappa + 4\chi)\ldots\ldots\ldots\ldots\ldots [\text{by Theorem } 6.1]$$

$$[E_p(CSF_n)] \leq \sqrt{l(5\kappa + 4\chi)}.$$

Thus, the upper bound of the pathway energy of the closed sunflower graph concluded by, $E_p(CSF_n) \leq \sqrt{l(5\kappa + 4\chi)}$.

**Case 2:** Proof for the pathway energy improved upper bound.

Using the Weighted Cauchy-Schwarz inequality from Eq. (5), we deduce the improved upper bound: $[E_p(CSF_n)]^2 \leq \left(\sum_{i=1}^{l} a_i b_i w_i\right)^2$ Substituting $a_i = \frac{1}{l^{3/2}}|\mu_i|$, $b_i = 1$, and $w_i = \sqrt{5}$ in RHS gives the result.

$$[E_p(CSF_n)]^2 \leq \left(\sum_{i=1}^{l} \frac{\mu_i^2}{(l)^3} . \sqrt{5}\right) \left(\sum_{i=1}^{l} 1.\sqrt{5}\right)$$

$$= 5(5\kappa + 4\chi) \ldots\ldots\ldots\ldots\ldots [\text{by Theorem } 6.1]$$

$$[E_p(CSF_n)] \leq \sqrt{5(5\kappa + 4\chi)}.$$

The improved upper bound of the path energy for the closed sunflower graph is:

$$[E_p(CSF_n)] \leq \sqrt{5(5\kappa + 4\chi)}.$$

$\square$

**Observation 6.7.** The pathway energy of upper bounds, improved upper bounds and time complexity analysis $O(n^3)$ for closed sunflower graphs with order of $n$ varying from 5 to 100 are depicted in Figure 3: (a) and (b).

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**Algorithm 3** Bounds of a Closed Sunflower Graph with Time Complexity

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**Require:** Number of vertices $n \geq 5$
 1: **Step 1: Adjacency Matrix Setup**
 2: $A \leftarrow \text{zeros}(2n+1, 2n+1)$
 3: **for** $i, j \leftarrow 0$ to $2n$ **do**
 4:     **if** $i = j$ **then**
 5:         $A[i,j] \leftarrow 0$                                                        ▷ Diagonal element
 6:     **else if** $i \leq n \wedge j \leq n$ **then**
 7:         $A[i,j] \leftarrow 5$
 8:     **else**
 9:         $A[i,j] \leftarrow 4$
10:     **end if**
11: **end for**
12: **Step 2: characteristic value and Energy Calculation**
13: $\mu \leftarrow \text{chval}(A)$
14: $E \leftarrow \sum |\mu_i|$
15: **Step 3: Bound Calculations**
16: $P \leftarrow (9n-4) + |(n-10)(0.47) + 0.47| + |(n-10)(8.53) + 85.53|$
17: $R \leftarrow \sum \mu_i^2$ (sum of the squares of the characteristic values)
18: $T \leftarrow \sqrt{(2n+1) \times R}$
19: $K \leftarrow \sqrt{R + (2n+1) \times 2n \times |\det(A)|^{2/(2n+1)}}$
20: $T_{improved} \leftarrow \sqrt{5 \times R}$
21: $K_{improved} \leftarrow \sqrt{R + \frac{2}{(2n+1)(2n)} \prod_{i=1}^{2n+1} |\mu_i|^2}$
22: **Step 4: Time Complexity Analysis**
23: Time complexity: $O(f(n))$, where $f(n) = (2n+1)^3 +$ computation time
24: **Output**: $E$, Bounds $P$, $T$, $K$, $T_{improved}$, $K_{improved}$, Time Complexity $O(f(n))$

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## 7.   The Pathway Energy Bounds on Blossom Graph

This section explores the pathway energy constraints for the blossom graph $Bl_n$, including upper and lower bounds, improved bounds, and exact energy.

**Proposition 7.1.** *If* $\mu_1, \mu_2, \ldots, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$ *are the characteristic values of* $A_p(Bl_n)$ *then*

$$\sum_{i=1}^{2n+1} \mu_i^2 = 5(\varrho_1 + \varrho_2),$$

**Proof.** The configuration of the vertex-disjoint path adjacency matrix $A_p(Bl_n)$ for a blossom graph is outlined as follows,

$$A_p(Bl_n) = \begin{cases} \alpha_{ij} = 0, & \text{whenever } i = j, \\ \beta_{ij} = 5, & \text{whenever. } i \neq j, \ i,j \leq 2n+1. \end{cases} \tag{8}$$

here     $\varrho_1 = \sum_{i=j}^{2n+1} (\alpha_{ij}), \quad \varrho_2 = \sum_{i<j} (\beta_{ij}) + \sum_{i>j} (\beta_{ij}).$

For this $2n+1 \times 2n+1$ matrix, the characteristic equation is,

$$a_0 \mu^{2n+1} + a_1 \mu^{2n} + a_2 \mu^{2n-1} + \cdots + a_n = 0.$$

All the characteristic values are represented by $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n, \mu_{n+1}, \ldots, \mu_{2n+1}$.

The total of the squared characteristic values can be outlined as,

$$\sum_{i=1}^{2n+1} \mu_i^2 = \sum_{i=j}^{2n+1} (a_{ij}) + \sum_{i<j} (b_{ij}) + \sum_{i>j} (b_{ij})$$

$$\sum_{i=1}^{2n+1} \mu_i^2 = 5\left[\varrho_1 + \varrho_2\right]$$

where $\varrho_1$ and $\varrho_2$ are specific terms related to the matrix components. $\square$

### 7.1. The pathway energy lower bound for the blossom graph

**Theorem 7.2.** *Let be a $Bl_n$ blossom graph comprising $l=2n+1$ vertices and $6n$ edges.*

*(i) The lower bounds of pathway energy is given by,*

$$E_p(Bl_n) \geq \sqrt{5(\varrho_1 + \varrho_2) + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2},$$

*where $w_i$ are the weights, $\sum_{i=1}^{l} w_i = 1$ gives the total sum of the characteristic value weights.*

*(ii) The improved pathway energy lower bound is given by,*

$$E_p(Bl_n) \geq \sqrt{5(\varrho_1 + \varrho_2) + l(l-1)D^{\frac{2}{l}}} \quad where \quad D = |\det A_p(Bl_n)|.$$

**Proof.** In general, the blossom graph $Bl_n$ has a vertex-disjoint adjacency matrix of order $l \times l$, with the number of vertices denoted by $l = 2n + 1$. The characteristic values $\mu_1, \mu_2, \ldots, \mu_l$ can be determined by solving $|A_p(Bl_n) - \mu I| = 0$. As similar to the method earlier in Theorem 4.2, the pathway energy $E_p(Bl_n)$, is the total of the absolute values of these characteristic values.

$$[E_p(Bl_n)]^2 = \left(\sum_{i=1}^{l} |\mu_i|\right)^2 + \left(\sum_{i \neq j} |\mu_i||\mu_j|\right).$$

**Case 1:** Proof for the pathway energy lower bound of blossom graph

From the Chebyshev's Inequality (3), we derive that,

$$[E_p(Bl_n)]^2 \geq \sum_{i=1}^{l} |\mu_i|^2 + \frac{2}{l(l-1)} \left(\sum_{i=1}^{l} |\mu_i|\right) \left(\sum_{i=1}^{l} |\mu_i|\right).$$

We derive the following from the Weighted AM–GM inequality (4):

$$[E_p(Bl_n)]^2 \geq \sum_{i=1}^{l} |\mu_i|^2 + \frac{2}{l(l-1)} \left(\sum_{i=1}^{l} \mu_i w_i\right) \left(\sum_{i=1}^{l} \mu_i w_i\right) \quad where \quad \sum_{i=1}^{l} w_i = 1,$$

$$\geq 5(\varrho_1 + \varrho_2) + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2 \quad \text{by Theorem 7.1.}$$

Thus, the lower bound of the pathway energy for blossom graph is,

$$E_p(Bl_n) \geq \sqrt{5(\varrho_1 + \varrho_2) + \frac{2}{l(l-1)} \prod_{i=1}^{l} (\mu_i^{w_i})^2}.$$

**Case (ii):** Proof for the pathway energy improved lower bound.

We obtain that from the AM-GM inequality in Eq. (2).

$$[E_p(Bl_n)]^2 = \sum_{i=1}^{l} |\mu_i|^2 + l(l-1) \left| \prod_{i=1}^{l} (\mu_i) \right|^{\frac{2}{l}}$$
$$= 5(\varrho_1 + \varrho_2) + l(l-1)D^{\frac{2}{l}} \quad \text{by Theorem 7.1.}$$

Thus, the improved lower bound of the pathway energy for blossom graph is,

$$E_p(Bl_n) \geq \sqrt{5(\varrho_1 + \varrho_2) + l(l-1)D^{\frac{2}{l}}}.$$

□

**Observation 7.3.** The pathway energy of lower bounds, improved lower bounds and time complexity analysis $O(n^3)$ for blossom graphs with order of $n$ varying from 5 to 100 are plotted in Figure 4: (a) and (b).



(a) The path energy bounds of blossom graphs       (b) Algorithm execution time for blossom graphs
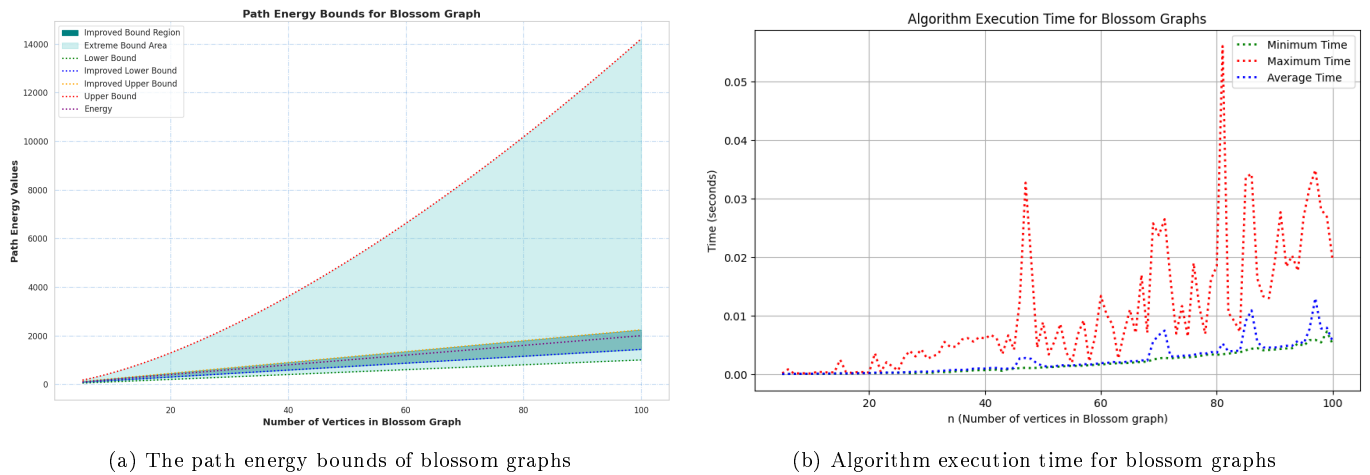
**Fig. 4.** The comparison of path energy bounds (a) and time complexity execution (b) for blossom graphs

### 7.2.   *The pathway energy exact bound for blossom graph*

**Theorem 7.4.** *The defined bounds for the pathway energy in the blossom graph $Bl_n$, under the condition that $n \geq 5$, are $E_p(Bl_n) = 10(2n)$.*

**Proof.** The Eq. (8) provides a detailed description of the pathway configuration for the vertex-disjoint matrix of the blossom graph. Solving the characteristic equation $(-\mu)^{2n+1} + tr(-\mu)^{2n} + \cdots + det\, A_p(Bl_n) = 0$ derives the $2n + 1$ characteristic values in this way: $\mu_1 = \mu_2 = \mu_3 = \cdots = \mu_{2n} = 5$ and $\mu_{2n+1} = 5(2n)$.

The spectrum can be articulated as: $\text{Spec}\,(\mathbb{A}_p(Bl_n)) = \begin{bmatrix} -5 & 5(2n) \\ 2n & 1 \end{bmatrix}.$

The exact energy path of the blossom graph provided by,

$$E_p(Bl_n) = \sum_{i=1}^{2n+1} |\mu_i| = \quad |-5|(2n) + 5(2n) = 10(2n).$$

The term $|-5|$ appears $2n$ times, while $5(2n)$ appears once. Consequently, the exact pathway energy bound for the blossom graph is given by: $E_p(Bl_n) = 10(2n).$ □

**Observation 7.5.** The exact path energy and time complexity analysis $O(n^3)$ for blossom graphs with $n$ varying from 5 to 100 are plotted in Figure 4: (a) and (b).

### 7.3. The pathway energy upper bound for blossom graph

**Theorem 7.6.** *For blossom graph $Bl_n$ with $n \geq 5$,*
  (i)  *The upper bound of pathway energy is given by:* $E_p(Bl_n) \leq \sqrt{5l(\varrho_1 + \varrho_2)}$
  (ii) *The improved upper bound of pathway energy is:* $E_p(Bl_n) \leq \sqrt{5(5(\varrho_1 + \varrho_2))}.$

**Proof.** This section of the proof establishes the upper bounds using an analogous method of determining characteristic values as in Theorem 4.2.

**Case 1:** Proof for the path energy upper bound of blossom graph.

For deriving the upper bound Cauchy-Schwarz Inequality (1) leads that, $[E_p(Bl_n)]^2 \leq \left(\sum_{i=1}^{l} \alpha_i \beta_i\right)^2$, where the pathway energy is constrained by the right-hand side. Substituting $\alpha_i = 1$ and $\beta_i = |\mu_i|$ leads to the desired conclusion,

$$[E_p(Bl_n)]^2 \leq \left(\sum_{i=1}^{l} 1.|\mu_i|\right)^2 \leq \left(\sum_{i=1}^{l} 1\right)^2 \cdot \left(\sum_{i=1}^{l} |\mu_i|^2\right) \ldots\ldots\ldots \text{by (1)}$$
$$= 5l(\varrho_1 + \varrho_2) \ldots\ldots\ldots\ldots\ldots \text{by Theorem 7.1}$$
$$[E_p(Bl_n)] \leq \sqrt{5l(\varrho_1 + \varrho_2)}.$$

Thus, the upper bounds of the pathway energy of the blossom graph concluded by

$$E_p(Bl_n) \leq \sqrt{5l(\varrho_1 + \varrho_2)}.$$

**Case 2:** Proof for the pathway energy improved upper bound.

The Weighted Cauchy Schwarz inequality 5 allows us to deduce that,

$$[E_p(Bl_n)]^2 \leq \left(\sum_{i=1}^{l} a_i.b_i.w_i\right)^2.$$

By substituting $a_i = \frac{1}{(l)^{\frac{3}{2}}}|\mu_i|$, $b_i = 1$ and $w_i = \sqrt{5}$, in the above inequality,

$$[E_p(Bl_n)]^2 \leq \left(\sum_{i=1}^{l} \frac{\mu_i^2}{(l)^3}.\sqrt{5}\right)\left(\sum_{i=1}^{l} 1.\sqrt{5}\right)$$
$$= 5(5(\varrho_1 + \varrho_2)) \ldots\ldots\ldots\ldots\ldots \text{by Theorem 7.1}$$
$$[E_p(Bl_n)] \leq \sqrt{5(5(\varrho_1 + \varrho_2))}.$$

Thus, the improved upper bound of the pathway energy for blossom graph is, $[E_p(Bl_n)] \leq \sqrt{5(5(\varrho_1 + \varrho_2))}$. □

**Observation 7.7.** The pathway energy of upper bounds, improved upper bounds and time complexity analysis $O(n^3)$ for blossom graphs with order of $n$ varying from 5 to 100 are plotted in Figure 4: (a) and (b).

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**Algorithm 4** Bounds of a Blossom Graph with Time Complexity

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**Require:** Number of vertices $n \geq 5$
 1: **Step 1: Adjacency Matrix Setup**
 2: Initialize $A \leftarrow \text{zeros}(2n + 1, 2n + 1)$
 3: **for** $i \leftarrow 1$ to $2n + 1$ **do**
 4:     **for** $j \leftarrow 1$ to $2n + 1$ **do**
 5:         **if** $i \neq j$ **then**
 6:             $A[i, j] \leftarrow 5$
 7:         **end if** $i = j$ $A[i, j] \leftarrow 0$                                      ▷ diagonal element
 8:     **end for**
 9: **end for**
10: **Step 2: characteristic value and Energy Calculation**
11: Compute characteristic values $\mu \leftarrow \text{chval}(A)$
12: Calculate total energy $E \leftarrow \sum |\mu_i|$
13: **Step 3: Exact, Upper, and Lower Bounds Calculation**
14: Exact Bound $P \leftarrow 10 \times (2n)$
15: Sum of squared characteristic values $R \leftarrow \sum \mu_i^2$
16: Upper Bound $T \leftarrow \sqrt{(2n + 1) \times R}$
17: Lower Bound $K \leftarrow \sqrt{R + (2n + 1) \times 2n \times |\det(A)|^{\frac{2}{2n+1}}}$
18: **Step 4: Improved Upper and Lower Bound Calculation**
19: Improved Upper Bound $T_{improved} \leftarrow \sqrt{5 \times R}$
20: Improved Lower Bound $K_{improved} \leftarrow \sqrt{R + \frac{2}{(2n+1)(2n)} \times \prod_{i=1}^{2n+1} |\mu_i|^2}$
21: **Step 5: Time Complexity Analysis**
22: Time complexity of characteristic value computation is $O((2n + 1)^3)$
23: Overall time complexity:   $O(f(n))$,   where   $f(n) = (2n + 1)^3 +$ computation time
24: **Output**: Energy $E$, Bounds $P, T, K, T_{improved}, K_{improved}$, Time Complexity $O(f(n))$

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## 8. Complexity Analysis of Flower Families

We undertook a comprehensive computational analysis of flower families, employing algorithmic techniques to quantify the energy of these graphs alongside the time taken for each calculation. By plotting energy against computation time, we meticulously assessed the complexity and efficiency inherent in each graph type. Our results reveal that sunflower graphs exhibit a significantly higher complexity compared to the flower families under investigation. Notably, the computational time for sunflower graphs adheres to a complexity of $O(n^3)$, underscoring its critical nature in relation to the flower families discussed in this study.
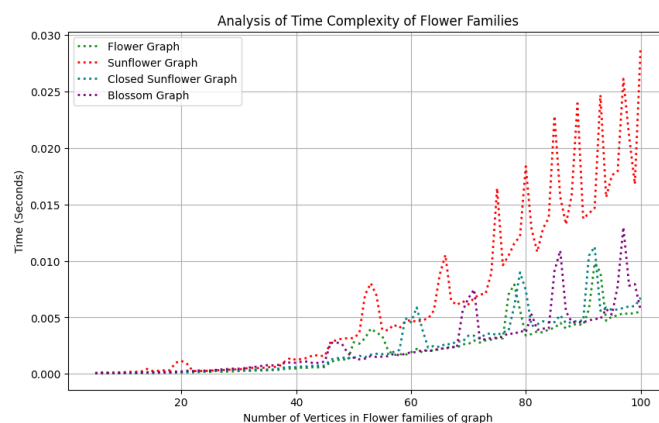
**Fig. 5.** Analysis of time complexity of flower families

# 9.  Conclusion

This paper presents an innovative algorithmic framework for analyzing pathway energy within flower families, encompassing a wide range of graph types. We established precise limits for pathway energies and conducted an in-depth analysis of time complexity, illuminating significant computational challenges in this domain. The hallmark of our work is the enhanced efficiency and versatility of our pathway energy calculations, providing a robust tool for addressing complex spectral graph problems. These findings pave the way for future research focused on optimizing algorithms and exploring new classes of graphs. Moreover, this research has substantial practical implications for network optimization, improving algorithmic efficiency, and advancing spectral graph theory. Ultimately, our contributions propel the progress of computational graph theory and its related fields.

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