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Decomposition of A Complete Equipartite Graph into Gregarious $\mathcal{Y}_5$ Tree

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ABSTRACT

A $\mathcal{Y}$ tree on k vertices is denoted by $\mathcal{Y}_k$. To decompose a graph into $\mathcal{Y}_k$ trees, it is necessary to create a collection of subgraphs that are isomorphic to $\mathcal{Y}_k$ tree and are all distinct. It is possible to acquire the necessary condition to decompose $K_m(n)$ into $\mathcal{Y}_k$ trees ($k \geq 5$), which has been obtained as $n^2m(m-1) \equiv 0 \pmod{2(k-1)}$. It has been demonstrated in this document that, a gregarious $\mathcal{Y}_5$ tree decomposition in $K_m(n)$ is possible only if $n^2m(m-1) \equiv 0 \pmod 8$.

Keywords: Decomposition, Complete equipartite graph, $\mathcal{Y}_5$ tree, Gregarious $\mathcal{Y}_5$ tree

1. Introduction

To create a $\mathcal{Y}_k$ *tree* $(v_1\ v_2\ \ldots\ v_{k-1}; v_{k-2}\ v_k)$, its edges are represented as $\{(v_1v_2, v_2v_3, \ldots, v_{k-2}v_{k-1}) \cup (v_{k-2}v_k)\}$ while the vertices are represented as $\{v_1, v_2, \ldots, v_k\}$. A $\mathcal{Y}_5$ *tree* $(v_1\ v_2\ v_3\ v_4; v_3\ v_5)$ can be seen in Figure 1.

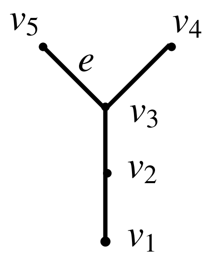

Fig. 1. $\mathcal{Y}_5$ *tree*

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The *wreath product* $(G \otimes H)$ of G and H be defined in this way: $V(G \otimes H) = \{(u, v) \mid u \in V(G), v \in V(H)\}$ and $E(G \otimes H) = \{(u, v)(x, y) \mid u = x$ and $vy \in E(H)$, or $ux \in E(G)\}$. I_r is the term used to describe the set of r vertices. *The extended graph* $(G \otimes I_r)$ of G is also a multipartite graph which is described in the following manner: $V(G \otimes I_r) = \{p_q \mid p \in V(G), q \in I_r\}$ and $E(G \otimes I_r) = \{p_q s_t \mid ps \in E(G)$ and $q, t \in I_r\}$. To make it easier for us, the extented graph is denoted by $\mathcal{E}_r(G)$. Here $K_m \otimes I_n$ is referred as the *complete equipartite graph* and is also identified by $K_m(n)$. Here, the extended graph $\mathcal{E}_r(K_m(n))$ can be considered as the extended graph $\mathcal{E}_{nr}(K_m)$, ie., $K_m(nr)$.

Decomposition of a graph G can be partitioned into subgraphs $\{G_i, 1 \le i \le n\}$, where each G_i is distinct by its edges, in addition with, the edge set of G is the union of the edge set of all subgraphs. In such a case that, if there is an isomorphism between each subgraph G_i and a graph $\mathcal{H}$, then G is said to decompose into $\mathcal{H}$.

However, a $\mathcal{Y}_5$ tree decomposition in $\mathcal{E}_r(G)$ is termed as gregarious, if for every $\mathcal{Y}_5$ tree, all its vertices are assigned to various partite sets.

Numerous authors have investigated tree decompositions and their special characteristic, in particular gregarious tree decompositions. C. Huang and A. Rosa [10] demonstrated that the complete graph K_m admits a $\mathcal{Y}_5$ tree decomposition when $m \equiv 0, 1 \pmod 8$. The study of G-decomposition of complete graphs, with G having 5 vertices, is detailed in [2]. According to the conjecture by Ringel [16], it is proposed that K_{2m+1} has been decomposed into a tree with precisely m edges. *Ja′nos Bara′t* and *Da′niel Gerbner* [1] show that 191-edge connected graph admits a $\mathcal{Y}$ tree decomposition. To know more about tree decompositions, refer [3, 4, 17, 14, 9, 12, 11, 13, 15]. A gregarious kite decomposition in $K_m \times K_n$ is demonstrated to exist by A. Tamil Elakkiya and A. Muthusamy [5], with the condition that $mn(m-1)(n-1) \equiv 0 \pmod 8$ being necessary and sufficient, where $\times$ denotes tensor product of graph. In [6], A. Tamil Elakkiya and A. Muthusamy established the conditions for a gregarious kite factorization of $K_m \times K_n$, stating that this factorization is only possible when $mn \equiv 0 \pmod 4$ and $(m-1)(n-1) \equiv 0 \pmod 2$ are present. A kite decomposition for $K_m(n)$ is gregarious is not possible unless $m \equiv 0, 1 \pmod 8$ for odd n and $m \ge 4$ for even n are present, which has been investigated in [7]. In [8], S. Gomathi and A. Tamil Elakkiya established the conditions of a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m \times K_n$, stating that this decomposition exists only if $mn(m-1)(n-1) \equiv 0 \pmod 8$ is present.

Our main concern is, to decompose a complete equipartite graph as gregarious $\mathcal{Y}_5$ trees. This paper proves that a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m(n)$ is only possible if $n^2m(m-1) \equiv 0 \pmod 8$. By the notion of a gregarious $\mathcal{Y}_5$ tree decomposition, the number of partite sets must be at least 5 ($m \ge 5$). Moreover, a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m(n)$ falls on the following cases:

(i) $m \equiv 0, 1 \pmod 8$, for all n, $n \ge 2$.

(ii) $m \equiv 5, 6, 7, 10, 11, 12 \pmod 8$, for even n.

To establish our key result, the following result is necessary:

Theorem 1.1. [10] *For $m \equiv 0, 1 \pmod 8$, a $\mathcal{Y}_5$ tree decomposition is possible in K_m.*

2. Gregarious $\mathcal{Y}_5$ tree Decomposition of $K_m(n)$

Remark 2.1. A Latin square of order r, denoted as $L = (a_{ij})$, is an $r \times r$ array where every row and every column contains only the elements $\{1, 2, 3, \ldots, r\}$ once, in which each cell a_{ij} would satisfies the arithmetic operation such as $a_{ij} = i + j - 1 \pmod r$. If $a_{ij} = a_{(i+h)(j+k)}$ and $a_{i(j+k)} = a_{(i+h)j}$,

then the set $\{a_{ij}, a_{i(j+k)}, a_{(i+h)j}, a_{(i+h)(j+k)}\}$ is called as $\mathcal{Y}$ tree cell. Here h and k are integers, which are equal to $\frac{r}{2}$, r is even. It provides the following three disjoint $\mathcal{Y}_5$ trees:
(i) $(1_{i+h}\ 2_{j+k}\ 1_i\ 3_{a_{(i+h)(j+k)}}; 1_i\ 2_j)$
(ii) $(2_{j+k}\ 3_{a_{(i+h)(j+k)}}\ 1_{i+h}\ 2_j; 1_{i+h}\ 3_{a_{i(j+k)}})$
(iii) $(3_{a_{ij}}\ 2_j\ 3_{a_{(i+h)j}}\ 2_{j+k}; 3_{a_{(i+h)j}}\ 1_i)$,
where the subscripts are considered to be divisible by r and their remainders must be taken as $1, 2, 3, \ldots, r$.

For example, let us consider the Latin square of order 4 as given in Table 1.

$\boxed{1}$	2	$\boxed{3}$	4
2	3	4	1
$\boxed{3}$	4	$\boxed{1}$	2
4	1	2	3

Table 1. Latin square of order 4 (L_4)

Here h, $k = 2$, and i, $j = 1$, so we get $a_{11} = a_{33}$ and $a_{13} = a_{31}$. Now, the $\mathcal{Y}$ tree cell $(a_{11}, a_{13}, a_{31}, a_{33})$ gives the following: $(1_3\ 2_3\ 1_1\ 3_{a_{33}}; 1_1\ 2_1)$, $(2_3\ 3_{a_{33}}\ 1_3\ 2_1; 1_3\ 3_{a_{13}})$ and $(3_{a_{11}}\ 2_1\ 3_{a_{31}}\ 2_3; 3_{a_{31}}\ 1_1)$. Then $a_{11} = a_{33} = 1$ and $a_{13} = a_{31} = 3$ implies the disjoint $\mathcal{Y}_5$ trees $(1_3\ 2_3\ 1_1\ 3_1; 1_1\ 2_1)$, $(2_3\ 3_1\ 1_3\ 2_1; 1_3\ 3_3)$ and $(3_1\ 2_1\ 3_3\ 2_3; 3_3\ 1_1)$.

Lemma 2.2. *For any $\mathcal{Y}_5$ tree, there is a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_r(\mathcal{Y}_5)$, $r \geq 2$.*

Proof. By taking $V(\mathcal{E}_r(\mathcal{Y}_5)) = \{\bigcup_{p=1}^{5} p_q, 1 \leq q \leq r\}$ and by using the latin square L of order r, the set $\{1_i\ 2_j\ 3_{a_{ij}}\ 4_i; 3_{a_{ij}}\ 5_j\}, 1 \leq i, j \leq r, r \geq 2$, provides a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_r(\mathcal{Y}_5)$. □

Lemma 2.3. *A gregarious $\mathcal{Y}_5$ tree decomposition is admissible in $\mathcal{E}_r(H)$, $r \geq 2$, if a $\mathcal{Y}_5$ tree decomposition is possible in H.*

Proof. If there is a collection $\mathcal{S}$ of $\mathcal{Y}_5$ trees in the decomposition of H, then by applying Lemma 2.2 to each $\mathcal{Y}_5 \in \mathcal{S}$, we will get a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_r(\mathcal{Y}_5)$. Consequently, we can attain a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_r(H)$, $r \geq 2$. □

Lemma 2.4. *A gregarious $\mathcal{Y}_5$ tree decomposition is admissible in $K_m(n)$, when $m \equiv 0, 1 \pmod 8$ and for every n, $n \geq 2$.*

Proof. In Theorem 1.1, stating that, $\mathcal{Y}_5$ tree decomposition is possible for K_m when $m \equiv 0, 1 \pmod 8$. Thus, according to the Lemma 2.3, we can attain a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_r(K_m)$, $r \geq 2$. □

Lemma 2.5. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_{6,6,6}, K_{8,8,8}, K_{10,10,10}, K_{12,12,12}\}$ is admissible in $\mathcal{E}_2(G)$.*

Proof. Let us consider $V(K_{2,2,2}) = \{\bigcup_{p=1}^{3} p_q, 1 \leq q \leq 2\}$. The set given below contains a $\mathcal{Y}_5$ tree decomposition for $K_{2,2,2}$: $\{(3_2\, 1_2\, 3_1\, 1_1;\, 3_1\, 2_1), (2_2\, 3_2\, 2_1\, 1_1;\, 2_1\, 1_2), (3_2\, 1_1\, 2_2\, 1_2; 2_2\, 3_1)\}$. Consequently, we may derive a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_r(K_{2,2,2})$ if $r = 3, 4, 5, 6$, according to the Lemma 2.3. That is, a gregarious $\mathcal{Y}_5$ tree decomposition exists for the graphs $\mathcal{G} = \{K_{6,6,6}, K_{8,8,8}, K_{10,10,10}, K_{12,12,12}\}$, since $\mathcal{E}_r(K_m(n)) \simeq K_m(nr)$. Moreover, by repeating the same process to each graph $G \in \mathcal{G}$, we can acquire a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(G)$. □

Lemma 2.6. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_{8,8,5},\ K_{8,8,6},\ K_{8,8,7}\}$ is admissible in $\mathcal{E}_2(G)$.*

Proof. (1) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,5})$ can be derived as follows:
Let $V(K_{8,8,5}) = \{(\bigcup_{p=1}^{2} p_q, 1 \leq q \leq 8) \cup (3_q, 1 \leq q \leq 5)\}$. By removing the entires 6, 7 and 8 from Table 2, we can attain a latin square L in Table 3. By using Table 3, we can produce a set $\mathcal{S}_1$ of $\mathcal{Y}_5$ tree decomposition for $K_{8,8,5}$ as follows:

1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	1
3	4	5	6	7	8	1	2
4	5	6	7	8	1	2	3
5	6	7	8	1	2	3	4
6	7	8	1	2	3	4	5
7	8	1	2	3	4	5	6
8	1	2	3	4	5	6	7

Table 2. Latin square of order 8 (L_8)

1	2	3	4	5	×	×	×
2	3	4	5	×	×	×	1
3	4	5	×	×	×	1	2
4	5	×	×	×	1	2	3
5	×	×	×	1	2	3	4
×	×	×	1	2	3	4	5
×	×	1	2	3	4	5	×
×	1	2	3	4	5	×	×

Table 3. Dropped the entries 6, 7, 8 from L_8

- As discussed in Remark 2.1, if $a_{ij} = a_{(i+h)(j+k)} = 5$ and $a_{i(j+k)} = a_{(i+h)j} = 1$, we get the following $\mathcal{Y}$ tree cells $\{(a_{15}, a_{11}, a_{55}, a_{51}), (a_{24}, a_{28}, a_{64}, a_{68}), (a_{33}, a_{37}, a_{73}, a_{77}), (a_{42}, a_{46}, a_{82}, a_{86})\}$. It follows that these $\mathcal{Y}$ tree cells yield 12 copies of $\mathcal{Y}_5$ trees, in which each are isomorphic and disjoint mutually.
- For all $(i, j) \in \{(1, 2), (2, 1), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3)\}$, we have $a_{ij} = 2$ and for all $(i, j) \in \{(1, 3), (2, 2), (3, 1), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4)\}$, we have $a_{ij} = 3$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{k+6-i}; 1_i\ 3_{k+1})$, if $a_{ij} = k$, $k = 2, 3$. Thus we may include these 16 disjoint copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• Similarly, for all $(i,j) \in \{(1,4),(2,3),(3,2),(4,1),(5,8),(6,7),(7,6),(8,5)\}$, $a_{ij} = k$, $k = 4$, it is possible to obtain a $\mathcal{Y}_5$ tree. We then place the 8 disjoint copies of $\mathcal{Y}_5$ trees follows from $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{j+2}; 1_i\ 3_{k-2})$ in $\mathcal{S}_1$. All together implies a $\mathcal{Y}_5$ tree decomposition for $K_{8,8,5}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,5})$ may derived through the use of Lemma 2.3.

(2) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,6})$ can be derived as follows:
Let $V(K_{8,8,6}) = \{(\bigcup_{p=1}^{2} p_q, 1 \le q \le 8) \cup (3_q, 1 \le q \le 6)\}$. By removing the entiries 7 and 8 from Table 2, we can attain a latin square L in Table 4. By using Table 4, we can produce a set $\mathcal{S}_2$ of $\mathcal{Y}_5$ tree decomposition for $K_{8,8,6}$ as follows:

1	2	3	4	5	6	×	×
2	3	4	5	6	×	×	1
3	4	5	6	×	×	1	2
4	5	6	×	×	1	2	3
5	6	×	×	1	2	3	4
6	×	×	1	2	3	4	5
×	×	1	2	3	4	5	6
×	1	2	3	4	5	6	×

Table 4. Dropped the entries 7, 8 from L_8

• As discussed in Remark 2.1, if $a_{ij} = a_{(i+h)(j+k)} = 5$ and $a_{i(j+k)} = a_{(i+h)j} = 1$, we get the following $\mathcal{Y}$ tree cells $\{(a_{15}, a_{11}, a_{55}, a_{51}), (a_{24}, a_{28}, a_{64}, a_{68}), (a_{33}, a_{37}, a_{73}, a_{77}), (a_{42}, a_{46}, a_{82}, a_{86})\}$. It follows that the $\mathcal{Y}$ tree cells yield 12 copies of $\mathcal{Y}_5$ trees, in which each are isomorphic and disjoint mutually.

• As discussed in Remark 2.1, if $a_{ij} = a_{(i+h)(j+k)} = 2$ and $a_{i(j+k)} = a_{(i+h)j} = 6$, we get the following $\mathcal{Y}$ tree cells $\{(a_{12}, a_{16}, a_{52}, a_{56}), (a_{21}, a_{25}, a_{61}, a_{65}), (a_{38}, a_{34}, a_{78}, a_{74}), (a_{47}, a_{43}, a_{87}, a_{83})\}$. It follows that the $\mathcal{Y}$ tree cells yield 12 copies of $\mathcal{Y}_5$ trees.

• For all $(i,j) \in \{(1,3),(2,2),(3,1),(4,8),(5,7),(6,6),(7,5),(8,4)\}$, $a_{ij} = k$, $k = 3$, it is possible to obtain a $\mathcal{Y}_5$ tree. We then place the 8 copies of $\mathcal{Y}_5$ trees follows from $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{k+6-i}; 1_i\ 3_{k+1})$ in $\mathcal{S}_2$.

• For all $(i,j) \in \{(1,4),(2,3),(3,2),(4,1),(5,8),(6,7),(7,6),(8,5)\}$, $a_{ij} = k$, $k = 4$, it is possible to obtain a $\mathcal{Y}_5$ tree. We then place the 8 copies of $\mathcal{Y}_5$ trees follows from $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{j+3}; 1_i\ 3_{k-1})$ in $\mathcal{S}_2$. All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{8,8,6}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,6})$ may derived through use of Lemma 2.3.

(3) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,6})$ can be derived as follows:
Let $V(K_{8,8,7}) = \{(\bigcup_{p=1}^{2} p_q, 1 \le q \le 8) \cup (3_q, 1 \le q \le 7)\}$. By removing the backword diagonal entries from Table 2, we can attain a latin square L in Table 5. By using Table 5, we can produce a set $\mathcal{S}_3$ of $\mathcal{Y}_5$ tree decomposition for $K_{8,8,7}$ as per the following:

• Consider the set of $\mathcal{Y}$ tree cells $\{(a_{ij}, a_{i(j+4)}, a_{(i+4)j}, a_{(i+4)(j+4)})\}, (i,j) \in \{(1,1),(1,2),(1,3),(2,1),(2,2)(2,4),(3,1),(3,3),(3,4),(4,2),(4,3),(4,4)\}$. It is possible to obtain a $\mathcal{Y}$ tree cell corresponding to each (i,j). All together gives 12 copies of $\mathcal{Y}$ tree cells. These $\mathcal{Y}$ tree cells provides the following 36 copies of $\mathcal{Y}_5$ trees.

(i) When $i = j$

$$1_{i+h}\ 2_{j+k}\ 1_i\ 3_{a_{(i+h)(j+k)}}; 1_i\ 3_4,\ 2_{j+k}\ 3_{a_{(i+h)(j+k)}}\ 1_{i+h}\ 3_4; 1_{i+h}\ 3_{a_{i(j+k)}},\ 3_{a_{ij}}\ 2_j\ 3_{a_{(i+h)j}}\ 2_{j+k}; 3_{a_{(i+h)j}}\ 1_i.$$

1	2	3	4	5	6	7	$\times$
2	3	4	5	6	7	$\times$	1
3	4	5	6	7	$\times$	1	2
4	5	6	7	$\times$	1	2	3
5	6	7	$\times$	1	2	3	4
6	7	$\times$	1	2	3	4	5
7	$\times$	1	2	3	4	5	6
$\times$	1	2	3	4	5	6	7

Table 5. Dropped the backword diagonal entries from L_8

(ii) When $i \neq j$

$1_{i+h}\ 2_{j+k}\ 1_i\ 3_{a_{(i+h)(j+k)}}; 1_i\ 2_j,\ 2_{j+k}\ 3_{a_{(i+h)(j+k)}}\ 1_{i+h}\ 2_j; 1_{i+h}\ 3_{a_{i(j+k)}},\ 3_{a_{ij}}\ 2_j\ 3_{a_{(i+h)j}}\ 2_{j+k}; 3_{a_{(i+h)j}}\ 1_i.$

• For all $(i,j) \in \{(1,4),(2,3),(3,2),(4,1)\}$, $a_{ij} = 4$, it is possible to obtain a $\mathcal{Y}_5$ tree. We then place the 4 copies of $\mathcal{Y}_5$ trees follows from $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{9-i}; 1_i\ 2_i)$ in $\mathcal{S}_3$.

• For all $(i,j) \in \{(5,8),(6,7),(7,6),(8,5)\}$, $a_{ij} = 4$, it is possible to obtain a $\mathcal{Y}_5$ tree. We then place the 4 copies of $\mathcal{Y}_5$ trees follows from $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{9-i}; 1_i\ 2_{i-4})$ in $\mathcal{S}_3$. All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{8,8,7}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,7})$ may derived through use of Lemma 2.3.

From Cases 1, 2 and 3, we can concluded that, a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(G)$, $G \in \mathcal{G} = \{K_{8,8,5}, K_{8,8,6}, K_{8,8,7}\}$.

□

Lemma 2.7. *A gregarious $\mathcal{Y}_5$ tree decomposition for each $G \in \mathcal{G} = \{K_{8,8,10},\ K_{8,8,11},\ K_{8,8,12}\}$ is admissible in $\mathcal{E}_2(G)$.*

Proof. **(1)** A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,10})$ can be derived as follows:

Let $V(K_{8,8,10}) = \{(\bigcup_{p=1}^{2} p_q, 1 \leq q \leq 8) \cup (3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, \infty_1, \infty_2)\}$. By using the latin square in Table 2, we can produce a set $\mathcal{S}_1$ of $\mathcal{Y}_5$ tree decomposition for $K_{8,8,10}$ as follows:

• As discussed in Remark 2.1, if $a_{ij} = a_{(i+h)(j+k)} = 5$ and $a_{i(j+k)} = a_{(i+h)j} = 1$, we get the following $\mathcal{Y}$ tree cells $\{(a_{15}, a_{11}, a_{55}, a_{51}), (a_{24}, a_{28}, a_{64}, a_{68}), (a_{33}, a_{37}, a_{73}, a_{77}), (a_{42}, a_{46}, a_{82}, a_{86})\}$. It follows that the $\mathcal{Y}$ tree cells yield 12 copies of $\mathcal{Y}_5$ trees.

• For all $(i,j) \in \{(1,2),(2,1),(3,8),(4,7),(5,6),(6,5),(7,4),(8,3)\}$, we have $a_{ij} = 2$ and for all $(i,j) \in \{(1,3),(2,2),(3,1),(4,8),(5,7),(6,6),(7,5),(8,4)\}$, we have $a_{ij} = 3$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ \infty_k; 1_i\ 3_{a_{ij}+1})$, if $a_{ij} = k+1$, $k = 1, 2$. Thus we may include these 16 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• For all $(i,j) \in \{(1,4),(2,3),(3,2),(4,1),(5,8),(6,7),(7,6),(8,5)\}$, we have $a_{ij} = 4$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ 3_{k+5}; 1_i\ 3_{k-1})$, if $a_{ij} = k+1$, $k = 3$. Thus we may include these 8 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• For all $(i,j) \in \{(1,6),(2,5),(3,4),(4,3)\}$, we have $a_{ij} = 6$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 1_i\ 2_j\ \infty_{k-3}; 2_j\ 3_{a_{ij}+1})$, if $a_{ij} = k+2$, $k = 4$. Thus we may include these 4 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• For all $(i,j) \in \{(5,2),(6,1),(7,8),(8,7)\}$, we have $a_{ij} = 6$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(2_{j+2}\ 1_i\ 2_j\ \infty_1; 2_j\ 3_{a_{ij}+1})$, if $a_{ij} = k + 2$, $k = 4$. Thus we may include these 4 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• For all $(i, j) \in \{(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1), (8, 8)\}$, we have $a_{ij} = 7$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 1_i\ 2_j\ \infty_{k-3}; 2_j\ 3_{k+3})$, if $a_{ij} = k + 2$, $k = 5$. Thus we may include these 8 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• For all $(i, j) \in \{(1, 8), (2, 7), (3, 6), (4, 5)\}$, we have $a_{ij} = 8$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(1_i\ 2_j\ 3_{(a_{ij}-2)}\ 2_i; 3_{(a_{ij}-2)}\ 1_{i+4})$, if $a_{ij} = k + 2$, $k = 6$. Thus we may include these 4 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{8,8,10}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,10})$ may derived through the use of Lemma 2.3.

(2) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,11})$ can be derived as follows:

Let $V(K_{8,8,11}) = \{(\bigcup_{p=1}^{2} p_q, 1 \leq q \leq 8) \cup (3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, \infty_1, \infty_2, \infty_3)\}$. By using the latin square in Table 2, we can produce a set $\mathcal{S}_2$ of $\mathcal{Y}_5$ tree decomposition for $K_{8,8,11}$ as follows:

• As discussed in Remark 2.1, if $a_{ij} = a_{(i+h)(j+k)} = 5$ and $a_{i(j+k)} = a_{(i+h)j} = 1$, we get the following $\mathcal{Y}$ tree cells $\{(a_{15}, a_{11}, a_{55}, a_{51}), (a_{24}, a_{28}, a_{64}, a_{68}), (a_{33}, a_{37}, a_{73}, a_{77}), (a_{42}, a_{46}, a_{82}, a_{86})\}$. It follows that the $\mathcal{Y}$ tree cells yield 12 copies of $\mathcal{Y}_5$ trees.

• For all $(i, j) \in \{(1, 2), (2, 1), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3)$, we have $a_{ij} = 2$ and for all $(i, j) \in \{(1, 3), (2, 2), (3, 1), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4)\}$, we have $a_{ij} = 3$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ \infty_k; 1_i\ 3_{k+2})$, if $a_{ij} = k + 1$, $k = 1, 2$. Thus we may include these 16 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_2$.

• For all $(i, j) \in \{(1, 4), (2, 3), (3, 2), (4, 1), (5, 8), (6, 7), (7, 6), (8, 5)\}$, we have $a_{ij} = 4$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ \infty_k; 1_i\ 3_{k-1})$, if $a_{ij} = k+1$, $k = 3$. Thus we may include these 8 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_2$.

• For all $(i, j) \in \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (7, 8), (8, 7)\}$, we have $a_{ij} = 6$ and for all $(i, j) \in \{(1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1), (8, 8)\}$, we have $a_{ij} = 7$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 1_i\ 2_j\ \infty_k; 2_j\ 3_{(a_{ij}+1)})$, if $a_{ij} = k + 5$, $k = 1, 2$. Thus we may include these 16 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_2$.

• For all $(i, j) \in \{(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1)\}$, we have $a_{ij} = 8$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 1_i\ 2_j\ \infty_k; 2_j\ 3_{(a_{ij}-2)})$, if $a_{ij} = k + 5$, $k = 3$. Thus we may include these 8 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_2$.

All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{8,8,11}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,11})$ may derived through use of Lemma 2.3.

(3) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,12})$ can be derived as follows:

Let $V(K_{8,8,12}) = \{(\bigcup_{p=1}^{2} p_q, 1 \leq q \leq 8) \cup (3_1, 3_2, 3_3, 3_4, 3_5, 3_6, 3_7, 3_8, \infty_1, \infty_2, \infty_3, \infty_4)\}$. By using the latin square L in Table 2, we can produce a set $\mathcal{S}_3$ for $\mathcal{Y}_5$ tree decomposition of $K_{8,8,12}$ as follows:

• For all $(i, j) \in \{(1, 1), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2)\}$, we have $a_{ij} = 1$, for all $(i, j) \in \{(1, 2), (2, 1), (3, 8), (4, 7), (5, 6), (6, 5), (7, 4), (8, 3)\}$, we have $a_{ij} = 2$ and for all $(i, j) \in \{(1, 3), (2, 2), (3, 1), (4, 8), (5, 7), (6, 6), (7, 5), (8, 4)\}$, we have $a_{ij} = 3$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ \infty_k; 1_i\ 3_{k+1})$, if $a_{ij} = k$, $k = 1, 2, 3$. Thus we may include these 24 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_3$.

• For all $(i, j) \in \{(1, 4), (2, 3), (3, 2), (4, 1), (5, 8), (6, 7), (7, 6), (8, 5)\}$, we have $a_{ij} = 4$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i, j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ \infty_k; 1_i\ 3_{k-3})$, if $a_{ij} = k$, $k = 4$. Thus we may include these 8 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_3$.

• For all $(i,j) \in \{(1,5),(2,4),(3,3),(4,2),(5,1),(6,8),(7,7),(8,6)\}$, we have $a_{ij} = 5$, for all $(i,j) \in \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(7,8),(8,7)\}$, we have $a_{ij} = 6$ and for all $(i,j) \in \{(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1),(8,8)\}$, we have $a_{ij} = 7$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 1_i\ 2_j\ \infty_k; 2_j\ 3_{(a_{ij}+1)})$, if $a_{ij} = k+4$, $k = 1,2,3$. Thus we may include these 24 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_3$.

• For all $(i,j) \in \{(1,8),(2,7),(3,6),(4,5),(5,4),(6,3),(7,2),(8,1),\}$, we have $a_{ij} = 8$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 1_i\ 2_j\ \infty_k; 2_j\ 3_{(k+1)})$, if $a_{ij} = k+4$, $k = 4$. Thus we may include these 8 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_3$.

All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{8,8,12}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,12})$ may derived through use of Lemma 2.3.

From Cases 1, 2 and 3, we can cocluded that, a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(G)$, $G \in \mathcal{G} = \{K_{8,8,10}, K_{8,8,11}, K_{8,8,12}\}$. □

Lemma 2.8. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_{12,12,7},\ K_{12,12,10}\}$ is admissible in $\mathcal{E}_2(G)$.*

Proof. (1) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{12,12,7})$ can be derived as follows:
Let $V(K_{12,12,7}) = \{(\bigcup_{p=1}^{2} p_q, 1 \le q \le 12) \cup (3_q, 1 \le q \le 7)\}$. By removing the entries 8, 9, 10, 11 and 12 from Table 6, we can attain a latin square L in Table 7. By using Table 7, we can produce a set $\mathcal{S}_1$ of $\mathcal{Y}_5$ tree decomposition for $K_{12,12,7}$ as follows:

1	2	3	4	5	6	7	8	9	10	11	12
2	3	4	5	6	7	8	9	10	11	12	1
3	4	5	6	7	8	9	10	11	12	1	2
4	5	6	7	8	9	10	11	12	1	2	3
5	6	7	8	9	10	11	12	1	2	3	4
6	7	8	9	10	11	12	1	2	3	4	5
7	8	9	10	11	12	1	2	3	4	5	6
8	9	10	11	12	1	2	3	4	5	6	7
9	10	11	12	1	2	3	4	5	6	7	8
10	11	12	1	2	3	4	5	6	7	8	9
11	12	1	2	3	4	5	6	7	8	9	10
12	1	2	3	4	5	6	7	8	9	10	11

Table 6. Latin square of order 12 (L_{12})

• Consider the set of $\mathcal{Y}$ tree cells $\{a_{ij}, a_{i(j+6)}, a_{(i+6)j}, a_{(i+6)(j+6)}\}$, where $(i,j) \in \{(1,1),(2,6),(3,5),(4,4),(5,3),(6,2)\}$. It is possible to obtain a $\mathcal{Y}$ tree cell corresponding to each (i,j). All together gives 6 copies of $\mathcal{Y}$ tree cells. As discussed in Remark 2.1, these $\mathcal{Y}$ tree cells will provide 18 copies of $\mathcal{Y}_5$ trees.

• For all $(i,j) \in \{(1,2),(2,1),(3,12),(4,11),(5,10),(6,9),(7,8),(8,7),(9,6),(10,5),(11,4),(12,3)\}$, we have $a_{ij} = 2$. For all $(i,j) \in \{(1,3),(2,2),(3,1),(4,12),(5,11),(6,10),(7,9),(8,8),(9,7),(10,6),(11,5),(12,4)\}$, we have $a_{ij} = 3$. For all $(i,j) \in \{(1,4),(2,3),(3,2),(4,1),(5,12),(6,11),(7,10),(8,9),(9,8),(10,7),(11,6),(12,5)\}$, we have $a_{ij} = 4$. And for all $(i,j) \in \{(1,5),(2,4),(3,3),(4,2),(5,1),$

1	2	3	4	5	6	7	×	×	×	×	×
2	3	4	5	6	7	×	×	×	×	×	1
3	4	5	6	7	×	×	×	×	×	1	2
4	5	6	7	×	×	×	×	×	1	2	3
5	6	7	×	×	×	×	×	1	2	3	4
6	7	×	×	×	×	×	1	2	3	4	5
7	×	×	×	×	×	1	2	3	4	5	6
×	×	×	×	×	1	2	3	4	5	6	7
×	×	×	×	1	2	3	4	5	6	7	×
×	×	×	1	2	3	4	5	6	7	×	×
×	×	1	2	3	4	5	6	7	×	×	×
×	1	2	3	4	5	6	7	×	×	×	×

Table 7. Dropped the entries 8, 9, 10, 11, 12 from L_{12}

$(6,12),(7,11),(8,10),(9,9),(10,8),(11,7),(12,6)\}$, we have $a_{ij}=5$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{k+7-i}; 1_i\ 3_{k+1})$, if $a_{ij}=k$, $k=2,3,4,5$. Thus we may include these 48 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$.

• For all $(i,j)\in\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(7,12),(8,11),(9,10),(10,9),(11,8),(12,7)\}$, we have $a_{ij}=6$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{k+7-i};$ $1_i\ 3_{k-4})$, if $a_{ij}=k$, $k=6$. Thus we may include these 12 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_1$. All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{12,12,7}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{12,12,7})$ may derived through use of Lemma 2.3.

(2) A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{12,12,10})$ can be derived as follows:

Let $V(K_{12,12,10})=\{(\bigcup_{p=1}^{2} p_q, 1\le q\le 12)\cup(3_q, 1\le q\le 10)\}$. By removing the entries 11 and 12 from Table 6, we can attain a latin square L in Table 8. By using Table 8, we can produce a set $\mathcal{S}_2$ of $\mathcal{Y}_5$ tree decomposition for $K_{12,12,10}$ as follows:

1	2	3	4	5	6	7	8	9	10	×	×
2	3	4	5	6	7	8	9	10	×	×	1
3	4	5	6	7	8	9	10	×	×	1	2
4	5	6	7	8	9	10	×	×	1	2	3
5	6	7	8	9	10	×	×	1	2	3	4
6	7	8	9	10	×	×	1	2	3	4	5
7	8	9	10	×	×	1	2	3	4	5	6
8	9	10	×	×	1	2	3	4	5	6	7
9	10	×	×	1	2	3	4	5	6	7	8
10	×	×	1	2	3	4	5	6	7	8	9
×	×	1	2	3	4	5	6	7	8	9	10
×	1	2	3	4	5	6	7	8	9	10	×

Table 8. Dropped the entries 11, 12 from L_{12}

• Consider the set of $\mathcal{Y}$ tree cells $\{a_{ij}, a_{i(j+6)}, a_{(i+6)j}, a_{(i+6)(j+6)}\}$, where $(i,j)\in\{(1,1),(1,2),$ $(1,3),(1,4),(2,1),(2,2),(2,3),(2,6),(3,1),(3,2),(3,5),(3,6),(4,1),(4,4),(4,5),(4,6),(5,3),(5,4),$

$(5,5), (5,6), (6,2), (6,3)(6,4), (6,5)\}$. It is possible to obtain a $\mathcal{Y}$ tree cell for each (i,j). All together gives 24 copies of $\mathcal{Y}$ tree cells. As discussed in Remark 2.1, the $\mathcal{Y}$ tree cells will provide 72 copies of $\mathcal{Y}_5$ trees.

• For all $(i,j) \in \{(1,5), (2,4), (3,3), (4,2), (5,1), (6,12), (7,11), (8,10), (9,9), (10,8), (11,7), (12,6)\}$, we have $a_{ij} = 5$. It is possible to obtain a $\mathcal{Y}_5$ tree corresponding to each (i,j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{k+7-i}; 1_i\ 3_{k+1})$, if $a_{ij} = k$, $k = 5$. Thus we may include these 12 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_2$.

• For all $(i,j) \in \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (7,12), (8,11), (9,10), (10,9), (11,8), (12,7)\}$, we have $a_{ij} = 6$. It is possible to obtain a $\mathcal{Y}_5$ tree for each (i,j), such as $(3_{a_{ij}}\ 2_j\ 1_i\ 2_{k+7-i}; 1_i\ 3_{k-1})$, if $a_{ij} = k$, $k = 6$. Thus we may include these 12 copies of $\mathcal{Y}_5$ trees in $\mathcal{S}_2$.

All together leads a $\mathcal{Y}_5$ tree decomposition for $K_{12,12,10}$. As a consequence of it, a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{12,12,10})$ may derived through use of Lemma 2.3.

From Cases 1 and 2, we can concluded that, a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(G)$, $G \in \mathcal{G} = \{K_{12,12,7}, K_{12,12,10}\}$. □

Lemma 2.9. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_{10} \setminus K_7, K_{11} \setminus K_6, K_{12} \setminus K_5, K_{13} \setminus K_5, K_{14} \setminus K_6, K_{15} \setminus K_7, K_{16} \setminus K_8, K_{19} \setminus K_{14}, K_{20} \setminus K_{13}, K_{29} \setminus K_{20}, K_{35} \setminus K_{30}\}$ is admissible in $\mathcal{E}_2(G)$.*

Proof. **(1)** Let us consider $V(K_{10} \setminus K_7) = \{1, 2, 3, \ldots, 10\}$ and

$$\mathcal{S}_1 = \{(1\ 4\ 3\ 6; 3\ 10), (2\ 6\ 1\ 8; 1\ 5), (2\ 7\ 3\ 1; 3\ 8), (7\ 1\ 2\ 4; 2\ 5), (5\ 3\ 2\ 8; 2\ 10), (10\ 1\ 9\ 2; 9\ 3)\}.$$

It follows that, the set $\mathcal{S}_1$ provides a $\mathcal{Y}_5$ tree decomposition of $K_{10} \setminus K_7$.

(2) By taking $V(K_{11} \setminus K_6) = \{1, 2, 3, \ldots, 11\}$, the set $\mathcal{S}_2$ given bellow derives a $\mathcal{Y}_5$ tree decomposition of $K_{11} \setminus K_6$.

$$\begin{aligned}\mathcal{S}_2 =& \{(1\ 2\ 8\ 4; 8\ 5), (2\ 5\ 1\ 6; 1\ 7), (3\ 4\ 9\ 5; 9\ 2), (2\ 3\ 1\ 8; 1\ 11), (2\ 6\ 4\ 10; 4\ 7),\\ & (5\ 6\ 3\ 8; 3\ 11), (7\ 2\ 10\ 5; 10\ 1), (7\ 5\ 11\ 4; 11\ 2), (3\ 5\ 4\ 2; 4\ 1), (1\ 9\ 3\ 7; 3\ 10)\}.\end{aligned}$$

(3) By taking $V(K_{12} \setminus K_5) = \{1, 2, 3, \ldots, 12\}$, a $\mathcal{Y}_5$ tree decomposition for $K_{12} \setminus K_5$ is contained with in the set $\mathcal{S}_3$ mentioned bellow.

$$\begin{aligned}\mathcal{S}_3 =& \{(11\ 1\ 12\ 5; 12\ 6), (12\ 2\ 1\ 6; 1\ 7), (1\ 3\ 2\ 7; 2\ 8), (2\ 4\ 3\ 8; 3\ 9), (3\ 5\ 4\ 9; 4\ 10),\\ & (4\ 8\ 5\ 10; 5\ 11), (1\ 4\ 12\ 7; 12\ 3), (6\ 5\ 1\ 8; 1\ 9), (3\ 6\ 2\ 9; 2\ 10), (8\ 7\ 3\ 10; 3\ 11),\\ & (7\ 6\ 4\ 11; 4\ 7), (5\ 7\ 10\ 6; 10\ 1), (8\ 6\ 9\ 7; 9\ 5), (5\ 2\ 11\ 6; 11\ 7)\}.\end{aligned}$$

(4) Let $V(K_{13} \setminus K_5) = \{1, 2, 3, \ldots, 13\}$, the collection $\mathcal{S}_4$ gives a $\mathcal{Y}_5$ tree decomposition of $K_{13} \setminus K_5$.

$$\begin{aligned}\mathcal{S}_4 =& \{(1\ 3\ 4\ 10; 4\ 13), (6\ 4\ 5\ 11; 5\ 1), (3\ 5\ 6\ 12; 6\ 8), (4\ 2\ 7\ 13; 7\ 3), (5\ 7\ 8\ 1; 8\ 3),\\ & (6\ 13\ 2\ 3; 2\ 5), (7\ 6\ 1\ 13; 1\ 10), (9\ 3\ 6\ 10; 6\ 11), (9\ 4\ 7\ 11; 7\ 12), (10\ 5\ 8\ 12; 8\ 4),\\ & (11\ 4\ 12\ 5; 12\ 3), (12\ 1\ 11\ 8; 11\ 2), (13\ 8\ 9\ 6; 9\ 1), (1\ 7\ 9\ 2; 9\ 5), (5\ 13\ 3\ 10; 3\ 11),\\ & (12\ 2\ 10\ 7; 10\ 8), (4\ 1\ 2\ 6; 2\ 8)\}.\end{aligned}$$

(5) By considering $V(K_{14} \setminus K_6) = \{1, 2, 3, \ldots, 14\}$, the set $\mathcal{S}_5$ must be a $\mathcal{Y}_5$ tree decomposition of

$K_{14} \setminus K_6$.

$$\begin{aligned}\mathcal{S}_5 =\{&(14\ 3\ 4\ 10; 4\ 1), (14\ 4\ 5\ 11; 5\ 1), (3\ 5\ 6\ 12; 6\ 14), (4\ 2\ 7\ 14; 7\ 3), (13\ 7\ 8\ 1; 8\ 3),\\ &(6\ 13\ 2\ 3; 2\ 5), (7\ 6\ 1\ 13; 1\ 10), (9\ 3\ 6\ 10; 6\ 11), (9\ 4\ 7\ 11; 7\ 12), (10\ 5\ 8\ 12; 8\ 4),\\ &(11\ 4\ 12\ 5; 12\ 3), (12\ 1\ 11\ 8; 11\ 2), (13\ 8\ 9\ 6; 9\ 1), (1\ 7\ 9\ 2; 9\ 5), (5\ 13\ 3\ 10; 3\ 11),\\ &(12\ 2\ 10\ 7; 10\ 8), (3\ 1\ 2\ 8; 2\ 14), (13\ 4\ 6\ 8; 6\ 2), (7\ 5\ 14\ 8; 14\ 1)\}.\end{aligned}$$

(6) Let $V(K_{15} \setminus K_7) = \{1, 2, 3, \ldots, 15\}$. The set $\mathcal{S}_6$ leads a $\mathcal{Y}_5$ tree decomposition of $K_{15} \setminus K_7$.

$$\begin{aligned}\mathcal{S}_6 =\{&(15\ 3\ 4\ 10; 4\ 1), (15\ 4\ 5\ 11; 5\ 10), (3\ 5\ 6\ 12; 6\ 14), (15\ 2\ 7\ 14; 7\ 3),\\ &(13\ 7\ 8\ 15; 8\ 2), (6\ 13\ 2\ 5; 2\ 14), (7\ 6\ 1\ 13; 1\ 10), (9\ 3\ 6\ 10; 6\ 15), (9\ 4\ 7\ 11; 7\ 12),\\ &(15\ 5\ 8\ 12; 8\ 4), (11\ 4\ 12\ 5; 12\ 3), (12\ 1\ 11\ 8; 11\ 6), (13\ 8\ 9\ 6; 9\ 1), (1\ 7\ 9\ 2; 9\ 5),\\ &(5\ 13\ 3\ 10; 3\ 11), (12\ 2\ 10\ 7; 10\ 8), (2\ 1\ 3\ 8; 3\ 14), (13\ 4\ 6\ 8; 6\ 2), (7\ 5\ 14\ 8; 14\ 1),\\ &(7\ 15\ 1\ 8; 1\ 5), (14\ 4\ 2\ 3; 2\ 11)\}.\end{aligned}$$

(7) By considering $V(K_{16} \setminus K_8) = \{1, 2, 3, \ldots, 16\}$, the set $\mathcal{S}_7$ provides a $\mathcal{Y}_5$ tree decomposition of $K_{16} \setminus K_8$.

$$\begin{aligned}\mathcal{S}_7 =\{&(3\ 6\ 15\ 1; 15\ 7), (15\ 5\ 4\ 8; 4\ 9), (13\ 7\ 5\ 1; 5\ 14), (13\ 3\ 2\ 16; 2\ 6),\\ &(8\ 13\ 4\ 16; 4\ 12), (14\ 1\ 6\ 16; 6\ 5), (14\ 2\ 5\ 16; 5\ 13), (14\ 4\ 3\ 16; 3\ 15), (14\ 6\ 8\ 16; 8\ 12),\\ &(11\ 7\ 1\ 16; 1\ 13), (15\ 2\ 7\ 16; 7\ 14), (9\ 3\ 8\ 1; 8\ 14), (4\ 7\ 9\ 1; 9\ 8), (10\ 6\ 9\ 2; 9\ 5),\\ &(15\ 8\ 2\ 1; 2\ 13), (5\ 8\ 10\ 1; 10\ 4), (6\ 7\ 10\ 3; 10\ 5), (11\ 5\ 3\ 1; 3\ 14), (4\ 2\ 12\ 5; 12\ 7),\\ &(10\ 2\ 11\ 1; 11\ 3), (7\ 8\ 11\ 4; 11\ 6), (13\ 6\ 4\ 1; 4\ 15), (7\ 3\ 12\ 1; 12\ 6)\}.\end{aligned}$$

(8) Let $V(K_{19} \setminus K_{14}) = \{1, 2, 3, \ldots, 19\}$. A $\mathcal{Y}_5$ tree decomposition of $K_{19} \setminus K_{14}$ belongs to $\mathcal{S}_8$.

$$\begin{aligned}\mathcal{S}_8 =\{&(3\ 18\ 5\ 4; 5\ 2), (2\ 19\ 1\ 4; 1\ 6), (3\ 1\ 2\ 6; 2\ 14), (4\ 2\ 3\ 7; 3\ 16), (5\ 6\ 4\ 3; 4\ 16),\\ &(6\ 3\ 19\ 5; 19\ 4), (7\ 1\ 12\ 3; 12\ 5), (1\ 10\ 2\ 18; 2\ 13), (16\ 2\ 9\ 4; 9\ 1), (5\ 15\ 2\ 8; 2\ 7),\\ &(5\ 11\ 4\ 10; 4\ 17), (17\ 2\ 11\ 3; 11\ 1), (4\ 7\ 5\ 8; 5\ 1), (10\ 5\ 13\ 3; 13\ 1), (1\ 16\ 5\ 9; 5\ 14),\\ &(2\ 12\ 4\ 13; 4\ 14), (5\ 3\ 15\ 4; 15\ 1), (4\ 18\ 1\ 8; 1\ 14), (10\ 3\ 17\ 1; 17\ 5), (4\ 8\ 3\ 9; 3\ 14)\}.\end{aligned}$$

(9) Let $V(K_{20} \setminus K_{13}) = \{1, 2, 3, \ldots, 20\}$. A $\mathcal{Y}_5$ tree decomposition of $K_{20} \setminus K_{13}$ is contained in $\mathcal{S}_9$.

$$\begin{aligned}\mathcal{S}_9 =\{&(3\ 6\ 15\ 1; 15\ 7), (20\ 5\ 4\ 19; 4\ 9), (13\ 7\ 5\ 1; 5\ 14), (13\ 3\ 2\ 20; 2\ 6),\\ &(6\ 13\ 4\ 18; 4\ 12), (20\ 1\ 6\ 12; 6\ 5), (8\ 2\ 5\ 16; 5\ 13), (20\ 4\ 15\ 3; 15\ 5), (19\ 6\ 7\ 16; 7\ 12),\\ &(11\ 7\ 1\ 16; 1\ 13), (15\ 2\ 7\ 19; 7\ 20), (20\ 3\ 19\ 1; 19\ 5), (18\ 7\ 9\ 1; 9\ 2), (10\ 6\ 9\ 3; 9\ 5),\\ &(16\ 3\ 18\ 5; 18\ 6), (6\ 17\ 2\ 13; 2\ 18), (4\ 10\ 1\ 12; 1\ 18), (4\ 7\ 10\ 3; 10\ 5), (11\ 5\ 3\ 1; 3\ 14),\\ &(4\ 2\ 12\ 5; 12\ 3), (10\ 2\ 11\ 1; 11\ 3), (20\ 6\ 4\ 1; 4\ 3), (7\ 3\ 8\ 1; 8\ 6), (8\ 7\ 14\ 6; 14\ 1),\\ &(8\ 4\ 16\ 2; 16\ 6), (8\ 5\ 17\ 7; 17\ 3), (6\ 11\ 4\ 14; 4\ 17), (17\ 1\ 2\ 14; 2\ 19)\}.\end{aligned}$$

(10) Let $V(K_{29} \setminus K_{20}) = \{1, 2, 3, \ldots, 29\}$. A $\mathcal{Y}_5$ tree decomposition of $K_{29} \setminus K_{20}$ has been constructed as follows:

$$\begin{aligned}\mathcal{S}_{10} =\{&(1\ 9\ 5\ 12; 5\ 17), (2\ 10\ 6\ 13; 6\ 18), (7\ 11\ 3\ 14; 3\ 19), (4\ 12\ 8\ 15; 8\ 20),\\ &(5\ 13\ 9\ 16; 9\ 21), (6\ 14\ 1\ 20; 1\ 22), (5\ 27\ 2\ 21; 2\ 23), (6\ 27\ 3\ 22; 3\ 24),\\ &(7\ 3\ 4\ 23; 4\ 25), (8\ 29\ 5\ 24; 5\ 26), (10\ 1\ 6\ 25; 6\ 28), (11\ 2\ 24\ 4; 24\ 9),\\ &(12\ 7\ 14\ 5; 14\ 4), (13\ 4\ 15\ 3; 15\ 2), (10\ 5\ 16\ 6; 16\ 3), (11\ 6\ 7\ 8; 7\ 26),\\ &(12\ 3\ 10\ 9; 10\ 8), (13\ 8\ 11\ 9; 11\ 4), (8\ 27\ 7\ 18; 7\ 29), (9\ 28\ 8\ 19; 8\ 18),\\ &(29\ 1\ 12\ 9; 12\ 2), (6\ 2\ 13\ 3; 13\ 1), (22\ 7\ 4\ 29; 4\ 26), (7\ 28\ 3\ 23; 3\ 25),\\ &(17\ 9\ 7\ 23; 7\ 20), (25\ 1\ 8\ 24; 8\ 21), (26\ 2\ 9\ 25; 9\ 22), (15\ 5\ 1\ 26; 1\ 23),\\ &(26\ 6\ 5\ 25; 5\ 4), (3\ 8\ 9\ 14; 9\ 29), (14\ 2\ 7\ 10; 7\ 17), (8\ 4\ 9\ 20; 9\ 19),\\ &(22\ 2\ 1\ 11; 1\ 21), (3\ 6\ 15\ 1; 15\ 7), (28\ 4\ 16\ 2; 16\ 8), (5\ 8\ 17\ 3; 17\ 6),\\ &(6\ 9\ 18\ 4; 18\ 1), (7\ 1\ 19\ 5; 19\ 2), (8\ 2\ 20\ 6; 20\ 3), (9\ 3\ 21\ 7; 21\ 4),\\ &(10\ 4\ 22\ 8; 22\ 5), (11\ 5\ 23\ 9; 23\ 6), (12\ 6\ 24\ 1; 24\ 7), (13\ 7\ 25\ 2; 25\ 8),\\ &(14\ 8\ 26\ 3; 26\ 9), (15\ 9\ 27\ 4; 27\ 1), (16\ 1\ 28\ 5; 28\ 2), (17\ 2\ 29\ 6; 29\ 3),\\ &(18\ 3\ 1\ 17; 1\ 4), (19\ 4\ 2\ 18; 2\ 5), (20\ 5\ 7\ 19; 7\ 16), (21\ 6\ 4\ 20; 4\ 17),\\ &(2\ 3\ 5\ 21; 5\ 18), (23\ 8\ 6\ 22; 6\ 19)\}.\end{aligned}$$

(11) Let $V(K_{35} \setminus K_{30}) = \{1, 2, 3, \ldots, 35\}$. A $\mathcal{Y}_5$ tree decomposition of $K_{35} \setminus K_{30}$ is shown bellow:

$$\begin{aligned}\mathcal{S}_{11} =\{&(3\ 18\ 5\ 1; 5\ 2), (4\ 19\ 1\ 2; 1\ 3), (5\ 20\ 2\ 3; 2\ 4), (1\ 21\ 3\ 4; 3\ 24),\\ &(2\ 22\ 4\ 13; 4\ 6), (3\ 23\ 5\ 14; 5\ 7), (4\ 24\ 1\ 15; 1\ 8), (4\ 25\ 2\ 16; 2\ 9),\\ &(1\ 26\ 3\ 19; 3\ 10), (2\ 27\ 4\ 18; 4\ 11), (3\ 28\ 5\ 25; 5\ 12), (4\ 29\ 1\ 20; 1\ 13),\\ &(5\ 30\ 2\ 13; 2\ 14), (1\ 31\ 3\ 14; 3\ 15), (2\ 32\ 4\ 15; 4\ 16), (3\ 33\ 5\ 16; 5\ 17),\\ &(4\ 34\ 1\ 17; 1\ 18), (5\ 35\ 2\ 18; 2\ 19), (3\ 17\ 2\ 23; 2\ 7), (5\ 26\ 4\ 21; 4\ 23),\\ &(1\ 27\ 5\ 22; 5\ 24), (2\ 28\ 1\ 23; 1\ 25), (3\ 29\ 2\ 24; 2\ 26), (4\ 30\ 3\ 25; 3\ 27),\\ &(5\ 31\ 4\ 28; 4\ 20), (1\ 32\ 5\ 29; 5\ 19), (2\ 33\ 1\ 30; 1\ 14), (3\ 34\ 2\ 31; 2\ 15),\\ &(4\ 35\ 3\ 32; 3\ 16), (6\ 1\ 4\ 33; 4\ 17), (20\ 3\ 5\ 34; 5\ 4), (8\ 3\ 6\ 2; 6\ 5), (9\ 4\ 7\ 3; 7\ 1),\\ &(10\ 5\ 8\ 4; 8\ 2), (11\ 1\ 9\ 5; 9\ 3), (12\ 2\ 10\ 1; 10\ 4), (13\ 3\ 11\ 2; 11\ 5),\\ &(14\ 4\ 12\ 3; 12\ 1), (3\ 22\ 1\ 16; 1\ 35), (2\ 21\ 5\ 13; 5\ 15)\}.\end{aligned}$$

From (1) - (11), we have got a $\mathcal{Y}_5$ tree decomposition for each $G \in \mathcal{G} = \{K_{10} \setminus K_7, K_{11} \setminus K_6, K_{12} \setminus K_5, K_{13} \setminus K_5, K_{14} \setminus K_6, K_{15} \setminus K_7, K_{16} \setminus K_8, K_{19} \setminus K_{14}, K_{20} \setminus K_{13}, K_{29} \setminus K_{20}, K_{35} \setminus K_{30}\}$. By applying Lemma 2.3, we can produce a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(G)$, $G \in \mathcal{G} = \{K_{10} \setminus K_7, K_{11} \setminus K_6, K_{12} \setminus K_5, K_{13} \setminus K_5, K_{14} \setminus K_6, K_{15} \setminus K_7, K_{16} \setminus K_8, K_{19} \setminus K_{14}, K_{20} \setminus K_{13}, K_{29} \setminus K_{20}, K_{35} \setminus K_{30}\}$. □

Lemma 2.10. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_i\}$ is admissible in $\mathcal{E}_2(K_i)$, for all $i \in \{5, 6, 7\}$.*

Proof. A gregarious $\mathcal{Y}_5$ tree decomposition of $\mathcal{E}_2(K_i)$, $i \in \{5,6,7\}$ have been discribed as follows:

$$\begin{aligned}\mathcal{E}_2(K_5) =\{&(1_2\ 2_2\ 3_1\ 4_2; 3_1\ 5_2) \oplus (1_1\ 2_1\ 3_2\ 4_1; 3_2\ 5_1) \oplus (2_1\ 4_2\ 1_1\ 3_2; 1_1\ 5_2) \oplus \\ &(2_2\ 4_1\ 1_2\ 3_1; 1_2\ 5_1) \oplus (1_1\ 3_1\ 2_1\ 4_1; 2_1\ 5_1) \oplus (1_2\ 3_2\ 2_2\ 4_2; 2_2\ 5_2) \oplus \\ &(2_1\ 1_2\ 5_2\ 4_1; 5_2\ 3_2) \oplus (2_2\ 1_1\ 5_1\ 4_2; 5_1\ 3_1) \oplus (2_1\ 5_2\ 4_2\ 3_2; 4_2\ 1_2) \oplus \\ &(2_2\ 5_1\ 4_1\ 3_1; 4_1\ 1_1)\}.\end{aligned}$$

$$\begin{aligned}\mathcal{E}_2(K_6) =\{&(1_2\ 2_1\ 6_2\ 3_2; 6_2\ 4_1) \oplus (1_1\ 6_2\ 3_1\ 2_1; 3_1\ 5_1) \oplus (3_2\ 5_2\ 2_2\ 4_1; 2_2\ 6_2) \oplus \\ &(1_2\ 5_1\ 3_2\ 2_2; 3_2\ 4_2) \oplus (2_1\ 5_2\ 3_1\ 4_1; 3_1\ 1_2) \oplus (2_1\ 4_2\ 5_2\ 6_2; 5_2\ 1_2) \oplus (4_1\ 5_1\ 2_2\ 6_1; 2_2\ 1_1) \oplus \\ &(2_1\ 6_1\ 1_1\ 4_1; 1_1\ 3_1) \oplus (2_2\ 4_2\ 5_1\ 6_1; 5_1\ 1_1) \oplus (3_2\ 4_1\ 5_2\ 6_1; 5_2\ 1_1) \oplus (5_1\ 2_1\ 1_1\ 4_2; 1_1\ 3_2) \oplus \\ &(3_1\ 2_2\ 1_2\ 4_2; 1_2\ 6_1) \oplus (3_1\ 6_1\ 4_1\ 1_2; 4_1\ 2_1) \oplus (3_1\ 4_2\ 6_2\ 5_1; 6_2\ 1_2) \oplus (4_2\ 6_1\ 3_2\ 2_1; 3_2\ 1_2)\}.\end{aligned}$$

$$\begin{aligned}\mathcal{E}_2(K_7) =\{&(3_1\ 2_1\ 1_1\ 6_1; 1_1\ 4_2) \oplus (4_2\ 7_1\ 6_1\ 5_1; 6_1\ 1_2) \oplus (2_1\ 7_2\ 6_2\ 5_1; 6_2\ 4_1) \oplus \\ &(1_2\ 7_1\ 6_2\ 5_2; 6_2\ 2_2) \oplus (1_2\ 2_1\ 6_1\ 5_2; 6_1\ 4_1) \oplus (2_1\ 7_1\ 5_1\ 3_1; 5_1\ 4_2) \oplus (1_1\ 7_2\ 5_2\ 4_2; 5_2\ 2_1) \oplus \\ &(2_2\ 4_2\ 6_2\ 3_1; 6_2\ 1_2) \oplus (7_1\ 3_2\ 5_1\ 4_1; 5_1\ 2_2) \oplus (2_2\ 7_1\ 5_2\ 4_1; 5_2\ 1_1) \oplus (7_1\ 4_1\ 1_1\ 2_2; 1_1\ 5_1) \oplus \\ &(3_2\ 5_2\ 1_2\ 2_2; 1_2\ 4_2) \oplus (4_2\ 3_2\ 7_2\ 6_1; 7_2\ 5_1) \oplus (2_2\ 5_2\ 3_1\ 1_2; 3_1\ 4_1) \oplus (4_1\ 2_1\ 3_2\ 6_1; 3_2\ 1_2) \oplus \\ &(4_2\ 2_1\ 6_2\ 3_2; 6_2\ 1_1) \oplus (2_1\ 5_1\ 1_2\ 7_2; 1_2\ 4_1) \oplus (4_2\ 7_2\ 3_1\ 1_1; 3_1\ 6_1) \oplus (4_2\ 6_1\ 2_2\ 3_2; 2_2\ 7_2) \oplus \\ &(1_1\ 3_2\ 4_1\ 7_2; 4_1\ 2_2) \oplus (1_1\ 7_1\ 3_1\ 2_2; 3_1\ 4_2)\}.\end{aligned}$$

□

Lemma 2.11. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_i\}$ is admissible in $\mathcal{E}_2(K_i)$, for all $i \in \{10,11,12,13,14,15,18\}$.*

Proof. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_i)$, $i \in \{10,11,12,13,14,15,18\}$ can be derived as follows:

(1) Let $K_{10} = K_7 \oplus K_{10} \setminus K_7$. We then write $\mathcal{E}_2(K_{10}) = \mathcal{E}_2(K_7) \oplus \mathcal{E}_2(K_{10} \setminus K_7)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{10} \setminus K_7)$ and $\mathcal{E}_2(K_7)$ have been respectively derived in Lemmas 2.9 and 2.10. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{10})$.

(2) Let $K_{11} = K_6 \oplus K_{11} \setminus K_6$. We then write $\mathcal{E}_2(K_{11}) = \mathcal{E}_2(K_6) \oplus \mathcal{E}_2(K_{11} \setminus K_6)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{11} \setminus K_6)$ and $\mathcal{E}_2(K_6)$ have been respectively derived in Lemmas 2.9 and 2.10. Hence, we concluded that, $\mathcal{E}_2(K_{11})$ has a gregarious $\mathcal{Y}_5$ tree decomposition.

(3) Let $K_{12} = K_5 \oplus K_{12} \setminus K_5$. We then write $\mathcal{E}_2(K_{12}) = \mathcal{E}_2(K_5) \oplus \mathcal{E}_2(K_{12} \setminus K_5)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{12} \setminus K_5)$ and $\mathcal{E}_2(K_5)$ have been respectively derived in Lemmas 2.9 and 2.10. Our conclusion was that, $\mathcal{E}_2(K_{12})$ has a gregarious $\mathcal{Y}_5$ tree decomposition.

(4) Let $K_{13} = K_5 \oplus K_{13} \setminus K_5$. We then write $\mathcal{E}_2(K_{13}) = \mathcal{E}_2(K_5) \oplus \mathcal{E}_2(K_{13} \setminus K_5)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{13} \setminus K_5)$ and $\mathcal{E}_2(K_5)$ have been respectively derived in Lemmas 2.9 and 2.10. A gregarious $\mathcal{Y}_5$ tree decomposition is obtained for $\mathcal{E}_2(K_{13})$.

(5) Let $K_{14} = K_6 \oplus K_{14} \setminus K_6$. We then write $\mathcal{E}_2(K_{14}) = \mathcal{E}_2(K_6) \oplus \mathcal{E}_2(K_{14} \setminus K_6)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{14} \setminus K_5)$ and $\mathcal{E}_2(K_6)$ have been respectively derived in Lemmas 2.9 and 2.10. Consequently, $\mathcal{E}_2(K_{14})$ is decomposed into a gregarious $\mathcal{Y}_5$ tree.

(6) Let $K_{15} = K_7 \oplus K_{15} \setminus K_7$. We then write $\mathcal{E}_2(K_{15}) = \mathcal{E}_2(K_7) \oplus \mathcal{E}_2(K_{15} \setminus K_7)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{15} \setminus K_7)$ and $\mathcal{E}_2(K_7)$ have been respectively derived in Lemmas 2.9 and 2.10. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{15})$.

(7) Let $K_{18} = 3K_6 \oplus K_{6,6,6}$. We then write $\mathcal{E}_2(K_{18}) = 3\ \mathcal{E}_2(K_6) \oplus \mathcal{E}_2(K_{6,6,6})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{6,6,6})$ and $\mathcal{E}_2(K_6)$ have been respectively derived in Lemmas 2.5 and 2.10. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{18})$.

For $\mathcal{E}_2(K_i)$, $i \in \{10, 11, 12, 13, 14, 15, 18\}$, a gregarious $\mathcal{Y}_5$ tree decomposition was obtained from (1) - (7). □

Lemma 2.12. *A gregarious $\mathcal{Y}_5$ tree decomposition for each graph $G \in \mathcal{G} = \{K_i\}$ is admissible in $\mathcal{E}_2(K_i)$, for all $i \in \{19, 20, 29, 30, 31, 34, 35, 36\}$.*

Proof. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_i)$, $i \in \{19, 20, 29, 30, 31, 34, 35, 36\}$ can be derived as follows:

(1) Let $K_{19} = K_{14} \oplus K_{19} \setminus K_{14}$. We then write $\mathcal{E}_2(K_{19}) = \mathcal{E}_2(K_{14}) \oplus \mathcal{E}_2(K_{19} \setminus K_{14})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{19} \setminus K_{14})$ and $\mathcal{E}_2(K_{14})$ have been respectively derived in Lemmas 2.9 and 2.11. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{19})$.

(2) Let $K_{20} = K_{13} \oplus K_{20} \setminus K_{13}$. We then write $\mathcal{E}_2(K_{20}) = \mathcal{E}_2(K_{13}) \oplus \mathcal{E}_2(K_{20} \setminus K_{13})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{20} \setminus K_{13})$ and $\mathcal{E}_2(K_{13})$ have been respectively derived in Lemmas 2.9 and 2.11. Hence, we concluded that, $\mathcal{E}_2(K_{20})$ has a gregarious $\mathcal{Y}_5$ tree decomposition.

(3) Let $K_{29} = K_{20} \oplus K_{29} \setminus K_{20}$. We then write $\mathcal{E}_2(K_{29}) = \mathcal{E}_2(K_{20}) \oplus \mathcal{E}_2(K_{29} \setminus K_{20})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{29} \setminus K_{20})$ and $\mathcal{E}_2(K_{20})$ have been respectively derived in Lemma 2.9 and in the above Case 2. Our conclusion was that, $\mathcal{E}_2(K_{29})$ has a gregarious $\mathcal{Y}_5$ tree decomposition.

(4) Let $K_{30} = 3K_{10} \oplus K_{10,10,10}$. We then write $\mathcal{E}_2(K_{30}) = 3\ \mathcal{E}_2(K_{10}) \oplus \mathcal{E}_2(K_{10,10,10})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{10})$ and $\mathcal{E}_2(K_{10,10,10})$ have been respectively derived in Lemmas 2.11 and 2.5. A gregarious $\mathcal{Y}_5$ tree decomposition is obtained for $\mathcal{E}_2(K_{30})$.

(5) Let $K_{31} = 2K_{12} \oplus K_7 \oplus K_{12,12,7}$. We then write $\mathcal{E}_2(K_{31}) = 2\ \mathcal{E}_2(K_{12}) \oplus \mathcal{E}_2(K_7) \oplus \mathcal{E}_2(K_{12,12,7})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_7)$, $\mathcal{E}_2(K_{12})$ and $\mathcal{E}_2(K_{12,12,7})$ have been respectively derived in Lemmas 2.10, 2.11 and 2.8. Consequently, $\mathcal{E}_2(K_{31})$ is decomposed into a gregarious $\mathcal{Y}_5$ tree.

(6) Let $K_{34} = 2K_{12} \oplus K_{10} \oplus K_{12,12,10}$. We then write $\mathcal{E}_2(K_{34}) = 2\ \mathcal{E}_2(K_{12}) \oplus \mathcal{E}_2(K_{10}) \oplus \mathcal{E}_2(K_{12,12,10})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{10})$, $\mathcal{E}_2(K_{12})$ and $\mathcal{E}_2(K_{12,12,10})$ have been derived in Lemmas 2.11 and 2.8. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{34})$.

(7) Let $K_{35} = K_{30} \oplus K_{35} \setminus K_{30}$. We then write $\mathcal{E}_2(K_{35}) = \mathcal{E}_2(K_{30}) \oplus \mathcal{E}_2(K_{35} \setminus K_{30})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{35} \setminus K_{30})$ and $\mathcal{E}_2(K_{30})$ have been respectively derived in Lemma 2.9 and in the above Case 4. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{35})$.

(8) Let $K_{36} = 3K_{12} \oplus K_{12,12,12}$. Then, we write $\mathcal{E}_2(K_{36}) = 3\ \mathcal{E}_2(K_{12}) \oplus \mathcal{E}_2(K_{12,12,12})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{12})$ and $\mathcal{E}_2(K_{12,12,12})$ have been respectively derived in Lemmas 2.11 and 2.5. A gregarious $\mathcal{Y}_5$ tree decomposition has been found for $\mathcal{E}_2(K_{36})$.

For $\mathcal{E}_2(K_i)$, $i \in \{19, 20, 29, 30, 31, 34, 35, 36\}$, a gregarious $\mathcal{Y}_5$ tree decomposition was obtained from (1) - (8). $\square$

Note 2.13. Further, in order to prove $K_m(n)$ has a gregarious $\mathcal{Y}_5$ tree decomposition when $m \equiv 5, 6, 7, 10, 11, 12 \pmod 8$ and n is even, it is enough to prove that $K_m(2)$ admits a gregarious $\mathcal{Y}_5$ tree decomposition. It is clearly stated in Lemma 2.3.

Lemma 2.14. *A gregarious $\mathcal{Y}_5$ tree decomposition is admissible in $K_m(2)$ when $m \equiv a \pmod 8$, $a \in \{5, 6, 7, 10, 11, 12\}$.*

Proof. Consider the graph $K_m(2) \simeq \mathcal{E}_2(K_m)$ and let $m = 8s + a$, $a \in \{5, 6, 7, 10, 11, 12\}$.

A non negative integer s can be categorized into 4 Cases: (i) $s \equiv 0, 2 \pmod 6$, (ii) $s \equiv 4 \pmod 6$, (iii) $s \equiv 1, 5 \pmod 6$ and (iv) $s \equiv 3 \pmod 6$.

Case i: For $s \equiv 0, 2 \pmod 6$, the graph K_m decomposes as a copy of K_a, s copies of K_8, $\frac{s}{2}$ copies of $K_{8,8,a}$ and $\frac{s^2-2s}{6}$ copies of $K_{8,8,8}$. Therefore, $\mathcal{E}_2(K_m) = \mathcal{E}_2(K_a \oplus s\,K_8 \oplus \frac{s}{2}\,K_{8,8,a} \oplus \frac{s^2-2s}{6}\,K_{8,8,8}) = \mathcal{E}_2(K_a) \oplus s\,\mathcal{E}_2(K_8) \oplus \frac{s}{2}\,\mathcal{E}_2(K_{8,8,a}) \oplus \frac{s^2-2s}{6}\,\mathcal{E}_2(K_{8,8,8})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_a)$ and $\mathcal{E}_2(K_8)$ have been obtained from the Lemmas 2.10, 2.11 and 2.4. Further more, the Lemma 2.5 has yielded a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,8})$. Also, the Lemmas 2.6 and 2.7 have been used to obtain a gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,a})$. Consequently, it proves that a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(K_m)$.

Case ii: For $s \equiv 4 \pmod 6$, the graph K_m decomposes as a copy of K_a, $s-4$ copies of K_8, $\frac{s}{2}$ copies of $K_{8,8,a}$, $\frac{s^2-2s-8}{6}$ copies of $K_{8,8,8}$ and 4 copies of $K_{16} \setminus K_8$. Therefore, $\mathcal{E}_2(K_m) = \mathcal{E}_2(K_a \oplus (s-4)\,K_8 \oplus \frac{s}{2}\,K_{8,8,a} \oplus \frac{s^2-2s-8}{6}\,K_{8,8,8} \oplus 4\,(K_{16} \setminus K_8)) = \mathcal{E}_2(K_a) \oplus (s-4)\,\mathcal{E}_2(K_8) \oplus \frac{s}{2}\,\mathcal{E}_2(K_{8,8,a}) \oplus \frac{s^2-2s-8}{6}\,\mathcal{E}_2(K_{8,8,8}) \oplus 4\,\mathcal{E}_2(K_{16} \setminus K_8)$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,8})$ have been obtained from the Lemma 2.5. Further more, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{8,8,a})$ from the Lemmas 2.6 and 2.7. Also, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{16} \setminus K_8)$ from the Lemma 2.9. In addition with, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_a)$ and $\mathcal{E}_2(K_8)$ from the Lemmas 2.10, 2.11 and 2.4. Consequently, it proves that a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(K_m)$.

Case iii: For $s \equiv 1, 5 \pmod 6$, the graph K_m decomposes as a copy of K_{a+8}, $\frac{s-1}{2}$ copies of K_{16}, $\frac{s-1}{2}$ copies of $K_{8,8,a}$ and $\frac{s^2-3s+2}{6}$ copies of $K_{8,8,8}$. Therefore, $\mathcal{E}_2(K_m) = \mathcal{E}_2(K_{a+8} \oplus \frac{s-1}{2}\,K_{16} \oplus \frac{s-1}{2}\,K_{8,8,a} \oplus \frac{s^2-3s+2}{6}\,K_{8,8,8}) = \mathcal{E}_2(K_{a+8}) \oplus \frac{s-1}{2}\,\mathcal{E}_2(K_{16}) \oplus \frac{s-1}{2}\,\mathcal{E}_2(K_{8,8,a}) \oplus \frac{s^2-3s+2}{6}\,\mathcal{E}_2(K_{8,8,8})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,8})$ have been obtained from the Lemma 2.5. Further more, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{8,8,a})$ from the Lemmas 2.6 and 2.7. Also, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{a+8})$ from the Lemmas 2.11 and 2.12. In addition with, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{16})$ from the Lemma 2.4. Consequently, it proves that a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(K_m)$.

Case iv: For $s \equiv 3 \pmod 6$, the graph K_m decomposes as a copy of K_{a+24}, $\frac{s-3}{2}$ copies of K_{16}, $\frac{s-3}{2}$ copies of $K_{8,8,a}$ and $\frac{s^2-3s}{6}$ copies of $K_{8,8,8}$. Therefore, $\mathcal{E}_2(K_m) = \mathcal{E}_2(K_{a+24} \oplus \frac{s-3}{2}\,K_{16} \oplus \frac{s-3}{2}\,K_{8,8,a} \oplus \frac{s^2-3s}{6}\,K_{8,8,8}) = \mathcal{E}_2(K_{a+24}) \oplus \frac{s-3}{2}\,\mathcal{E}_2(K_{16}) \oplus \frac{s-3}{2}\,\mathcal{E}_2(K_{8,8,a}) \oplus \frac{s^2-3s}{6}\,\mathcal{E}_2(K_{8,8,8})$. A gregarious $\mathcal{Y}_5$ tree decomposition for $\mathcal{E}_2(K_{8,8,8})$ have been obtained from the Lemma 2.5. Further more, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{8,8,a})$ from the Lemmas 2.6 and 2.7. Also, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{a+24})$ from the Lemma 2.12. In addition with, a gregarious $\mathcal{Y}_5$ tree decomposition have been obtained for $\mathcal{E}_2(K_{16})$ from the Lemma 2.4. Consequently, it proves that a gregarious $\mathcal{Y}_5$ tree decomposition exists for $\mathcal{E}_2(K_m)$.

The Cases (i) - (iv) mentioned earlier will provide a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m(2)$. □

Theorem 2.15. *The occurence of a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m(n)$ is possible only if $n^2m(m-1) \equiv 0 \pmod 8$.*

Proof. *Necessity:* Given that $\mid E(K_m(n)) \mid = \frac{n^2m(m-1)}{2}$ and $\mid E(\mathcal{Y}_5) \mid = 4$. To determine a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m(n)$, the necessary condition for edge divisibility has been expressed as $\frac{|E(K_m(n)|}{|E(\mathcal{Y}_5)|} = \frac{n^2m(m-1)}{2\times 4}$. That is, $8|n^2m(m-1)$. It can be written as $n^2m(m-1) \equiv 0 \pmod 8$.

Sufficiency: The occurence of a gregarious $\mathcal{Y}_5$ tree decomposition for $K_m(n)$ has been described in Lemmas 2.4 and 2.14. □

3. Conclusion

In this document, we present a complete and detailed solution to the problem of identifying the presence of a gregarious $\mathcal{Y}_5$ tree decomposition in $K_m(n)$. Decomposing $K_m(n)$ into a $\mathcal{Y}_k$ tree (gregarious $\mathcal{Y}_k$ tree) is generally a challanging task for $k \geq 6$.

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