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e-Injective Coloring: 2-Distance and Injective Coloring Conjectures

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ABSTRACT

An injective coloring of a given graph $G = (V, E)$ is a vertex coloring of G such that any two vertices with common neighbor receive distinct colors. An e-injective coloring of a graph G is a vertex coloring of G in which, any two vertices v, u with common edge e ($e \neq uv$) receive distinct colors, in other words, any two end vertices of a path P_4 of G achieve different colors. With this new definition, we want to take a review at injective coloring of a graph from the new point of view. For this purpose, we review the conjectures raised so far in the literature of injective coloring and 2-distance coloring, from the new approach, e-injective coloring. As well, we prove that, for disjoint graphs G, H, with $E(G) \neq \emptyset$ and $E(H) \neq \emptyset$, $\chi_{ei}(G \cup H) = max\{\chi_{ei}(G), \chi_{ei}(H)\}$ and $\chi_{ei}(G \vee H) = |V(G)|+|V(H)|$. The e-injective chromatic number of G versus of the maximum degree and packing number of G is investigated, and we denote $max\{\chi_{ei}(G), \chi_{ei}(H)\} \leq \chi_{ei}(G \square H) \leq \chi_2(G)\chi_2(H)$. Finally, we prove that, for any tree T (T is not a star), $\chi_{ei}(T) = \chi(T)$, and we obtain the exact value of e-injective chromatic number of some specified graphs.

Keywords: Injective coloring conjecture, 2-distance coloring conjecture, e-injective coloring

1. Introduction

Graph coloring has many applications in various fields of life, such as timetabling, scheduling daily life activities, scheduling computer processes, registering allocations to different institutions and libraries, manufacturing tools, printed circuit testing, routing and wavelength, bag rationalization for a food manufacturer, satellite range scheduling, and frequency assignment. These are many applications that are out there right now and many more come in the follow.

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A proper k-coloring (hereafter k-coloring) of a graph G is a function $f : V(G) \to \{1, 2, 3, \dots, k\}$ such that for all edge $xy \in E$, $f(x) \neq f(y)$. The chromatic number of G, denoted by $\chi(G)$, is the minimum integer k such that G has a k-coloring. There are many research works on the graph coloring parameters that is not possible to name all of them here. For instance see [4, 17].

In 1969, Kramer and Kramer introduced the notion of 2-*distance coloring* of a graph G or the usual proper coloring of G^2 [9], we can some of its applications in [10]. A 2-*distance k-coloring* of a graph G is a function $f : V \to \{1, 2, 3, \dots, k\}$, such that no pair of vertices at distance at most 2, receive the same color, in the other words, the colors of the vertices of all P_3 paths in the graph are distinct. The 2-distance chromatic number of G, denoted by $\chi_2(G) = \chi(G^2)$, is the minimum positive integer k such that G has a 2-distance k-coloring. The 2-distance coloring of G, is a proper coloring [3, 2, 5].

For a graph G, the subset S of $V(G)$ is said to be a dominating set if any vertex $x \in V \setminus S$ is adjacent to a vertex y in S. A dominating set of G with minimum cardinality is called the domination number of G and is denoted by $\gamma(G)$. A subset D of $V(G)$ is said to be 2-distance dominating set if any vertex $d \in V \setminus D$, is in at most 2-distance of to a vertex in D. A 2-distance dominating set of G with minimum cardinality is called the 2-distance domination number of G and is denoted by $\gamma_2(G)$.

The injective coloring was first introduced in 2002 by Hahn et al. [6] and it was also further studied in [1, 8, 11, 12, 15, 16, 18]. An injective k-coloring of a graph G is a function $f : V \to \{1, 2, 3, \dots, k\}$ such that no vertex v is adjacent to two vertices u and w with $f(u) = f(w)$, in the other words, for any path $P_3 = xyz$, we have $f(x) \neq f(z)$. The injective chromatic number of G, denoted by $\chi_i(G)$, is the minimum positive integer k such that G has an injective k-coloring. The injective chromatic number of the hypercube has important applications in the theory of error-correcting codes. As it is well known, the injective coloring of G, is not necessarily proper coloring. Injective coloring of a graph G is related to the usual coloring of the square G^2. The inequality $\chi_i(G) \leq \chi_2(G)$ obviously holds.

There are several results related to injective coloring that review the usual coloring results in graph theory from a new point of view, in particular on the injective chromatic number of planar graphs. As well, many conjectures on planar graphs have been posed and studied by authors so far. In this regard, we bring up some of them as follows.

From the relation between the injective coloring of a graph G and the usual coloring of the square G^2, Wegner [19] posed the following conjecture.

Conjecture 1.1. [19] *Let G be a planar graph with maximum degree Δ. Then,*

1. $\chi(G^2) \leq 7$ *if* $\Delta = 3$,
2. $\chi(G^2) \leq \Delta + 5$ *if* $4 \leq \Delta \leq 7$,
3. $\chi(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ *otherwise.*

Lužar and Škrekovski in [11] showed that:

Theorem 1.2. ([11] Theorem 2.1) *There exist planar graphs G of maximum degree $\Delta \geq 3$ satisfying the following,*

1. $\chi_i(G) = 5$ *if* $\Delta = 3$,
2. $\chi_i(G) = \Delta + 5$ *if* $4 \leq \Delta \leq 7$,
3. $\chi_i(G) = \lfloor \frac{3}{2}\Delta \rfloor + 1$ *if* $\Delta \geq 8$.

Adapted from Theorem 1.2, they proposed the following Wegner type conjecture for the injective chromatic number of planar graphs.

Conjecture 1.3. [11] *Let G be a planar graph with maximum degree Δ. Then,*
(i). $\chi_i(G) \leq 5$ *if* $\Delta = 3$,
(ii). $\chi_i(G) \leq \Delta + 5$ *if* $4 \leq \Delta \leq 7$,
(iii). $\chi_i(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ *otherwise.*

By the relation between injective chromatic number and 2-distance chromatic number of a graph; showing the truth of Conjecture 1.1 (parts (2), (3)), will deduce the truth of Conjecture 1.3 (parts (ii), (iii)).

Now we introduce a new concept of vertex coloring (near to injective coloring) as an e-injective coloring of a graph. The motivation of the alleging is to study, how it behaves against of the injective graph coloring, usual graph coloring, 2-distance graph coloring, packing set, dominating set and 2-distance dominating set of graphs. As well, in particular we investigate the posed conjectures from the point of view of e-injective colorings. Also since the notion of e-injective coloring is near to injective coloring, one can predict, it has applications in various fields of life in real world and would also be useful in coding theory as so did injective coloring.

This concept is introduced in next definition.

Hereafter, we say that, two vertices u, v have a common edge neighbor if there exist an edge e in which, u is adjacent to one end vertex of e and v is adjacent to another end vertex of e.

Definition 1.4. Let G be a graph. A function $f : V(G) \rightarrow \{1, 2, 3, \ldots, k\}$ is an e-injective k-coloring function if any two vertices u and v are the ends of a path $P_4 = uxyv$ in G, then $f(u) \neq f(v)$.

The *e-injective chromatic number* of G, denoted by $\chi_{ei}(G)$, is the minimum positive integer k such that G has an e-injective k-coloring.

The e-injective coloring of G, is not necessarily proper coloring. This concept can be expressed as new discourse.

Definition 1.5. For a given graph G, *the three-step graph* $S_3(G) = G^{(3)}$ of a graph G is the graph having the same vertex set as G with an edge joining two vertices in $S_3(G)$ if and only if there is a path of length 3 between them in G.

Taking into account, the fact that a vertex subset S is independent in $S_3(G)$ if and only if there is no path of length 3 between any two vertices corresponding of S in G, we can readily observe that:

$$\chi_{ei}(G) = \chi(S_3(G)).$$

From the point of view of e-injective coloring, the type of Conjectures 1.1 and 1.3 can be declared as a problem, which can be argued in next section.

Problem 1.6. *Let G be a planar graph with maximum degree Δ. Then,*
1. $\chi_{ei}(G) \leq 5$ *if* $\Delta = 3$,
2. $\chi_{ei}(G) \leq \Delta + 5$ *if* $4 \leq \Delta \leq 7$,
3. $\chi_{ei}(G) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$ *otherwise.*

In the sequence, we assume that all graphs in this paper are finite, simple, and undirected. We use [4, 20] as a reference for terminology and notation which are not explicitly defined here. Throughout the paper, we consider $G = (V, E)$ be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The open neighborhood of a vertex $v \in V$ is the set $N(v) = \{u|uv \in E\}$, and its closed neighborhood is the set $N[v] = N(v) \cup \{v\}$. The cardinality of $|N(v)|$ is called the degree of v, denoted by $\deg(v)$. The minimum degree of G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. A vertex v of degree 1 is called a pendant vertex or a leaf, and its neighbor is called a support vertex. A vertex of degree $n-1$ is called a full or universal vertex while a vertex of degree 0 is called an isolated vertex.

For any two vertices u and v of G, we denote by $d_G(u, v)$ the distance between u and v, that is the length of a shortest path joining u and v. The path, cycle and complete graph with n vertices are denoted by P_n, C_n and K_n respectively. The complete bipartite graph with n and m vertices in their partite sets is denoted by $K_{n,m}$, while the wheel graph with $n+1$ vertices is denoted by W_n. A star graph with $n+1$ vertices, denoted by S_n, consists of n leaves and one support vertex. A double star graph is a graph consisting of the union of two star graphs S_m and S_n, with one edge joining their support vertices; the double star graph with $m+n+2$ vertices is denoted by $S_{m,n}$.

The join of two graphs G and H, denoted by $G \vee H$, is the graph obtained from the disjoint union of G and H with vertex set $V(G \vee H) = V(G) \cup V(H)$ and edge set $E(G \vee H) = E(G) \cup E(H) \cup \{xy \mid x \in V(G),\ y \in V(H)\}$. A fan graph is a simple graph consisting of joining $\overline{K}_m$ and P_n; the fan graph with $m+n$ vertices is denoted by $F_{m,n}$. For two sets of vertices X and Y, the set $[X, Y]$ denotes the set of edges $e = uv$ such that $u \in X$ and $v \in Y$.

The square graph G^2 is a graph with the same vertex set as G and with its edge set given by $E(G^2) = \{uv|\ dist(u, v) \leq 2\}$. The chromatic number $\chi(G^2)$ of G^2 (or 2-distance chromatic number $\chi_2(G)$ of G) has been studied extensively in planar graph [7, 14].

A subset $B \subseteq V(G)$ is a packing set in G if for every pair of distinct vertices $u, v \in B$, $N_G[u] \bigcap N_G[v] = \emptyset$. The packing number $\rho(G)$ is the maximum cardinality of a packing set in G.
A subset $B \subseteq V(G)$ is an open packing set in G if for every pair of distinct vertices $u, v \in B$, $N_G(u) \bigcap N_G(v) = \emptyset$. The open packing number $\rho^\circ(G)$ is the maximum cardinality among all open packing sets in G.

Let G and H be simple graphs. For three standard products of graphs G and H, the vertex set of the product is $V(G) \times V(H)$ and their edge set is defined as follows:

- In the *Cartesian product* $G \Box H$, two vertices are adjacent if they are adjacent in one coordinate and equal in the other.
- In the *direct product* $G \times H$, two vertices are adjacent if they are adjacent in both coordinates.
- The edge set of the *strong product* $G \boxtimes H$, is the union of $E(G \Box H)$ and $E(G \times H)$.

In the end of this section, we explore the purpose of the paper as follows. In Section 2, we study $\chi_{ei}(G)$ versus to the $\chi(G)$, $\chi_2(G)$ and $\chi_i(G)$, as well, we review the conjectures raised so far in the literature of injective and 2-distance colorings, from the new approach, e-injective coloring, and by disproving the Problem 1.6, we show that the Conjectures 1.1 and 1.3 maybe wrong under the conditions. For disjoint graphs G, H, with non-empty edge sets, $\chi_{ei}(G \cup H) = max\{\chi_{ei}(G), \chi_{ei}(H)\}$ and $\chi_{ei}(G \vee H) = |V(G)|+|V(H)|$. The e-injective chromatic number of G versus of the maximum degree and packing number of G is investigated, and denote $max\{\chi_{ei}(G), \chi_{ei}(H)\} \leq \chi_{ei}(G \Box H) \leq \chi_2(G)\chi_2(H)$ in Section 3. In Section 4, we prove that, for any tree T $(T \neq S_n)$, $\chi_{ei}(T) = \chi(T)$, and we obtain the exact value of e-injective chromatic number of some special graphs and finally, we end the paper with discussion on research problems.

2. On the Two Conjectures

We maybe cannot compare $\chi_{ei}(G)$ with $\chi(G)$, $\Delta(G)$, $\chi_i(G)$ or $\chi_2(G)$ in general. For instance, $\chi_{ei}(K_3) = 1$ while $\chi(K_3) = 3 = \chi_i(K_3) = \chi_2(G)$; or $\chi_{ei}(K_{1,n}) = 1$ while $\chi(K_{1,n}) = 2$, $\chi_i(K_{1,n}) = n$ and $\chi_2(K_{1,n}) = n + 1$. But in the Figure 1, $\chi_{ei}(G) = 12$, $\chi_i(G) \leq 7$, $\chi_2(G) \leq 7$, $\chi(G) = 3$.

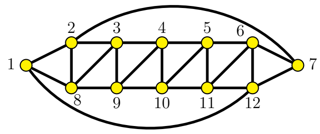

Fig. 1. The graph G with $\chi_{ei}(G) \geq max\{\chi_i(G), \chi_2(G), \chi(G)\}$

Also let $H = K_m \circledcirc K_n$ be a graph obtain from two complete graphs K_m and K_n $(m \geq n \geq 4)$ with joining one vertex of K_m to one vertex of K_n. Then $\chi(H) = m = \chi_i(H)$, $\chi_2(H) = m + 1$ and $\chi_{ei}(H) = m + n - 1$ (see Proposition 4.4). On the other hand, for Complete graph K_n $(n \geq 4)$, odd cycle C_{3k} for odd $k \geq 1$, we have $\chi_{ei}(K_n) = n = \chi(K_n) = \chi_i(K_n) = \chi_2(K_n)$, and $\chi_{ei}(C_{3k}) = 3 = \chi(C_{3k}) = \chi_i(C_{3k}) = \chi_2(C_{3k})$.

In the same way, we have $\Delta(K_3) = 2$, $\Delta(H) = m$, and for graph G in Figure 1, $\Delta(G) = 4$, while $\chi_{ei}(K_3) = 1$, $\chi_{ei}(H) = m + n - 1$, and $\chi_{ei}(G) = 12$. On the other hand, for graph $K_n \circledcirc K_1$ $(n \geq 4)$ and even cycle C_{2k}, we have $\chi_{ei}(K_n \circledcirc K_1) = n = \Delta(K_n \circledcirc K_1)$, and $\chi_{ei}(C_{2k}) = 2 = \Delta(C_{2k})$ (see Proposition 4.3).

However we have the following.

Proposition 2.1. *Let G be a graph in which any two adjacent vertices be the end vertices of a path P_4 in G. Then $\chi(G) \leq \chi_{ei}(G)$.*

Conversely, if any two end vertices of each path P_4 in G are adjacent, then $\chi_{ei}(G) \leq \chi(G)$.

Proof. Since any two adjacent vertices of given graph G are the end vertices of a path P_4 in G, these two vertices must be colored with distinct colors by any e-injective coloring. Therefore, any e-injective coloring for this graph can be a usual coloring. Then $\chi(G) \leq \chi_{ei}(G)$.

Conversely, by the construction of graph G, usual coloring of G deduces that any two end vertices of each path P_4 has different colors. Thus, the usual coloring of given G is an e-injective coloring. Therefore, $\chi_{ei}(G) \leq \chi(G)$. □

Proposition 2.2. *Let G be a graph and v in $V(G)$ be any vertex. If every two vertices in $N(v)$ are the end vertices of a path P_4, then $\chi_i(G) \leq \chi_{ei}(G)$.*

Conversely, if end vertices of each path P_4 in a graph G have a common neighbor, then $\chi_{ei}(G) \leq \chi_i(G)$.

Proof. Since any two vertices of graph G are the end vertices of a path P_4, these vertices have distinct colors by e-injective coloring. Therefore, an e-injective coloring for this graph is an injective coloring. Then $\chi_i(G) \leq \chi_{ei}(G)$.

Conversely, let any two end vertices of path P_4 have a common neighbor. Then by injective coloring of G, two end vertices of any path P_4 take different colors. Thus, this will be an e-injective coloring of G and $\chi_{ei}(G) \leq \chi_i(G)$. □

Also, we may have.

Proposition 2.3. *Let G be a graph with the property that, for any two adjacent vertices or two vertices with a common neighbor are the end vertices of a path P_4 in G. Then $\chi_2(G) \leq \chi_{ei}(G)$.*

Conversely, if G is a graph and any two end vertices of each path P_4 in G are adjacent or have a common neighbor, then $\chi_{ei}(G) \leq \chi_2(G)$.

Proof. Let v, u, w be three vertices of P_3 in G. Since both of them are the end vertices of a path P_4 in G, then e-injective coloring of the given graph G assign three distinct colors to v, u, w. This implies that, this coloring is a 2-distance coloring of G. Thus, $\chi_2(G) \leq \chi_{ei}(G)$.

Conversely, let any two end vertices of each path P_4 are adjacent or have a common neighbor. Then, from a 2-distance coloring of G, we deduce that, any two end vertices of the path P_4 are vertices of a P_2 or a P_3 in graph G and so their colors are distinct. Therefore this coloring is an e-injective coloring of the given graph G and $\chi_{ei}(G) \leq \chi_2(G)$. □

Now we discuss on the Problem 1.6. Below figures denote that the Problem 1.6 is not necessarily true. On the other hand the type of Conjectures 1.1, 1.3 are not true for e-injective coloring. But if we use the Propositions 2.2, 2.3, then maybe characterize graphs G in which, satisfy on the Conjectures 1.1, 1.3 and also characterize graphs G in which, the Conjectures 1.1, 1.3 are not true for them.

Disprove of Problem 1.6

Let G be a planar graph with maximum degree Δ. Then we present counterexample that denote the Problem 1.6 is not true.

As we observe the Figure 2, for $(3 \leq \Delta \leq 8)$ we have.

The graph M_3 denotes a planar graph in which $\Delta(M_3) = 3$ and $\chi_{ei}(M_3) = 6 > 5$.

The graph M_4 denotes a planar graph in which $\Delta(M_4) = 4$ and $\chi_{ei}(M_4) = 12 > \Delta(M_4) + 5$.

The graph M_5 denotes a planar graph in which $\Delta(M_5) = 5$ and $\chi_{ei}(M_5) = 13 > \Delta(M_5) + 5$.

The graph M_6 denotes a planar graph in which $\Delta(M_6) = 6$ and $\chi_{ei}(M_6) = 16 > \Delta(M_6) + 5$.

The graph M_7 denotes a planar graph in which $\Delta(M_7) = 7$ and $\chi_{ei}(M_7) = 16 > \Delta(M_7) + 5$.

For $\Delta(G) = 8$, consider the graph M_8 of order 16, which is seen $\Delta(G) = 8$ and $\chi_{ei}(M_n) = 16 > \lfloor \frac{24}{2} \rfloor + 1 = 13 = \lfloor \frac{3\Delta}{2} \rfloor + 1$.

For $\Delta(G) \geq 9$ consider, the graph M_n $(n \geq 9)$ of order $2n$, Figure 2, which is seen $\Delta(G) = n$ and $\chi_{ei}(M_n) = 2n > \lfloor \frac{3n}{2} \rfloor + 1 = \lfloor \frac{3\Delta}{2} \rfloor + 1$.

With this regard, the Problem 1.6 is disproved. In the other words the type of Conjecture 1.3 for e-injective coloring is rejected. However, from the Propositions 2.2 and 2.3, we can have.

Proposition 2.4. 1. *Let G be a graph with the property that the given data in Proposition, 2.3 (Conversely part) hold. Then, the Conjecture 1.1 is wrong.*

2. *Let G be a graph with the property that the given data in Proposition, 2.2 (Conversely part) hold. Then, the Conjecture 1.3 is wrong.*

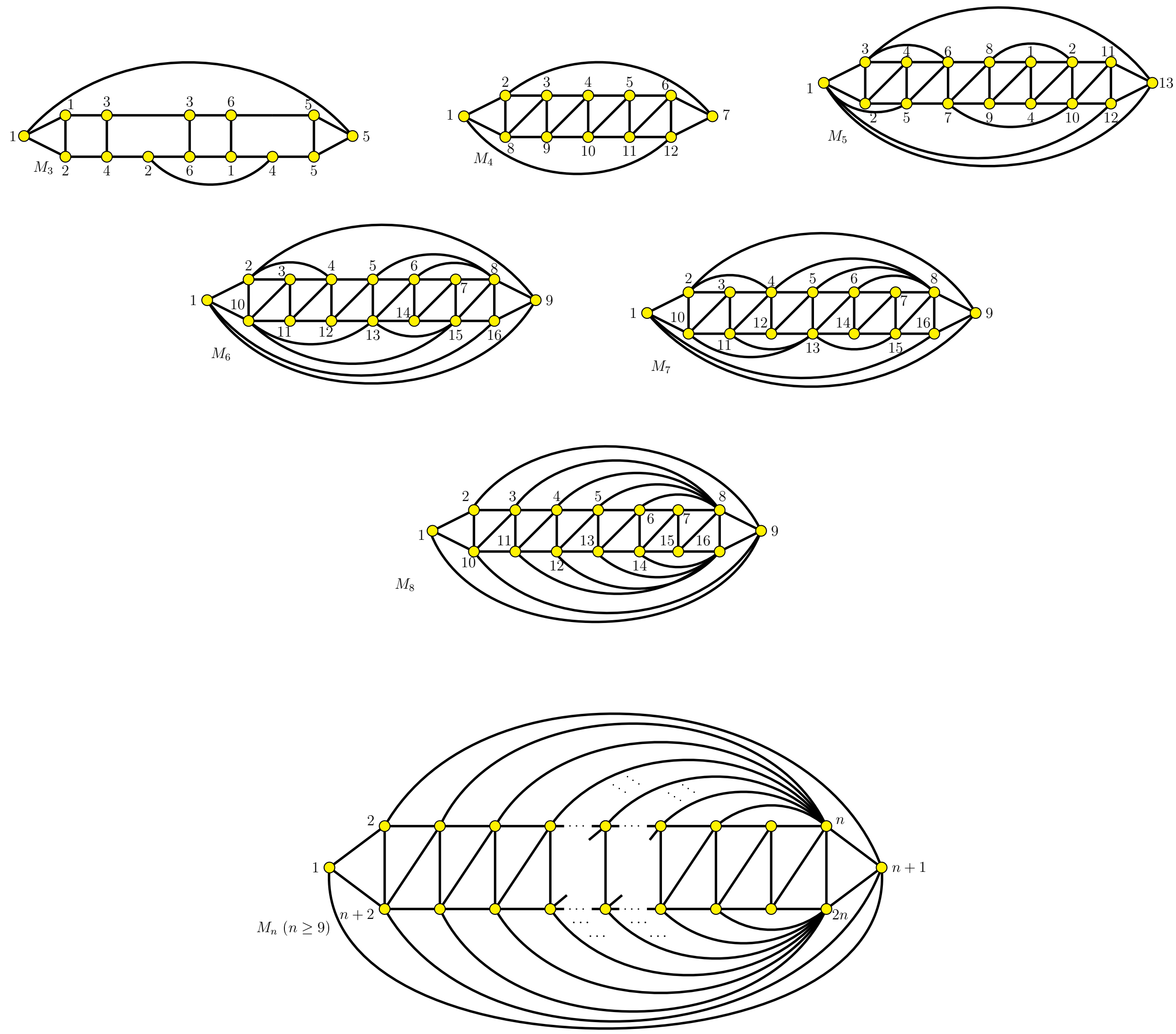

Fig. 2. The graphs G related to Problem 1.6 for $\Delta \geq 3$

3. e-Injective Chromatic Number on Operation of Graphs

In this section we prove some results on e-injective coloring using some operations.

For graphs G and H, let $G \cup H$ be the disjoint union of G and H. Then it is easy to see that $\chi_{ei}(G \cup H) = max\{\chi_{ei}(G), \chi_{ei}(H)\}$.

For the join of two graphs, we have the following.

Theorem 3.1. *Let G and H be two graphs of order m and n respectively, with the property that, $E(G)$ and $E(H)$ are non-empty sets. Then $\chi_{ei}(G \vee H) = m + n$*

Proof. Let $e_1 = v_1w_1 \in E(G)$ and $e_2 = v_2w_2 \in E(H)$ be two edges. We show that any two vertices x, y in $G \vee H$, there is a path of length 3, with end vertices x, y. For observing the result, we bring up five positions.

1. For $x, y \in V(G)$, consider the path xv_2w_2y in $G \vee H$.

2. For $x, y \in V(H)$, consider the path xv_1w_1y in $G \vee H$.
3. For $x \in V(G) \setminus \{v_1, w_1\}$ and $y \in V(H) \setminus \{v_2, w_2\}$, consider the path xv_2v_1y in $G \vee H$.
4. For $x \in \{v_1, w_1\}$, say v_1 and $y \in V(H) \setminus \{v_2, w_2\}$, consider the path $v_1v_2w_1y$ in $G \vee H$.
5. For $x \in \{v_1, w_1\}$ and $y \in \{v_2, w_2\}$ and without loss of generality, say $x = v_1$ and $y = v_2$, consider the path $v_1w_1w_2v_2$ in $G \vee H$.

The other positions are similar. Therefore, for any two vertices $x, y \in G \vee H$ there is a path of length 3, with end vertices x, y. Therefore the result is observed. □

Let G be a graph and B be a maximum packing set of G. If $v \in V(G) \setminus B$, then there is a vertex $u \in B$ such that $N(v) \cap N(u) \neq \emptyset$. This shows that, $d(v, u) \leq 2$. Thus B is a 2-distance dominating set. Therefore we have.

Theorem 3.2. *Let G be a graph of diameter 3. Then $\chi_{ei}(G) \geq \rho(G) \geq \gamma_2(G)$. One can have the equalities.*

Proof. Let B be maximum packing set of graph G. Since two vertices of B has distance 3, they are assigned with two distinct colors. Thus $\chi_{ei}(G) \geq \rho$.
For equalities, consider the cycles C_6 and C_8, (see Propositions 4.3). □

We now give an upper bound on $\chi_{ei}(G)$ that may be slightly important.

Theorem 3.3. *Let G have maximum degree Δ. Then, $\chi_{ei}(G) \leq \Delta(\Delta-1)^2+1$. This bound is sharp for odd cycle C_n $(n \geq 5)$.*

Proof. Let G be a graph and $v \in V(S_3(G)) = V(G)$. It is well known that there are at most $\Delta(\Delta-1)^2$ vertices in G such that any of them with v form two end vertices of path P_4. This shows that $\deg_{S_{3(G)}}(v) \leq \Delta(\Delta-1)^2$. On the other hand $\chi_{ei}(G) = \chi(S_3(G))$ and from Brooks Theorem in usual coloring of graphs, $\chi(S_3(G)) \leq \Delta(S_3(G))+1 \leq \Delta(\Delta-1)^2+1$. Therefore $\chi_{ei}(G) \leq \Delta(\Delta-1)^2+1$. For seeing the sharpness observe Proposition 4.3. □

Also we want to drive bounds for the e-injective coloring of Cartesian product of two graphs G, H in terms of 2-distance coloring of the of G and H. For this we explore a result from [13] and a lemma.

Theorem 3.4. ([13] Theorem 1) *For any graphs G and H with no isolated vertices,*

$$(\Delta(G)+1)(\Delta(H)+1) \leq \chi_2(G \boxtimes H) \leq \chi_2(G)\chi_2(H).$$

Lemma 3.5. *Let G and H be two graphs with no isolated vertices. If two end vertices of each path P_4 in G and H are adjacent or have a common neighbor, then so does $G \boxtimes H$.*

Proof. Suppose that the end vertices of each path P_4 in graphs G and H are adjacent or have a common neighbor. We would to be show any two end vertices of a path P_4 in $G \boxtimes H$ are adjacent or have a common neighbor. For this, we can bring up the possible paths P_4 in graph $G \boxtimes H$.

1.1. $(a,u)(a,v)(a,w)(a,t)$; 1.2. $(a,u)(a,v)(a,w)(b,w)$; 1.3. $(a,u)(a,v)(a,w)(b,t)$.
2.1. $(a,u)(a,v)(b,v)(b,w)$; 2.2. $(a,u)(a,v)(b,v)(c,v)$; 2.3 $(a,u)(a,v)(b,v)(c,w)$.

3.1. $(a,u)(a,v)(b,w)(b,t)$; 3.2. $(a,u)(a,v)(b,w)(c,w)$; 3.3. $(a,u)(a,v)(b,w)(c,t)$.
4.1. $(a,u)(b,u)(b,v)(b,w)$; 4.2. $(a,u)(b,u)(b,v)(c,v)$; 4.3. $(a,u)(b,u)(b,v)(c,w)$.
5.1. $(a,u)(b,u)(c,u)(c,v)$; 5.2. $(a,u)(b,u)(c,u)(d,u)$; 5.3. $(a,u)(b,u)(c,u)(d,v)$.
6.1. $(a,u)(b,u)(c,v)(c,w)$; 6.2. $(a,u)(b,u)(c,v)(d,v)$; 6.3. $(a,u)(b,u)(c,v)(d,w)$.
7.1. $(a,u)(b,v)(b,w)(b,t)$; 7.2. $(a,u)(b,v)(b,w)(c,w)$; 7.3. $(a,u)(b,v)(b,w)(c,t)$.
8.1. $(a,u)(b,v)(c,v)(c,w)$; 8.2. $(a,u)(b,v)(c,v)(d,v)$; 8.3. $(a,u)(b,v)(c,v)(d,w)$.
9.1. $(a,u)(b,v)(c,w)(d,w)$; 9.2. $(a,u)(b,v)(c,w)(c,t)$; 9.3. $(a,u)(b,v)(c,w)(d,t)$.

Now we observe that, all these paths type P_4 are adjacent or have a common neighbor.

1.1. Since $uvwt$ is a path P_4 in H, the vertices u and t are adjacent or have a common neighbor. If u and t are adjacent, then the vertices (a,u) and (a,t) are adjacent in $G \boxtimes H$. If u and t have a common neighbor s, then (a,s) is a common neighbor of (a,u) and (a,t).

1.2. (a,v) is a common neighbor of (a,u) and (b,w) in $G \boxtimes H$.

1.3. The $uvwt$ is a path P_4 in H. If u and t are adjacent, then the vertices (a,u) and (b,t) are adjacent in $G \boxtimes H$. If u and t have a common neighbor s, then (a,s) is a common neighbor of (a,u) and (b,t).

2.1. The vertex (b,v) is a common neighbor of (a,u) and (b,w) in $G \boxtimes H$.

2.2. The vertex (b,v) is a common neighbor of (a,u) and (c,v) in $G \boxtimes H$.

2.3. The vertex (b,v) is a common neighbor of (a,u) and (c,w) in $G \boxtimes H$.

3.1. Its proof is readily and similar to the proof of 1.3.

3.2. The vertex (b,v) is a common neighbor of (a,u) and (c,w) in $G \boxtimes H$.

3.3. Its proof is readily, and is similar to the proof of 1.3.

4.1. The vertex (b,v) is a common neighbor of (a,u) and (b,w) in $G \boxtimes H$.

4.2. The vertex (b,v) is a common neighbor of (a,u) and (c,v) in $G \boxtimes H$.

4.3. The vertex (b,v) is a common neighbor of (a,u) and (c,w) in $G \boxtimes H$.

5.1. The vertex (b,u) is a common neighbor of (a,u) and (c,v) in $G \boxtimes H$.

5.2. Since $abcd$ is a path P_4 in G, the vertices a and d are adjacent or have a common neighbor. If a and d are adjacent, then the vertices (a,u) and (d,u) are adjacent in $G \boxtimes H$. If a and d have a common neighbor r, then (r,u) is a common neighbor of (a,u) and (d,u).

5.3. Its proof is obvious and it is similar to the proof of 5.2.

6.1. The vertex (b,v) is a common neighbor of (a,u) and (c,v) in $G \boxtimes H$.

6.2. Its proof is obvious and it is similar to the proof of 5.2.

6.3. Its proof is obvious and it is similar to the proof of 5.2.

7.1. Its proof is similar to the proof of 1.3.

7.2. The vertex (b,v) is a common neighbor of (a,u) and (c,w) in $G \boxtimes H$.

7.3. Its proof is similar to the proof of 1.3.

8.1. The vertex (b,v) is a common neighbor of (a,u) and (c,w) in $G \boxtimes H$.

8.2. Its proof is similar to the proof of 5.2.

8.3. Its proof is similar to the proof of 5.2.

9.1. Its proof is similar to the proof of 5.2.

9.2. Its proof is similar to the proof of 1.3.

9.3. There are two paths $abcd$ and $uvwt$ in G and H respectively. If $ad \in E(G)$ and $ut \in E(H)$, then (a,d) and (u,t) are adjacent in $G \boxtimes H$. If $ad \in E(G)$ and s is a common neighbor of u and t, then (a,s) is a common neighbor of (a,u) and (d,t) in $G \boxtimes H$. If $ut \in E(H)$ and r is a common neighbor of a and d, then (r,u) is a common neighbor of (a,u) and (d,t) in $G \boxtimes H$. If a and d have a common neighbor r, and similarly, s is a common neighbor of u and t, then (r,s) is a common neighbor of (a,u) and (d,t) in $G \boxtimes H$.

It is observed that, both end vertices of every path P_4 in $G \boxtimes H$ are adjacent or have a common neighbor. Therefore the proof is complete. □

Now we have the following.

Theorem 3.6. *For any graphs G and H with no isolated vertices, with the property that, any two end vertices of each path P_4 in G and H are adjacent or have a common neighbor, we have*

$$Max\{\chi_{ei}(G), \chi_{ei}(H)\} \leq \chi_{ei}(G \square H) \leq \chi_2(G)\chi_2(H).$$

The bounds are sharp.

Proof. For the first inequality, since G and H have no isolated vertices, and any path of length 3 of G and H gives at least one path of length 3 of $G \square H$, thus the first inequality holds. For seeing the sharpness, consider $G = P_m$ and $H = P_n$ where $m \geq 4$ or $n \geq 4$.

We now prove the second inequality. From the definitions of Cartesian and strong products, we may have $G \square H$ as a subgraph of $G \boxtimes H$, and next any path P_4 of $(G \square H)$ is a path P_4 of $(G \boxtimes H)$. Therefore, $\chi_{ei}(G \square H) \leq \chi_{ei}(G \boxtimes H)$. As the same way, $\chi_2(G \square H) \leq \chi_2(G \boxtimes H)$. From Proposition 2.3 and Lemma 3.5 $\chi_{ei}(G \boxtimes H) \leq \chi_2(G \boxtimes H)$. On the other hand, from Theorem 3.4, $\chi_2(G \boxtimes H) \leq \chi_2(G)\chi_2(H)$. These deduce that $\chi_{ei}(G \square H) \leq \chi_2(G)\chi_2(H)$. It is easy to see that, this bound is sharp for $G = C_3$ and $H = C_5$ and also $G = C_3$ and $H = C_7$. □

4. e-Injective Chromatic Number of Special Graphs

In this section we investigate the e-injective coloring of some special graphs, such as trees, path, cycle, complete graphs, wheel graphs, star, complete bipartite graphs, k-regular bipartite graphs, multipartite graphs and fan graphs.

Theorem 4.1. *Let T be a tree. Then,*

1. $\chi_{ei}(T) = 1$ *if and only if* $diam(T) \leq 2$.
2. $\chi_{ei}(T) = 2$ *if and only if* $diam(T) \geq 3$.

Proof. 1. If $diam(T) = 1$, then $T = P_2$ and if $diam(T) = 2$, then T is a star and since there is no path of length 3 between any two vertices, $\chi_{ei}(T) = 1$.

Conversely, let $\chi_{ei}(T) = 1$. Then there is no path of length 3 in T. Thus $diam(T) \leq 2$.

2. Let $diam(T) \geq 3$ and v_0 be a vertex of maximum degree in T. We assign color 1 to the v_0 and to the vertex u if $d(u, v_0)$ is even, and color 2 to the vertex u if $d(u, v_0)$ is odd. Since there is only one path between any two vertices in any tree T, so if two vertices x, y are in distance 3 and two vertices x, z are in distance 3, then two vertices y, z are not in distance 3. This shows that, we can use color 1 for x and color 2 for y, z. Therefore $\chi_{ei}(T) \leq 2$. On the other hand, if $diam(T) \geq 3$, then $\chi_{ei}(T) \geq 2$. Therefore, if $diam(T) \geq 3$, then $\chi_{ei}(T) = 2$.

Conversely, if $\chi_{ei}(T) = 2$, there is two vertices v, w in T so that the path $vxyw$ is of length 3 in T. This shows that $diam(T) \geq 3$. □

As an immediate from Theorem 4.1 we have.

Proposition 4.2. *For Path P_n, we have*

$$\chi_{ei}(P_n) = \begin{cases} 1 & n \leq 3, \\ 2 & n \geq 4. \end{cases}$$

Proposition 4.3. *For cycle C_n, we have*

$$\chi_{ei}(C_n) = \begin{cases} 1 & n = 3, \\ 2 & n \geq 4, \ 2|n, \\ 3 & n \geq 4, \ 2 \nmid n. \end{cases}$$

Proof. Let $n = 3$. It is obvious that $\chi_{ei}(C_3) = 1$.

Let $n \geq 4$. There are two cases to be considered.

Case 1. If n is even.

We assign the color 1 to the odd vertices and color 2 to the even vertices. Therefore $\chi_{ei}(C_{2k}) = 2$.

Case 2. If n is odd.

Let $n = 5$. We assign the color 1 to the vertices v_1, v_2 and color 2 to the vertices v_3, v_4 and we assign color 3 to the vertex v_5. This assignments is an e-injective coloring of C_5.

Let $n \geq 7$. We assign the color 1 to the odd vertices v_is, for $i \leq n-4$ and color 2 to the even vertices v_is, for $i \leq n-3$ and we assign color 3 to the vertices v_{n-2}, v_{n-1}, v_n. This assignments is an e-injective coloring of C_n for odd $n \geq 7$. On the other hand, since there are two paths of length 3 between v_{n-2} with two vertices v_1, v_{n-5}, as well as v_{n-1} with two vertices v_2, v_{n-4} and also v_n with two vertices v_3, v_{n-3}. It is clear that, one cannot e-injective color to the vertices cycle C_n withe two colors for odd n. Therefore the result holds. □

Since for $n \geq 4$, any two vertices of K_n are end of a path P_4, then we have.

Proposition 4.4. *For complete graph K_n, we have*

$$\chi_{ei}(K_n) = \begin{cases} 1 & n \leq 3, \\ n & n \geq 4. \end{cases}$$

Proposition 4.5. *For wheel graph $W_n (n \geq 3)$, $\chi_{ei}(W_n) = n + 1$.*

Proof. Let v_1 be a universal vertex. For $i, j \geq 2$, there exists a path $v_j v_{j+1} v_1 v_i$ between v_i and v_j if $v_i \neq v_{j+1}$; and there exists a path $v_j v_1 v_{j+2} v_i$ between v_i and v_j if $v_i = v_{j+1}$. On the other hand, there exists a path $v_1 v_{i-2} v_{i-1} v_i$ between v_1 and v_i. Taking this account, there exist a path of length 3 between two vertices in W_n. Therefore $\chi_{ei}(W_n) = n + 1$. □

Proposition 4.6. *For complete bipartite graph $K_{n,m}$ with $m, n \geq 2$, $\chi_{ei}(K_{n,m}) = 2$.*

Proof. Let $n \geq 2$, $m \geq 2$. It is easy to observe that, there is a path of length 3 between two vertices of two different partite sets. Therefore one can assign color 1 to one partite set and 2 to another partite set. Thus, $\chi_{ei}(K_{n,m}) = 2$. □

Using the proof of Proposition 4.6, For regular bipartite graphs, we have the next result, which proof is similar.

Proposition 4.7. *For k-regular bipartite graph G ($k \geq 2$), $\chi_{ei}(G) = 2$.*

A complete r partite graph is a simple graph such that the vertices are partitioned to r independent vertex sets and every pair of vertices are adjacent if and only if they belong to different partite sets.

Proposition 4.8. *Let $G = K_{n_1,\dots,n_r}$ be a complete ($r \geq 3$) partite graph of order n. Then*

$$\chi_{ei}(G) = \begin{cases} 1, & r = 3 \text{ and } G = K_{1,1,1}, \\ n-1, & r = 3 \text{ and } G \in \{K_{n-2,1,1}, K_{1,n-2,1}, K_{1,1,n-2}\} \text{ with } n \geq 4, \\ n, & \textit{otherwise}. \end{cases}$$

Proof. 1. It is trivial.

2. Let $r = 3$ and $G \in \{K_{n-2,1,1}, K_{1,n-2,1}, K_{1,1,n-2}\}$. Without loss of generality, assume that $G = K_{n-2,1,1}$ with vertex set $V = \{v_1, v_2, \dots, v_{n-2}, u_1, w_1\}$. The path $v_i u_1 w_1 v_j$ is a path of length 3 between v_i, v_j. The paths $u_1 v_i w_1 v_j$ and $w_1 v_i u_1 v_j$ are paths of length 3 between u_1, v_j and between w_1, v_j respectively. On the other hand, there exists no path of length 3 from u_1 and w_1. Now we can assign same color to u_1, w_1 and $n-2$ other different colors to the v_is. Thus $\chi_{ei}(G) = n-1$.

3. If $r \geq 4$, and v_i, w_j are two vertices of two partite sets, then taking two vertices from two other partite sets x_m, y_l, one can construct a path v_i, x_m, y_l, w_j of length 3.

Let $r = 3$ and $G = K_{k,l,m}$ where two of k, l, m are at least 2. If $k = 1$ and $l, m \geq 2$ with $V(G) = \{v_1, u_1 \dots, u_l, w_1, \dots, w_m\}$, then $v_1 u_i w_s u_j$, $v_1 w_i u_s w_j$, $u_i v_1 u_j w_s$, $u_i w_t v_1 u_j$, $w_i u_x v_1 w_j$ with $i \neq j$, give us a path of length 3 between v_1, u_j, v_1, w_j, u_i, w_s, u_i, u_j and w_i, w_j respectively.

Let $r = 3$ and $G = K_{k,l,m}$ where $k, l, m \geq 2$. Then, similar to the second part of situation 3, there exist a path of length 3 between any two vertices of G. Therefore $\chi_{ei}(G) = n$. Thus the result holds. □

Proposition 4.9. *For fan graph $F_{m,n}$, we have*

$$\chi_{ei}(F_{m,n}) = \begin{cases} 1, & m = 1, n = 2, \\ m+1, & m \geq 2, n = 2, \\ m+n, & m = 1, n \geq 4 \text{ and } m \geq 2, n \geq 3. \end{cases}$$

Proof.

There are three situations to be considered.

1. If $m = 1, n = 2$. It is obvious.

2. If $m \geq 2, n = 2$ and $V(F_{m,2}) = \{v_1, v_2, \dots, v_m, u_1, u_2\}$, then by definition $F_{2,2} \cong F_{1,3}$ and $F_{m,2} = \overline{K}_m \vee P_2$. Two vertices u_1, u_2 receive same color because there is no path of length 3 between them. On the other hand there exist a path $v_i u_1 u_2 v_j$ of length 3 between any pair of vertices v_i, v_j, and there exist a path $v_i u_l v_j u_k$ with $l \neq k$ of length 3 between any pair of vertices v_i, u_k. Therefore $\chi_{ei}(F_{m,2}) = m+1$.

3.1. Let $m = 1, n \geq 4$ and $V(F_{1,n}) = \{v_1, u_1, u_2, \dots, u_n\}$. Then $v_1 u_{i+2} u_{i+1} u_i$, $(i \leq n-2)$, $v_1 u_{i-2} u_{i-1} u_i$, $(i \geq n-1)$, $u_i v_1 u_{j+1} u_j$, $(i < j < n)$, $u_i v_1 u_{n-1} u_n$ $(i < n-1)$ and $u_{n-1} u_{n-2} v_1 u_n$ are the paths of length 3 between any pair of vertices u_i, u_j and v_1, u_i. Thus $\chi_{ei}(F_{1,n}) = 1 + n$.

3.2. Let $m \geq 2, n \geq 3$. Using the reasons given in proof of part 3.1, one can easy to see that there is a path of length 3 between any pair of vertices of $F_{m,n}$.

All in all the proof is completed. □

5. Discussion and Conclusions

From Propositions 2.1, 2.2 and 2.3,

1. Characterize graphs G with $\chi_{ei}(G) = \chi(G)$; Characterize graphs G with $\chi_{ei}(G) = \chi_i(G)$; Characterize graphs G with $\chi_{ei}(G) = \chi_2(G)$.
2. After discussion on the solution of 1, one can revisit the Conjectures 1.1 and 1.3.

From Propositions 3.3 and 3.2, we can have the following.

3. Characterize graphs G with $\chi_{ei}(G) = \rho(G)$.
4. Characterize graphs G with $\chi_{ei}(G) = \Delta(G)(\Delta(G) - 1)^2 + 1$.

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