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Multi-Decomposition of Graphs into Paths and Y-Trees of Order Five

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ABSTRACT

Let K_n, P_n, and Y_n respectively denote a complete graph, a path, and a Y-tree on n vertices, and let $K_{m,n}$ denote a complete bipartite graph with m and n vertices in its parts. Graph decomposition is the process of breaking down a graph into a collection of edge-disjoint subgraphs. A graph G has a (H_1, H_2)-multi-decomposition if it can be decomposed into $\alpha \geq 0$ copies of H_1 and $\beta \geq 0$ copies of H_2, where H_1 and H_2 are subgraphs of G. In this paper, we derive the necessary and sufficient conditions for the (P_5, Y_5)-multi-decomposition of K_n and $K_{m,n}$.

Keywords: Path, Y-Tree, Multi-decomposition, Complete graph, Complete bipartite graph, Conjoined Twins

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Let K_n denote a complete graph on n vertices, $K_{m,n}$ a complete bipartite graph with vertex partite sets of cardinality m and n, and P_k a path on k vertices. A Y-tree on k vertices, denoted by Y_k, is a tree in which one edge is attached to a vertex v of the path P_{k-1} such that at least one of the adjacent vertices of v has degree 1.

A decomposition of a graph G is a set of edge-disjoint subgraphs $H_1, H_2, \ldots, H_r$ of G such that every edge of G belongs to exactly one H_i, $1 \leq i \leq r$. If all the subgraphs in the decomposition of G are isomorphic to a graph H, then G is said to be H-decomposable. If G can be decomposed into α copies of H_1 and β copies of H_2, then G is said to have an (H_1, H_2)-multi-decomposition or an $\{H_1^\alpha, H_2^\beta\}$-decomposition. The pair (α, β) is called admissible if it satisfies the necessary conditions for the existence of an $\{H_1^\alpha, H_2^\beta\}$-decomposition. If G has an (H_1, H_2)-multi-decomposition for all

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admissible pairs (α, β), it is said to have an $(H_1, H_2)_{\{\alpha,\beta\}}$-decomposition.

The necessary and sufficient condition for the existence of a P_5-decomposition of complete graphs was studied in [5], and for complete bipartite graphs in [1]. The path decomposition of various graphs was explored in [15, 11]. The graph Y_5 is one of the three non-isomorphic trees of order five, excluding paths and stars. Caterina and Antonio [3] named Y_5 as the *chair* and studied the stability number of chair-free graphs.

The Y_5-decomposition of complete graphs was obtained by C. Huang and A. Rosa [5]. J. Paulraj Joseph and A. Samuel Issacraj [8] referred to Y_5 as the *fork* and studied its decomposition in complete bipartite graphs. S. Gomathi and A. Tamil Elakkiya [4] defined this graph as a Y_5-tree and investigated its decomposition in the tensor product of complete graphs. The Y_5-decomposition of various graphs was further analyzed in [9, 10]. The concept of multi-decomposition was introduced by A. Abueida and M. Daven [2]. In recent years, multi-decomposition of graphs has emerged as a prominent research area in graph theory. T.-W. Shyu studied the multi-decomposition of complete graphs into paths with cycles and stars [12, 13]. S. Jeevadoss and A. Muthusamy established necessary and sufficient conditions for the multi-decomposition of complete bipartite graphs into paths and cycles [6]. Multi-decomposition of complete bipartite graphs into paths and stars was considered in [14].

In this paper, we establish the necessary and sufficient conditions for the existence of a (P_5, Y_5)-multi-decomposition of K_n and $K_{m,n}$. To prove our results, we recall the following theorems:

Theorem 1.1. [5] *The complete graph K_n is Y_5-decomposable if and only if $n \equiv 0 \pmod 8$.*

Theorem 1.2. [1] *Let k, m, and n be positive integers. The necessary conditions for a P_{k+1}-decomposition of $K_{m,n}$ are listed in Table 1, and these conditions are also sufficient.*

Case	k	m	n	Characterization
1.	even	even	even	$k \leq 2m, k \leq 2n$, not both equal
2.	odd	even	even	Equalities hold when k is even
3.	even	even	odd	$k \leq 2m - 2, k \leq 2n$
4.	even	odd	even	$k \leq 2m, k \leq 2n - 2$
5.	even	odd	odd	Decomposition impossible
6.	odd	even	odd	$k \leq 2m - 1, k \leq n$
7.	odd	odd	even	$k \leq m, k \leq 2n - 1$
8.	odd	odd	odd	$k \leq m, k \leq n$

Table 1. Necessary and sufficient conditions for a P_{k+1}-decomposition of $K_{m,n}$

Theorem 1.3. [8] *The complete bipartite graph $K_{m,n}$ is fork-decomposable if and only if $mn \equiv 0 \pmod 4$, except for $K_{2,4i+2}$, $(i = 1, 2, \dots)$.*

2. Multi-Decomposition of Complete Graphs into P_5 and Y_5

2.1. Preliminaries

Definition 2.1. [7] For a graph G, two disjoint subsets of vertices are called *twins* if they have the same order and induce subgraphs with the same number of edges.

Next, we introduce a new graph structure called *Conjoined Twins* in the following remark.

Remark 2.2. Consider the graph T with vertex set $\{v_i : 1 \leq i \leq 8\}$.

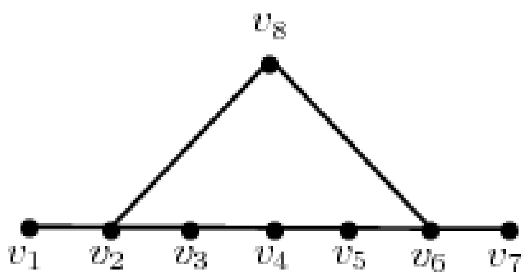

Fig. 1. Conjoined twins (T)

The subgraphs induced by A and B are isomorphic to P_5 when $A = \{v_1, v_2, v_6, v_7, v_8\}$ and $B = \{v_2, v_3, v_4, v_5, v_6\}$. Similarly, if $A = \{v_1, v_2, v_3, v_4, v_8\}$ and $B = \{v_4, v_5, v_6, v_7, v_8\}$, the corresponding induced subgraphs are isomorphic to Y_5. We call these subsets of vertices *Conjoined Twins* (T) because the subsets A and B are not disjoint (there are two common vertices), but the induced subgraphs are isomorphic.

It is interesting to note that the subgraph induced by A is isomorphic to P_5 when $A = \{v_1, v_2, v_3, v_8, v_6\}$, and if $B = \{v_3, v_4, v_5, v_6, v_7\}$, the corresponding induced subgraph is isomorphic to Y_5.

Thus, decomposing the graph G into a structure whose vertices are *Conjoined Twins* as in Figure 1 can be viewed as consisting of 2 copies of P_5, 2 copies of Y_5, or 1 copy each of P_5 and Y_5, which significantly simplifies the (P_5, Y_5)-multi-decomposition.

2.2. Notations

- For a subgraph H of G, $G\backslash H$ denotes a graph where $V(G\backslash H) = V(G)$ and $E(G\backslash H) = E(G) - E(H)$.
- rG denotes r disjoint copies of the graph G.
- $G = H_1 \oplus H_2$ means G can be decomposed into H_1 and H_2.
- Let v_i, $1 \leq i \leq n$, be the vertices of the complete graph K_n.
- In the complete bipartite graph $K_{m,n}$, the vertices of the first partite set with m vertices are denoted by v_{1i}, $1 \leq i \leq m$, and the second partite set with n vertices by v_{2j}, $1 \leq j \leq n$.
- A path P_5 with 5 vertices v_i, $1 \leq i \leq 5$, having v_1 and v_5 as pendant vertices is denoted by $P_5(v_1, v_2, v_3, v_4, v_5)$.
- The Y_5 graph with 5 vertices v_i, $1 \leq i \leq 5$, is denoted by $Y_5(v_1, v_2, \underline{v_3}, v_4; \underline{v_5})$, where v_i, $1 \leq i \leq 4$, form a path of length three, and the underlined vertices denote an edge v_3v_5.
- Suppose we have a graph whose vertices are *Conjoined Twins* (T) as in Figure 1. We denote it by $T(v_1, \underline{v_2}, v_3, v_4, v_5, \underline{v_6}, v_7; \underline{v_8})$, where v_i, $1 \leq i \leq 7$, form a path of length six, and the underlined vertices denote edges v_2v_8 and v_6v_8.

Remark 2.3. If two graphs G_1 and G_2 have an (H_1, H_2)-multi-decomposition, then $G_1 \oplus G_2$ also has such a decomposition.

2.3. Necessary condition

The following theorem gives the necessary condition for the existence of a multi-decomposition of the complete graph K_n into paths and Y-trees with 5 vertices.

Theorem 2.4. *If K_n has a (P_5, Y_5) - multi-decomposition, then $n \equiv 0\, or 1 (mod\ 8)$.*

Proof. Proof follows from the edge divisibility condition. □

2.4. Sufficient conditions

In this section, we show that the necessary condition obtained in Theorem 2.4 is also sufficient for the existence of a multi-decomposition of K_n, $(n \geq 8)$ into P_5 and Y_5.

Lemma 2.5. *The Complete graphs K_8 and K_9 have (P_5, Y_5) - multi-decomposition.*

Proof. We can see that $K_8 = 3T \oplus 1P_5$, where the $3T$'s and $1P_5$ are given by,

$$T(v_6, \underline{v_7}, v_5, v_1, v_4, \underline{v_8}, v_2; \underline{v_3}), T(v_3, \underline{v_6}, v_4, v_2, v_5, \underline{v_8}, v_7; \underline{v_1}),$$
$$T(v_8, \underline{v_6}, v_5, v_3, v_1, \underline{v_7}, v_4; \underline{v_2}), P_5(v_1, v_2, v_3, v_4, v_5).$$

Similarly, K_9 can be written as $K_9 = 4T \oplus 1P_5$, where the $4T$'s and $1P_5$ are as follows:

$$T(v_3, \underline{v_6}, v_4, v_1, v_8, \underline{v_7}, v_5; \underline{v_2}), T(v_3, \underline{v_9}, v_4, v_2, v_5, \underline{v_6}, v_1; \underline{v_7}),$$
$$T(v_4, \underline{v_8}, v_5, v_3, v_1, \underline{v_9}, v_2; \underline{v_6}), T(v_4, \underline{v_7}, v_1, v_5, v_9, \underline{v_8}, v_2; \underline{v_3}), P_5(v_1, v_2, v_3, v_4, v_5).$$

□

Lemma 2.6. *The Complete bipartite graphs $K_{7,8}$, $K_{8,8}$ and $K_{9,8}$ have (P_5, Y_5) - multi-decomposition.*

Proof. It is clear that $K_{7,8} = 7T$, where $7T$'s are given by,

$$T(v_{21}, \underline{v_{11}}, v_{23}, v_{12}, v_{24}, \underline{v_{13}}, v_{22}; \underline{v_{25}}), T(v_{25}, \underline{v_{12}}, v_{22}, v_{11}, v_{26}, \underline{v_{13}}, v_{23}; \underline{v_{21}}),$$
$$T(v_{24}, \underline{v_{11}}, v_{28}, v_{15}, v_{26}, \underline{v_{14}}, v_{25}; \underline{v_{27}}), T(v_{24}, \underline{v_{14}}, v_{28}, v_{13}, v_{27}, \underline{v_{15}}, v_{25}; \underline{v_{23}}),$$
$$T(v_{27}, \underline{v_{12}}, v_{28}, v_{17}, v_{24}, \underline{v_{16}}, v_{25}; \underline{v_{26}}), T(v_{28}, \underline{v_{16}}, v_{21}, v_{14}, v_{22}, \underline{v_{17}}, v_{26}; \underline{v_{27}}),$$
$$T(v_{24}, \underline{v_{15}}, v_{22}, v_{16}, v_{23}, \underline{v_{17}}, v_{25}; \underline{v_{21}}).$$

Similarly $K_{8,8} = 8T$, where $8T$'s are identified as,

$$T(v_{22}, \underline{v_{13}}, v_{24}, v_{12}, v_{23}, \underline{v_{18}}, v_{21}; \underline{v_{25}}), T(v_{28}, \underline{v_{12}}, v_{25}, v_{11}, v_{26}, \underline{v_{13}}, v_{23}; \underline{v_{27}}),$$
$$T(v_{26}, \underline{v_{12}}, v_{21}, v_{13}, v_{28}, \underline{v_{14}}, v_{25}; \underline{v_{22}}), T(v_{28}, \underline{v_{11}}, v_{27}, v_{14}, v_{26}, \underline{v_{15}}, v_{25}; \underline{v_{24}}),$$
$$T(v_{24}, \underline{v_{14}}, v_{21}, v_{16}, v_{27}, \underline{v_{15}}, v_{22}; \underline{v_{23}}), T(v_{24}, \underline{v_{16}}, v_{28}, v_{15}, v_{21}, \underline{v_{17}}, v_{26}; \underline{v_{22}}),$$
$$T(v_{21}, \underline{v_{11}}, v_{23}, v_{17}, v_{24}, \underline{v_{18}}, v_{27}; \underline{v_{22}}), T(v_{23}, \underline{v_{16}}, v_{26}, v_{18}, v_{28}, \underline{v_{17}}, v_{27}; \underline{v_{25}}).$$

Further $K_{9,8} = 9T$, the following are the required $9T$'s

$$T(v_{26}, \underline{v_{11}}, v_{23}, v_{12}, v_{24}, \underline{v_{13}}, v_{25}; \underline{v_{22}}), T(v_{25}, \underline{v_{12}}, v_{26}, v_{19}, v_{27}, \underline{v_{13}}, v_{23}; \underline{v_{28}}),$$
$$T(v_{22}, \underline{v_{12}}, v_{21}, v_{13}, v_{26}, \underline{v_{14}}, v_{23}; v_{27}), T(v_{28}, \underline{v_{11}}, v_{24}, v_{14}, v_{22}, \underline{v_{15}}, v_{25}; v_{27}),$$
$$T(v_{21}, \underline{v_{14}}, v_{25}, v_{16}, v_{23}, \underline{v_{15}}, v_{24}; \underline{v_{28}}), T(v_{27}, \underline{v_{16}}, v_{21}, v_{15}, v_{26}, \underline{v_{17}}, v_{22}; \underline{v_{24}}),$$
$$T(v_{21}, \underline{v_{18}}, v_{27}, v_{17}, v_{25}, \underline{v_{19}}, v_{22}; \underline{v_{24}}), T(v_{22}, \underline{v_{16}}, v_{26}, v_{18}, v_{23}, \underline{v_{17}}, v_{21}; \underline{v_{28}}),$$
$$T(v_{22}, \underline{v_{18}}, v_{25}, v_{11}, v_{21}, \underline{v_{19}}, v_{23}; \underline{v_{28}}).$$

□

Lemma 2.7. *The graph K_8 admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition when $\alpha + \beta = 7$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 7$ are {(0,7),(1,6),(2,5),(3,4),(4,3),(5,2), (6,1),(7,0)}.

Case 1. $\alpha \neq 0$.

From Lemma 2.5, $K_8 = 3T \oplus 1P_5$, which can be taken into any of the forms: $6Y_5 \oplus 1P_5, 5Y_5 \oplus 2P_5, 4Y_5 \oplus 3P_5, 3Y_5 \oplus 4P_5, 2Y_5 \oplus 5P_5, 1Y_5 \oplus 6P_5,$ and $7P_5$ using Remark 2.2.

Thus we have (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1),(7,0)\}$.

Case 2. $\alpha = 0$.

Theorem 1.1 gives the required decomposition for the admissible pair $(0, 7)$.

Hence the proof follows from Cases 1 & 2 for all admissible pairs (α, β). □

Lemma 2.8. *The graph K_9 admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition if $\alpha + \beta = 9$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 9$ are {(0,9),(1,8),(2,7),(3,6),(4,5),(5,4), (6,3),(7,2),(8,1),(9,0)}.

Case 1. $\alpha \neq 0$.

From Lemma 2.5, $K_9 = 4T \oplus 1P_5$, which can be taken into any of the forms: $8Y_5 \oplus 1P_5$, $7Y_5 \oplus 2P_5, 6Y_5 \oplus 3P_5, 5Y_5 \oplus 4P_5, 4Y_5 \oplus 5P_5, 3Y_5 \oplus 6P_5, 2Y_5 \oplus 7P_5, 1Y_5 \oplus 8P_5$ and $9P_5$ using Remark 2.2.

Thus we have (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(1,8),(2,7),(3,6),(4,5),(5,4),(6,3),(7,2),(8,1),(9,0)\}$.

Case 2. $\alpha = 0$.

Theorem 1.1 gives the required decomposition for the admissible pair $(0, 9)$.

Thus, K_9 admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition. □

Lemma 2.9. *The graph $K_{7,8}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition if $\alpha + \beta = 14$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 14$ are {(0,14),(1,13),(2,12),(3,11),(4,10),(5,9),(6,8), (7,7),(8,6),(9,5),(10,4),(11,3),(12,2),(13,1),(14,0)}.

From Lemma 2.6, $K_{7,8} = 7T$, which can be taken into any of the forms: $14Y_5, 13Y_5 \oplus 1P_5, 12Y_5 \oplus 2P_5, 11Y_5 \oplus 3P_5, 10Y_5 \oplus 4P_5, 9Y_5 \oplus 5P_5, 8Y_5 \oplus 6P_5, 7Y_5 \oplus 7P_5, 6Y_5 \oplus 8P_5, 5Y_5 \oplus 9P_5, 4Y_5 \oplus 10P_5, 3Y_5 \oplus 11P_5, 2Y_5 \oplus 12P_5, 1Y_5 \oplus 13P_5$ and $14P_5$ using Remark 2.2.

Thus we have (P_5, Y_5) - multi-decomposition for all the admissible pairs (α, β). □

Lemma 2.10. *The graph $K_{8,8}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition if $\alpha + \beta = 16$.*

Proof. The admissible pairs satisfying $\alpha+\beta = 16$ are $\{(0,16),(1,15),(2,14),(3,13),(4,12),(5,11),(6,10),(7,9),(8,8),(9,7),(10,6),(11,5),(12,4),(13,3),(14,2),(15,1),(16,0)\}$.

From Lemma 2.6 , $K_{8,8} = 8T$. Then we have (P_5, Y_5) - multi-decomposition for all the admissible pairs (α, β) using Remark 2.2. □

Lemma 2.11. *The graph $K_{9,8}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition when $\alpha + \beta = 18$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 18$ are $\{(0, 18), (1, 17), (2, 16), (3, 15), (4, 14), (5, 13), (6, 12), (7, 11), (8, 10), (9, 9), (10, 8), (11, 7), (12, 6), (13, 5), (14, 4), (15, 3), (16, 2), (17, 1), (18, 0)\}$.

From Lemma 2.6 , $K_{9,8} = 9T$. Then we have (P_5, Y_5) - multi-decomposition for all the admissible pairs (α, β) using Remark 2.2. □

Theorem 2.12. *(Sufficient conditions) For given non negative integers α, β and $n \geq 8$, K_n has $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $4(\alpha + \beta) = \binom{n}{2}$.*

Proof. From the given (Necessary conditions) edge divisibility condition,
we have $n \equiv 0$ or $1 \pmod 8$.

Case 1: $n \equiv 0 \pmod 8$.

Let $n = 4t$, t is even. We prove this theorem using induction on t. When $t = 2$, the proof follows from Lemma 2.7. We observe that for $t \geq 4$,

$$K_{4t} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-2)-1,8}. \tag{1}$$

Also for $t \geq 6$,

$$K_{4(t-2)-1,8} = K_{4(t-4)-1,8} \oplus K_{8,8}. \tag{2}$$

From (1) and (2),

$$K_{4t} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-4)-1,8} \oplus K_{8,8}, \quad t \geq 6. \tag{3}$$

Assume that the theorem is true for all even $k < t$. We have to prove for $t = k + 2$. From (3), we can write,

$$K_{4(k+2)} = K_{4k} \oplus K_9 \oplus K_{4(k-2)-1,8} \oplus K_{8,8}.$$

By induction hypothesis and from Lemmas 2.7, 2.8, 2.9 and 2.10 the proof follows.

Case 2: $n \equiv 1 \pmod 8$.

Let $n = 4t + 1$, t is even. When $t = 2$, the proof follows from Lemma 2.8. We observe that for $t \geq 4$,

$$K_{4t+1} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-2)+\frac{1}{2}(t-2),8} \tag{4}$$

Also for $t \geq 6$,

$$K_{4(t-2)+\frac{1}{2}(t-2),8} = K_{4(t-4)+\frac{1}{2}(t-2),8} \oplus K_{9,8}. \tag{5}$$

From (4) and (5),

$$K_{4t+1} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-4)+\frac{1}{2}(t-2),8} \oplus K_{9,8}, \quad t \geq 6. \tag{6}$$

Assume that the theorem is true for all even $k < t$. We have to prove for $t = k+2$. From (6), we can write,

$$K_{4(k+2)+1} = K_{4k} \oplus K_9 \oplus K_{4(k-2)+\frac{1}{2}k,8} \oplus K_{9,8}.$$

By induction hypothesis and from Lemmas 2.7, 2.8 and 2.11 the proof follows.

□

Theorem 2.13. *(Main Theorem) For non-negative integers α, β and $n \geq 8$, $K_n = \alpha P_5 \oplus \beta Y_5$ if and only if $4(\alpha+\beta) = \binom{n}{2}$.*

Proof. The proof follows from Theorems 2.4 and 2.12. □

3. Multi-Decomposition of Complete Bipartite Graphs into P_5 and Y_5

3.1. Necessary conditions

In this section, we derive the necessary conditions for the existence of multi-decomposition of $K_{m,n}$, $(m > 2, n \geq 2)$ into paths and Y-trees with 5 vertices.

Lemma 3.1. *Let k be even. If $K_{2k,2}$ has a (P_5, Y_5) - multi-decomposition for the admissible pair (α, β), then α is even.*

Proof. Let $V(K_{2k,2}) = V_1 \cup V_2$, where $|V_1| = 2k$, $|V_2| = 2$ and $|E(K_{2k,2})| = 4k$. P_5 has a degree sequence $(2,2,2,1,1)$. While decomposing $K_{2k,2}$ into P_5's and Y_5's, the two vertices of P_5 with degree 2 which are incident with a vertex of degree 1, should be formed using the vertex set $V_2 = \{v_{21}, v_{22}\}$. Y_5 has a degree sequence $(3,2,1,1,1)$. Here, the vertex with degree 3 and the vertex with degree 1 which is incident with a vertex of degree 2, should be formed using the vertex set V_2. Since each vertex in V_2 has degree $2k$, after decomposing $K_{2k,2}$ into α number of P_5, each vertex v_{2i}, $i = 1,2$ has degree $2k - 2\alpha$ and $|E(K_{2k,2} \backslash \alpha P_5)| = 4k - 4\alpha$. Since k is even, it is clear that

$$2(k-\alpha) \equiv \begin{cases} 0 (mod\ 4), & if \quad \alpha \ is \ even, \\ 2 (mod\ 4), & if \quad \alpha \ is \ odd. \end{cases}$$

Therefore, partitioning the remaining $4(k-\alpha)$ edges into $k-\alpha$ number of Y_5 is possible only when α is even. □

Lemma 3.2. *Let $k \geq 3$ be odd. If $K_{2k,2}$ has a (P_5, Y_5) - multi-decomposition for the admissible pair (α, β), then α is odd.*

Proof. The proof is same as Lemma 3.1 with the same argument. Since $k \geq 3$ is odd,

$$2(k-\alpha) \equiv \begin{cases} 2 (mod\ 4), & if \quad \alpha \ is \ even, \\ 0 (mod\ 4), & if \quad \alpha \ is \ odd. \end{cases}$$

Hence the proof follows. □

Theorem 3.3. *(Necessary conditions) If $K_{m,n}$ has $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition, then $mn = 4(\alpha+\beta)$ with $m > 2$ and $n > 1$ except*

1. *$m = 2k$, k even; $n = 2$ and α is odd*
2. *$m = 2k$, $k \geq 3$ odd; $n = 2$ and α is even*

Proof. The proof follows from edge divisibility condition and by Lemmas 3.1 and 3.2. □

3.2. Sufficient conditions

In the following lemmas we prove that the above necessary conditions are also sufficient.

Lemma 3.4.

$$K_{4,2} = \begin{cases} 2P_5, \\ or, \\ 2Y_5. \end{cases}$$

Proof. By Theorem 3.3, $\alpha+\beta = 2$. Hence the admissible pairs (α, β) are $(0, 2)$, $(1, 1)$ and $(2, 0)$. By Theorem 1.2, $K_{4,2}$ can be decomposed into $2P_5$ and by Theorem 1.3, $K_{4,2}$ can be decomposed into $2Y_5$. Hence there exists a (P_5, Y_5) - multi-decomposition for the admissible pairs $(0, 2)$ and $(2, 0)$. By Lemma 3.1, it is clear that there does not exist a (P_5, Y_5) - multi-decomposition for the admissible pair $(1, 1)$. Hence the proof. □

Lemma 3.5. *The graph $K_{6,2}$ has (P_5, Y_5) - multi-decomposition for some of the admissible pairs (α, β) where α is odd.*

Proof. The admissible pairs for which the decomposition exists are $(\alpha, \beta) \in \{(3,0), (1,2)\}$. For $(3, 0)$, Theorem 1.2 gives the required decomposition. For $(1, 2)$, we have the necessary breakdown is as follows:

$$P_5(v_{11}, v_{21}, v_{12}, v_{22}, v_{13}), Y_5(v_{21}, v_{15}, \underline{v_{22}}, v_{14}; \underline{v_{11}}), Y_5(v_{22}, v_{16}, \underline{v_{21}}, v_{14}; \underline{v_{13}}).$$

The desired decomposition does not exist for the admissible pairs $(2, 1)$ and $(0, 3)$ by Lemma 3.2. □

Lemma 3.6. *Let k be even. If α is even in the admissible pair (α, β), then $K_{2k,2}$ has a (P_5, Y_5) - multi-decomposition.*

Proof. Since k is even, $k = 2k_1$ for $k_1 \in \mathbb{N}$. we write, $K_{2k,2} = k_1 K_{4,2}$.

Therefore, by Lemma 3.4, for any even α such that $\alpha + \beta = k$, there exists a (P_5, Y_5) - multi-decomposition for the admissible pairs (α, β) with α, β are even. This completes the proof. □

Lemma 3.7. *Let $k \geq 3$ be odd. If α is odd, then $K_{2k,2}$ has (P_5, Y_5) - multi-decomposition.*

Proof. Since $k \neq 1$ is odd, $k = 2q + 1$ for $q \in \mathbb{N}$. we write, $K_{2k,2} = (q-1)K_{4,2} \oplus K_{6,2}$.
Therefore, by Lemmas 3.4 and 3.5, for any odd α such that $\alpha + \beta = k$, there exists a (P_5, Y_5) - multi-decomposition for the admissible pairs (α, β) with α is odd and β is even. This completes the proof. □

Lemma 3.8. *The graph $K_{4,3}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 3$.*

Proof. Case 1: (3,0).
Theorem 1.2 gives required $3P_5$'s.
Case 2: (2,1).

$$P_5(v_{21}, v_{12}, v_{22}, v_{11}, v_{23}), P_5(v_{11}, v_{21}, v_{13}, v_{22}, v_{14}), Y_5(v_{21}, v_{14}, \underline{v_{23}}, v_{13}; \underline{v_{12}}).$$

Case 3: (1,2).

$$P_5(v_{21}, v_{14}, v_{22}, v_{11}, v_{23}), Y_5(v_{22}, v_{12}, \underline{v_{23}}, v_{13}; \underline{v_{14}}), Y_5(v_{22}, v_{13}, \underline{v_{21}}, v_{12}; \underline{v_{11}}).$$

Case 4: (0,3).
Theorem 1.3 gives the required decomposition. □

Lemma 3.9. *The graph $K_{4,4}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 4$.*

Proof. Case 1: α is even i.e.,$(\alpha, \beta) \in \{(4,0), (2,2), (0,4)\}$.
Since $K_{4,4} = 2K_{4,2}$, Theorems 1.2 and 1.3 give the required decomposition.
Case 2: α is odd.
Subcase 1: (3,1).

$$P_5(v_{11}, v_{22}, v_{14}, v_{23}, v_{13}), P_5(v_{21}, v_{14}, v_{24}, v_{12}, v_{22}), P_5(v_{12}, v_{23}, v_{11}, v_{24}, v_{13}), Y_5(v_{22}, v_{13}, \underline{v_{21}}, v_{12}; \underline{v_{11}})$$

Subcase 2: (1,3).

$$P_5(v_{12}, v_{22}, v_{13}, v_{21}, v_{14}), Y_5(v_{13}, v_{23}, \underline{v_{14}}, v_{24}; \underline{v_{22}}), Y_5(v_{13}, v_{24}, \underline{v_{11}}, v_{23}; \underline{v_{22}}), Y_5(v_{11}, v_{21}, \underline{v_{12}}, v_{24}; \underline{v_{23}})$$

□

Lemma 3.10. *The graphs $K_{4,5}$ and $K_{4,6}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 5$ and $\alpha + \beta = 6$ respectively.*

Proof. We can write $K_{4,5} = K_{4,2} \oplus K_{4,3}$, $K_{4,6} = 2K_{4,3}$. Then the proof follows from Lemmas 3.4 and 3.8. □

Lemma 3.11. *The graph $K_{6,6}$ admits $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition whenever $\alpha + \beta = 9$.*

Proof. We can write $K_{6,6} = K_{4,6} \oplus K_{2,6}$. Since $K_{m,n} \cong K_{n,m}$, the proof follows from Lemmas 3.5 and 3.10. $\square$

Lemma 3.12. *If $k, n \in \mathbb{N}$, $n \geq 3$, then $K_{4k,n}$ can be decomposed into admissible pairs of P_5 and Y_5.*

Proof. Let $n = 4q + r$ for $q > 0$ and $r \in \{0, 1, 2, 3\}$.

If $r = 0$, $K_{4k,n} = K_{4k,4q} = kqK_{4,4}$.

For $r = 1$, $K_{4k,n} = K_{4k,4q+1} = K_{4k,4(q-1)+5} = k(q-1)K_{4,4} \oplus K_{4,5}$.

When $r = 2$, $K_{4k,n} = K_{4k,4q+2} = K_{4k,4(q-1)+6} = k(q-1)K_{4,4} \oplus K_{4,6}$.

When $r = 3$, $K_{4k,n} = K_{4k,4q+3} = kqK_{4,4} \oplus K_{4,3}$.

Then the proof follows from Lemmas 3.8, 3.9, 3.10 and by mathematical induction on k, n. $\square$

Lemma 3.13. *If $k_1, k_2 \geq 3$ be odd, then $K_{2k_1,2k_2}$ can be decomposed into admissible pairs of P_5 and Y_5.*

Proof. Since $k_1 \neq 1$, $k_2 \neq 1$ are odd, $k_1 = 2q_1 + 1$ and $k_2 = 2q_2 + 1$ for $q_1, q_2 \in \mathbb{N}$ and we write, $K_{2k_1,2k_2} = (q_1 - 1)(q_2 - 1)K_{4,4} \oplus (q_1 - 1)K_{4,6} \oplus (q_2 - 1)K_{6,4} \oplus K_{6,6}$.

Then the proof follows from Lemmas 3.9, 3.10, 3.11 and by mathematical induction on k_1, k_2. $\square$

Theorem 3.14. *(Sufficient Conditions) If m, n, α and β satisfy the necessary condition given in Theorem 3.3, then $K_{m,n}$ has $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition.*

Proof. Case 1: $m \equiv 0 \pmod 4$ or $n \equiv 0 \pmod 4$, w.l.o.g, let $m = 4k$ for $k \in \mathbb{N}$.

Subcase 1.1. $n = 2$.

Lemma 3.6 gives the required decomposition.

Subcase 1.2. $n \geq 3$.

Lemma 3.12 gives the required decomposition.

Case 2: $m \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, i.e., $m = 2k_1$, $n = 2k_2$ for k_1, $k_2 \in \mathbb{N}$.

Subcase 2.1. When one of them is 2, w.l.o.g, let $n = 2$.

When k_1 is even, this falls in Subcase 1.1. If $k_1 \neq 1$ is odd, Lemma 3.7 gives the required decomposition.

Subcase 2.2. $m, n > 2$.

When one of k_1 and k_2 or both of them are even, then the proof follows from Subcase 1.2. If both of them are odd, Lemma 3.13 gives the required decomposition. $\square$

Theorem 3.15. *(Main Theorem) There exists $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of $K_{m,n}$ if and only if any one of the following holds:*

1. *$m = 2k, k$ is even, $n = 2$ and α is even.*
2. *$m = 2k, k \geq 3$ is odd, $n = 2$ and α is odd.*
3. *$m = 4k$ and $n \geq 3$.*
4. *$m = 2k_1$ and $n = 2k_2$; where $k_1, k_2 \geq 3$ are odd.*

Proof. Proof follows from Theorems 3.3 and 3.14. □

4. Conclusion

In this paper, it is proved that the necessary and sufficient condition for the existence of the $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of the complete graph K_n $(n \geq 8)$ is $n \equiv 0$ *or* 1 $(mod\ 8)$. Also we have obtained the necessary and sufficient conditions for the $(P_5, Y_5)_{\{\alpha,\beta\}}$ - decomposition of the complete bipartite graph $K_{m,n}$ $(m > 2,\ n \geq 2)$ as $mn = 4(\alpha + \beta)$ whenever

(i) $m = 2k$, k even; $n = 2$ then α is even.

(ii) $m = 2k$, $k \geq 3$ odd; $n = 2$ then α is odd.

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