



Multi-decomposition of graphs into paths and Y -trees of order five

Chaadhanaa A¹, Hemalatha P^{1,✉}

¹ *Department of Mathematics, Vellalar College For Women, Tamil Nadu, India*

ABSTRACT

Let K_n , P_n , and Y_n respectively denote a complete graph, a path, and a Y -tree on n vertices, and let $K_{m,n}$ denote a complete bipartite graph with m and n vertices in its parts. Graph decomposition is the process of breaking down a graph into a collection of edge-disjoint subgraphs. A graph G has a (H_1, H_2) -multi-decomposition if it can be decomposed into $\alpha \geq 0$ copies of H_1 and $\beta \geq 0$ copies of H_2 , where H_1 and H_2 are subgraphs of G . In this paper, we derive the necessary and sufficient conditions for the (P_5, Y_5) -multi-decomposition of K_n and $K_{m,n}$.

Keywords: Path, Y -Tree, Multi-decomposition, Complete graph, Complete bipartite graph, Conjoined Twins

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Let K_n denote a complete graph on n vertices, $K_{m,n}$ a complete bipartite graph with vertex partite sets of cardinality m and n , and P_k a path on k vertices. A Y -tree on k vertices, denoted by Y_k , is a tree in which one edge is attached to a vertex v of the path P_{k-1} such that at least one of the adjacent vertices of v has degree 1.

A decomposition of a graph G is a set of edge-disjoint subgraphs H_1, H_2, \dots, H_r of G such that every edge of G belongs to exactly one H_i , $1 \leq i \leq r$. If all the subgraphs in the decomposition of G are isomorphic to a graph H , then G is said to be H -decomposable. If G can be decomposed into α copies of H_1 and β copies of H_2 , then G is said to have an (H_1, H_2) -multi-decomposition or an $\{H_1^\alpha, H_2^\beta\}$ -decomposition. The pair (α, β) is called admissible if it satisfies the necessary conditions for the existence of an $\{H_1^\alpha, H_2^\beta\}$ -decomposition. If G has an (H_1, H_2) -multi-decomposition for all

✉ Corresponding author.

E-mail addresses: math.chaad@gmail.com (Chaadhanaa A), dr.hemalatha@gmail.com (Hemalatha P).

Received 19 November 2024; accepted 05 December 2024; published 31 December 2024.

DOI: [10.61091/jcmcc123-09](https://doi.org/10.61091/jcmcc123-09)

© 2024 The Author(s). Published by Combinatorial Press. This is an open access article under the CC BY license (<https://creativecommons.org/licenses/by/4.0/>).

admissible pairs (α, β) , it is said to have an $(H_1, H_2)_{\{\alpha, \beta\}}$ -decomposition.

The necessary and sufficient condition for the existence of a P_5 -decomposition of complete graphs was studied in [5], and for complete bipartite graphs in [1]. The path decomposition of various graphs was explored in [15, 11]. The graph Y_5 is one of the three non-isomorphic trees of order five, excluding paths and stars. Caterina and Antonio [3] named Y_5 as the *chair* and studied the stability number of chair-free graphs.

The Y_5 -decomposition of complete graphs was obtained by C. Huang and A. Rosa [5]. J. Paulraj Joseph and A. Samuel Issacraj [8] referred to Y_5 as the *fork* and studied its decomposition in complete bipartite graphs. S. Gomathi and A. Tamil Elakkiya [4] defined this graph as a Y_5 -tree and investigated its decomposition in the tensor product of complete graphs. The Y_5 -decomposition of various graphs was further analyzed in [9, 10]. The concept of multi-decomposition was introduced by A. Abueida and M. Daven [2]. In recent years, multi-decomposition of graphs has emerged as a prominent research area in graph theory. T.-W. Shyu studied the multi-decomposition of complete graphs into paths with cycles and stars [12, 13]. S. Jeevadoss and A. Muthusamy established necessary and sufficient conditions for the multi-decomposition of complete bipartite graphs into paths and cycles [6]. Multi-decomposition of complete bipartite graphs into paths and stars was considered in [14].

In this paper, we establish the necessary and sufficient conditions for the existence of a (P_5, Y_5) -multi-decomposition of K_n and $K_{m,n}$. To prove our results, we recall the following theorems:

Theorem 1.1. [5] *The complete graph K_n is Y_5 -decomposable if and only if $n \equiv 0 \pmod{8}$.*

Theorem 1.2. [1] *Let k, m , and n be positive integers. The necessary conditions for a P_{k+1} -decomposition of $K_{m,n}$ are listed in Table 1, and these conditions are also sufficient.*

Table 1. Necessary and sufficient conditions for a P_{k+1} -decomposition of $K_{m,n}$

Case	k	m	n	Characterization
1.	even	even	even	$k \leq 2m, k \leq 2n$, not both equal
2.	odd	even	even	Equalities hold when k is even
3.	even	even	odd	$k \leq 2m - 2, k \leq 2n$
4.	even	odd	even	$k \leq 2m, k \leq 2n - 2$
5.	even	odd	odd	Decomposition impossible
6.	odd	even	odd	$k \leq 2m - 1, k \leq n$
7.	odd	odd	even	$k \leq m, k \leq 2n - 1$
8.	odd	odd	odd	$k \leq m, k \leq n$

Theorem 1.3. [8] *The complete bipartite graph $K_{m,n}$ is fork-decomposable if and only if $mn \equiv 0 \pmod{4}$, except for $K_{2,4i+2}$, ($i = 1, 2, \dots$).*

2. Multi-decomposition of complete graphs into P_5 and Y_5

2.1. Preliminaries

Definition 2.1. [7] For a graph G , two disjoint subsets of vertices are called *twins* if they have the same order and induce subgraphs with the same number of edges.

Next, we introduce a new graph structure called *Conjoined Twins* in the following remark.

Remark 2.2. Consider the graph T with vertex set $\{v_i : 1 \leq i \leq 8\}$.

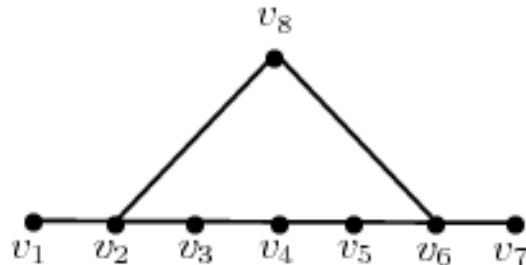


Fig. 1. Conjoined twins (T)

The subgraphs induced by A and B are isomorphic to P_5 when $A = \{v_1, v_2, v_6, v_7, v_8\}$ and $B = \{v_2, v_3, v_4, v_5, v_6\}$. Similarly, if $A = \{v_1, v_2, v_3, v_4, v_8\}$ and $B = \{v_4, v_5, v_6, v_7, v_8\}$, the corresponding induced subgraphs are isomorphic to Y_5 . We call these subsets of vertices *Conjoined Twins* (T) because the subsets A and B are not disjoint (there are two common vertices), but the induced subgraphs are isomorphic.

It is interesting to note that the subgraph induced by A is isomorphic to P_5 when $A = \{v_1, v_2, v_3, v_8, v_6\}$, and if $B = \{v_3, v_4, v_5, v_6, v_7\}$, the corresponding induced subgraph is isomorphic to Y_5 .

Thus, decomposing the graph G into a structure whose vertices are *Conjoined Twins* as in Figure 1 can be viewed as consisting of 2 copies of P_5 , 2 copies of Y_5 , or 1 copy each of P_5 and Y_5 , which significantly simplifies the (P_5, Y_5) -multi-decomposition.

2.2. Notations

- For a subgraph H of G , $G \setminus H$ denotes a graph where $V(G \setminus H) = V(G)$ and $E(G \setminus H) = E(G) - E(H)$.
- rG denotes r disjoint copies of the graph G .
- $G = H_1 \oplus H_2$ means G can be decomposed into H_1 and H_2 .
- Let $v_i, 1 \leq i \leq n$, be the vertices of the complete graph K_n .
- In the complete bipartite graph $K_{m,n}$, the vertices of the first partite set with m vertices are denoted by $v_{1i}, 1 \leq i \leq m$, and the second partite set with n vertices by $v_{2j}, 1 \leq j \leq n$.
- A path P_5 with 5 vertices $v_i, 1 \leq i \leq 5$, having v_1 and v_5 as pendant vertices is denoted by $P_5(v_1, v_2, v_3, v_4, v_5)$.
- The Y_5 graph with 5 vertices $v_i, 1 \leq i \leq 5$, is denoted by $Y_5(v_1, v_2, \underline{v_3}, v_4; \underline{v_5})$, where $v_i, 1 \leq i \leq 4$, form a path of length three, and the underlined vertices denote an edge v_3v_5 .
- Suppose we have a graph whose vertices are *Conjoined Twins* (T) as in Figure 1. We denote it by $T(v_1, \underline{v_2}, v_3, v_4, v_5, \underline{v_6}, v_7; \underline{v_8})$, where $v_i, 1 \leq i \leq 7$, form a path of length six, and the underlined vertices denote edges v_2v_8 and v_6v_8 .

Remark 2.3. If two graphs G_1 and G_2 have an (H_1, H_2) -multi-decomposition, then $G_1 \oplus G_2$ also has such a decomposition.

2.3. Necessary condition

The following theorem gives the necessary condition for the existence of a multi-decomposition of the complete graph K_n into paths and Y -trees with 5 vertices.

Theorem 2.4. *If K_n has a (P_5, Y_5) - multi-decomposition, then $n \equiv 0$ or $1 \pmod{8}$.*

Proof. Proof follows from the edge divisibility condition. □

2.4. Sufficient conditions

In this section, we show that the necessary condition obtained in Theorem 2.4 is also sufficient for the existence of a multi-decomposition of K_n , ($n \geq 8$) into P_5 and Y_5 .

Lemma 2.5. *The Complete graphs K_8 and K_9 have (P_5, Y_5) - multi-decomposition.*

Proof. We can see that $K_8 = 3T \oplus 1P_5$, where the $3T$'s and $1P_5$ are given by,

$$\begin{aligned} &T(v_6, \underline{v_7}, v_5, v_1, v_4, \underline{v_8}, v_2; \underline{v_3}), T(v_3, \underline{v_6}, v_4, v_2, v_5, \underline{v_8}, v_7; \underline{v_1}), \\ &T(v_8, \underline{v_6}, v_5, v_3, v_1, \underline{v_7}, v_4; \underline{v_2}), P_5(v_1, v_2, v_3, v_4, v_5). \end{aligned}$$

Similarly, K_9 can be written as $K_9 = 4T \oplus 1P_5$, where the $4T$'s and $1P_5$ are as follows:

$$\begin{aligned} &T(v_3, \underline{v_6}, v_4, v_1, v_8, \underline{v_7}, v_5; \underline{v_2}), T(v_3, \underline{v_9}, v_4, v_2, v_5, \underline{v_6}, v_1; \underline{v_7}), \\ &T(v_4, \underline{v_8}, v_5, v_3, v_1, \underline{v_9}, v_2; \underline{v_6}), T(v_4, \underline{v_7}, v_1, v_5, v_9, \underline{v_8}, v_2; \underline{v_3}), P_5(v_1, v_2, v_3, v_4, v_5). \end{aligned}$$

□

Lemma 2.6. *The Complete bipartite graphs $K_{7,8}$, $K_{8,8}$ and $K_{9,8}$ have (P_5, Y_5) - multi-decomposition.*

Proof. It is clear that $K_{7,8} = 7T$, where $7T$'s are given by,

$$\begin{aligned} &T(v_{21}, \underline{v_{11}}, v_{23}, v_{12}, v_{24}, \underline{v_{13}}, v_{22}; \underline{v_{25}}), T(v_{25}, \underline{v_{12}}, v_{22}, v_{11}, v_{26}, \underline{v_{13}}, v_{23}; \underline{v_{21}}), \\ &T(v_{24}, \underline{v_{11}}, v_{28}, v_{15}, v_{26}, \underline{v_{14}}, v_{25}; \underline{v_{27}}), T(v_{24}, \underline{v_{14}}, v_{28}, v_{13}, v_{27}, \underline{v_{15}}, v_{25}; \underline{v_{23}}), \\ &T(v_{27}, \underline{v_{12}}, v_{28}, v_{17}, v_{24}, \underline{v_{16}}, v_{25}; \underline{v_{26}}), T(v_{28}, \underline{v_{16}}, v_{21}, v_{14}, v_{22}, \underline{v_{17}}, v_{26}; \underline{v_{27}}), \\ &T(v_{24}, \underline{v_{15}}, v_{22}, v_{16}, v_{23}, \underline{v_{17}}, v_{25}; \underline{v_{21}}). \end{aligned}$$

Similarly $K_{8,8} = 8T$, where $8T$'s are identified as,

$$\begin{aligned} &T(v_{22}, \underline{v_{13}}, v_{24}, v_{12}, v_{23}, \underline{v_{18}}, v_{21}; \underline{v_{25}}), T(v_{28}, \underline{v_{12}}, v_{25}, v_{11}, v_{26}, \underline{v_{13}}, v_{23}; \underline{v_{27}}), \\ &T(v_{26}, \underline{v_{12}}, v_{21}, v_{13}, v_{28}, \underline{v_{14}}, v_{25}; \underline{v_{22}}), T(v_{28}, \underline{v_{11}}, v_{27}, v_{14}, v_{26}, \underline{v_{15}}, v_{25}; \underline{v_{24}}), \\ &T(v_{24}, \underline{v_{14}}, v_{21}, v_{16}, v_{27}, \underline{v_{15}}, v_{22}; \underline{v_{23}}), T(v_{24}, \underline{v_{16}}, v_{28}, v_{15}, v_{21}, \underline{v_{17}}, v_{26}; \underline{v_{22}}), \\ &T(v_{21}, \underline{v_{11}}, v_{23}, v_{17}, v_{24}, \underline{v_{18}}, v_{27}; \underline{v_{22}}), T(v_{23}, \underline{v_{16}}, v_{26}, v_{18}, v_{28}, \underline{v_{17}}, v_{27}; \underline{v_{25}}). \end{aligned}$$

Further $K_{9,8} = 9T$, the following are the required $9T$'s

$$\begin{aligned} &T(v_{26}, \underline{v_{11}}, v_{23}, v_{12}, v_{24}, \underline{v_{13}}, v_{25}; \underline{v_{22}}), T(v_{25}, \underline{v_{12}}, v_{26}, v_{19}, v_{27}, \underline{v_{13}}, v_{23}; \underline{v_{28}}), \\ &T(v_{22}, \underline{v_{12}}, v_{21}, v_{13}, v_{26}, \underline{v_{14}}, v_{23}; v_{27}), T(v_{28}, \underline{v_{11}}, v_{24}, v_{14}, v_{22}, \underline{v_{15}}, v_{25}; v_{27}), \\ &T(v_{21}, \underline{v_{14}}, v_{25}, v_{16}, v_{23}, \underline{v_{15}}, v_{24}; \underline{v_{28}}), T(v_{27}, \underline{v_{16}}, v_{21}, v_{15}, v_{26}, \underline{v_{17}}, v_{22}; \underline{v_{24}}), \\ &T(v_{21}, \underline{v_{18}}, v_{27}, v_{17}, v_{25}, \underline{v_{19}}, v_{22}; \underline{v_{24}}), T(v_{22}, \underline{v_{16}}, v_{26}, v_{18}, v_{23}, \underline{v_{17}}, v_{21}; \underline{v_{28}}), \\ &T(v_{22}, \underline{v_{18}}, v_{25}, v_{11}, v_{21}, \underline{v_{19}}, v_{23}; \underline{v_{28}}). \end{aligned}$$

□

Lemma 2.7. *The graph K_8 admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition when $\alpha + \beta = 7$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 7$ are $\{(0,7), (1,6), (2,5), (3,4), (4,3), (5,2), (6,1), (7,0)\}$.

Case 1. $\alpha \neq 0$.

From Lemma 2.5, $K_8 = 3T \oplus 1P_5$, which can be taken into any of the forms: $6Y_5 \oplus 1P_5, 5Y_5 \oplus 2P_5, 4Y_5 \oplus 3P_5, 3Y_5 \oplus 4P_5, 2Y_5 \oplus 5P_5, 1Y_5 \oplus 6P_5$, and $7P_5$ using Remark 2.2.

Thus we have (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1), (7, 0)\}$.

Case 2. $\alpha = 0$.

Theorem 1.1 gives the required decomposition for the admissible pair $(0, 7)$.

Hence the proof follows from Cases 1 & 2 for all admissible pairs (α, β) .

□

Lemma 2.8. *The graph K_9 admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition if $\alpha + \beta = 9$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 9$ are $\{(0,9), (1,8), (2,7), (3,6), (4,5), (5,4), (6,3), (7,2), (8,1), (9,0)\}$.

Case 1. $\alpha \neq 0$.

From Lemma 2.5, $K_9 = 4T \oplus 1P_5$, which can be taken into any of the forms: $8Y_5 \oplus 1P_5, 7Y_5 \oplus 2P_5, 6Y_5 \oplus 3P_5, 5Y_5 \oplus 4P_5, 4Y_5 \oplus 5P_5, 3Y_5 \oplus 6P_5, 2Y_5 \oplus 7P_5, 1Y_5 \oplus 8P_5$ and $9P_5$ using Remark 2.2.

Thus we have (P_5, Y_5) - multi-decomposition for the admissible pairs $(\alpha, \beta) \in \{(1, 8), (2, 7), (3, 6), (4, 5), (5, 4), (6, 3), (7, 2), (8, 1), (9, 0)\}$.

Case 2. $\alpha = 0$.

Theorem 1.1 gives the required decomposition for the admissible pair $(0, 9)$.

Thus, K_9 admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition.

□

Lemma 2.9. *The graph $K_{7,8}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition if $\alpha + \beta = 14$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 14$ are $\{(0,14), (1,13), (2,12), (3,11), (4,10), (5,9), (6,8), (7,7), (8,6), (9,5), (10,4), (11,3), (12,2), (13,1), (14,0)\}$.

From Lemma 2.6, $K_{7,8} = 7T$, which can be taken into any of the forms: $14Y_5, 13Y_5 \oplus 1P_5, 12Y_5 \oplus 2P_5, 11Y_5 \oplus 3P_5, 10Y_5 \oplus 4P_5, 9Y_5 \oplus 5P_5, 8Y_5 \oplus 6P_5, 7Y_5 \oplus 7P_5, 6Y_5 \oplus 8P_5, 5Y_5 \oplus 9P_5, 4Y_5 \oplus 10P_5, 3Y_5 \oplus 11P_5, 2Y_5 \oplus 12P_5, 1Y_5 \oplus 13P_5$ and $14P_5$ using Remark 2.2.

Thus we have (P_5, Y_5) - multi-decomposition for all the admissible pairs (α, β) . □

Lemma 2.10. *The graph $K_{8,8}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition if $\alpha + \beta = 16$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 16$ are $\{(0,16), (1,15), (2,14), (3,13), (4,12), (5,11), (6,10), (7,9), (8,8), (9,7), (10,6), (11,5), (12,4), (13,3), (14,2), (15,1), (16,0)\}$.

From Lemma 2.6 , $K_{8,8} = 8T$. Then we have (P_5, Y_5) - multi-decomposition for all the admissible pairs (α, β) using Remark 2.2. □

Lemma 2.11. *The graph $K_{9,8}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition when $\alpha + \beta = 18$.*

Proof. The admissible pairs satisfying $\alpha + \beta = 18$ are $\{(0, 18), (1, 17), (2, 16), (3, 15), (4, 14), (5, 13), (6, 12), (7, 11), (8, 10), (9, 9), (10, 8), (11, 7), (12, 6), (13, 5), (14, 4), (15, 3), (16, 2), (17, 1), (18, 0)\}$.

From Lemma 2.6 , $K_{9,8} = 9T$. Then we have (P_5, Y_5) - multi-decomposition for all the admissible pairs (α, β) using Remark 2.2. □

Theorem 2.12. *(Sufficient conditions) For given non negative integers α, β and $n \geq 8$, K_n has $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition whenever $4(\alpha + \beta) = \binom{n}{2}$.*

Proof. From the given (Necessary conditions) edge divisibility condition, we have $n \equiv 0$ or $1 \pmod{8}$.

Case 1: $n \equiv 0 \pmod{8}$.

Let $n = 4t$, t is even. We prove this theorem using induction on t . When $t = 2$, the proof follows from Lemma 2.7. We observe that for $t \geq 4$,

$$K_{4t} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-2)-1,8}. \tag{1}$$

Also for $t \geq 6$,

$$K_{4(t-2)-1,8} = K_{4(t-4)-1,8} \oplus K_{8,8}. \tag{2}$$

From (1) and (2),

$$K_{4t} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-4)-1,8} \oplus K_{8,8}, \quad t \geq 6. \tag{3}$$

Assume that the theorem is true for all even $k < t$. We have to prove for $t = k + 2$. From (3), we can write,

$$K_{4(k+2)} = K_{4k} \oplus K_9 \oplus K_{4(k-2)-1,8} \oplus K_{8,8}.$$

By induction hypothesis and from Lemmas 2.7, 2.8, 2.9 and 2.10 the proof follows.

Case 2: $n \equiv 1 \pmod{8}$.

Let $n = 4t + 1$, t is even. When $t = 2$, the proof follows from Lemma 2.8. We observe that for $t \geq 4$,

$$K_{4t+1} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-2)+\frac{1}{2}(t-2),8} \tag{4}$$

Also for $t \geq 6$,

$$K_{4(t-2)+\frac{1}{2}(t-2),8} = K_{4(t-4)+\frac{1}{2}(t-2),8} \oplus K_{9,8}. \tag{5}$$

From (4) and (5),

$$K_{4t+1} = K_{4(t-2)} \oplus K_9 \oplus K_{4(t-4)+\frac{1}{2}(t-2),8} \oplus K_{9,8}, \quad t \geq 6. \tag{6}$$

Assume that the theorem is true for all even $k < t$. We have to prove for $t = k + 2$. From (6), we can write,

$$K_{4(k+2)+1} = K_{4k} \oplus K_9 \oplus K_{4(k-2)+\frac{1}{2}k,8} \oplus K_{9,8}.$$

By induction hypothesis and from Lemmas 2.7, 2.8 and 2.11 the proof follows. □

Theorem 2.13. (Main Theorem) For non-negative integers α, β and $n \geq 8$, $K_n = \alpha P_5 \oplus \beta Y_5$ if and only if $4(\alpha + \beta) = \binom{n}{2}$.

Proof. The proof follows from Theorems 2.4 and 2.12. □

3. Multi-Decomposition of complete bipartite graphs into P_5 and Y_5

3.1. Necessary conditions

In this section, we derive the necessary conditions for the existence of multi-decomposition of $K_{m,n}$, ($m > 2, n \geq 2$) into paths and Y -trees with 5 vertices.

Lemma 3.1. Let k be even. If $K_{2k,2}$ has a (P_5, Y_5) - multi-decomposition for the admissible pair (α, β) , then α is even.

Proof. Let $V(K_{2k,2}) = V_1 \cup V_2$, where $|V_1| = 2k, |V_2| = 2$ and $|E(K_{2k,2})| = 4k$. P_5 has a degree sequence $(2, 2, 2, 1, 1)$. While decomposing $K_{2k,2}$ into P_5 's and Y_5 's, the two vertices of P_5 with degree 2 which are incident with a vertex of degree 1, should be formed using the vertex set $V_2 = \{v_{21}, v_{22}\}$. Y_5 has a degree sequence $(3, 2, 1, 1, 1)$. Here, the vertex with degree 3 and the vertex with degree 1 which is incident with a vertex of degree 2, should be formed using the vertex set V_2 . Since each vertex in V_2 has degree $2k$, after decomposing $K_{2k,2}$ into α number of P_5 , each vertex $v_{2i}, i = 1, 2$ has degree $2k - 2\alpha$ and $|E(K_{2k,2} \setminus \alpha P_5)| = 4k - 4\alpha$. Since k is even, it is clear that

$$2(k - \alpha) \equiv \begin{cases} 0 \pmod{4}, & \text{if } \alpha \text{ is even,} \\ 2 \pmod{4}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

Therefore, partitioning the remaining $4(k - \alpha)$ edges into $k - \alpha$ number of Y_5 is possible only when α is even. □

Lemma 3.2. Let $k \geq 3$ be odd. If $K_{2k,2}$ has a (P_5, Y_5) - multi-decomposition for the admissible pair (α, β) , then α is odd.

Proof. The proof is same as Lemma 3.1 with the same argument. Since $k \geq 3$ is odd,

$$2(k - \alpha) \equiv \begin{cases} 2 \pmod{4}, & \text{if } \alpha \text{ is even,} \\ 0 \pmod{4}, & \text{if } \alpha \text{ is odd.} \end{cases}$$

Hence the proof follows. \square

Theorem 3.3. (Necessary conditions) *If $K_{m,n}$ has $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition, then $mn = 4(\alpha + \beta)$ with $m > 2$ and $n > 1$ except*

1. $m = 2k$, k even; $n = 2$ and α is odd
2. $m = 2k$, $k \geq 3$ odd; $n = 2$ and α is even

Proof. The proof follows from edge divisibility condition and by Lemmas 3.1 and 3.2. \square

3.2. Sufficient conditions

In the following lemmas we prove that the above necessary conditions are also sufficient.

Lemma 3.4.

$$K_{4,2} = \begin{cases} 2P_5, \\ \text{or,} \\ 2Y_5. \end{cases}$$

Proof. By Theorem 3.3, $\alpha + \beta = 2$. Hence the admissible pairs (α, β) are $(0, 2)$, $(1, 1)$ and $(2, 0)$. By Theorem 1.2, $K_{4,2}$ can be decomposed into $2P_5$ and by Theorem 1.3, $K_{4,2}$ can be decomposed into $2Y_5$. Hence there exists a (P_5, Y_5) - multi-decomposition for the admissible pairs $(0, 2)$ and $(2, 0)$. By Lemma 3.1, it is clear that there does not exist a (P_5, Y_5) - multi-decomposition for the admissible pair $(1, 1)$. Hence the proof. \square

Lemma 3.5. *The graph $K_{6,2}$ has (P_5, Y_5) - multi-decomposition for some of the admissible pairs (α, β) where α is odd.*

Proof. The admissible pairs for which the decomposition exists are $(\alpha, \beta) \in \{(3, 0), (1, 2)\}$. For $(3, 0)$, Theorem 1.2 gives the required decomposition. For $(1, 2)$, we have the necessary breakdown is as follows:

$$P_5(v_{11}, v_{21}, v_{12}, v_{22}, v_{13}), Y_5(v_{21}, v_{15}, \underline{v_{22}}, v_{14}; \underline{v_{11}}), Y_5(v_{22}, v_{16}, \underline{v_{21}}, v_{14}; \underline{v_{13}}).$$

The desired decomposition does not exist for the admissible pairs $(2, 1)$ and $(0, 3)$ by Lemma 3.2. \square

Lemma 3.6. *Let k be even. If α is even in the admissible pair (α, β) , then $K_{2k,2}$ has a (P_5, Y_5) - multi-decomposition.*

Proof. Since k is even, $k = 2k_1$ for $k_1 \in \mathbb{N}$. we write, $K_{2k,2} = k_1 K_{4,2}$.

Therefore, by Lemma 3.4, for any even α such that $\alpha + \beta = k$, there exists a (P_5, Y_5) - multi-decomposition for the admissible pairs (α, β) with α, β are even. This completes the proof. \square

Lemma 3.7. *Let $k \geq 3$ be odd. If α is odd, then $K_{2k,2}$ has (P_5, Y_5) - multi-decomposition.*

Proof. Since $k \neq 1$ is odd, $k = 2q + 1$ for $q \in \mathbb{N}$. we write, $K_{2k,2} = (q - 1)K_{4,2} \oplus K_{6,2}$.

Therefore, by Lemmas 3.4 and 3.5, for any odd α such that $\alpha + \beta = k$, there exists a (P_5, Y_5) - multi-decomposition for the admissible pairs (α, β) with α is odd and β is even. This completes the proof. \square

Lemma 3.8. *The graph $K_{4,3}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition whenever $\alpha + \beta = 3$.*

Proof. Case 1: (3,0).

Theorem 1.2 gives required $3P_5$'s.

Case 2: (2,1).

$$P_5(v_{21}, v_{12}, v_{22}, v_{11}, v_{23}), P_5(v_{11}, v_{21}, v_{13}, v_{22}, v_{14}), Y_5(v_{21}, v_{14}, \underline{v_{23}}, v_{13}; \underline{v_{12}}).$$

Case 3: (1,2).

$$P_5(v_{21}, v_{14}, v_{22}, v_{11}, v_{23}), Y_5(v_{22}, v_{12}, \underline{v_{23}}, v_{13}; \underline{v_{14}}), Y_5(v_{22}, v_{13}, \underline{v_{21}}, v_{12}; \underline{v_{11}}).$$

Case 4: (0,3).

Theorem 1.3 gives the required decomposition. \square

Lemma 3.9. *The graph $K_{4,4}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition whenever $\alpha + \beta = 4$.*

Proof. Case 1: α is even i.e., $(\alpha, \beta) \in \{(4,0), (2,2), (0,4)\}$.

Since $K_{4,4} = 2K_{4,2}$, Theorems 1.2 and 1.3 give the required decomposition.

Case 2: α is odd.

Subcase 1: (3,1).

$$P_5(v_{11}, v_{22}, v_{14}, v_{23}, v_{13}), P_5(v_{21}, v_{14}, v_{24}, v_{12}, v_{22}), P_5(v_{12}, v_{23}, v_{11}, v_{24}, v_{13}), Y_5(v_{22}, v_{13}, \underline{v_{21}}, v_{12}; \underline{v_{11}})$$

Subcase 2: (1,3).

$$P_5(v_{12}, v_{22}, v_{13}, v_{21}, v_{14}), Y_5(v_{13}, v_{23}, \underline{v_{14}}, v_{24}; \underline{v_{22}}), Y_5(v_{13}, v_{24}, \underline{v_{11}}, v_{23}; \underline{v_{22}}), Y_5(v_{11}, v_{21}, \underline{v_{12}}, v_{24}; \underline{v_{23}})$$

\square

Lemma 3.10. *The graphs $K_{4,5}$ and $K_{4,6}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition whenever $\alpha + \beta = 5$ and $\alpha + \beta = 6$ respectively.*

Proof. We can write $K_{4,5} = K_{4,2} \oplus K_{4,3}$, $K_{4,6} = 2K_{4,3}$. Then the proof follows from Lemmas 3.4 and 3.8. \square

Lemma 3.11. *The graph $K_{6,6}$ admits $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition whenever $\alpha + \beta = 9$.*

Proof. We can write $K_{6,6} = K_{4,6} \oplus K_{2,6}$. Since $K_{m,n} \cong K_{n,m}$, the proof follows from Lemmas 3.5 and 3.10. \square

Lemma 3.12. *If $k, n \in \mathbb{N}$, $n \geq 3$, then $K_{4k,n}$ can be decomposed into admissible pairs of P_5 and Y_5 .*

Proof. Let $n = 4q + r$ for $q > 0$ and $r \in \{0, 1, 2, 3\}$.

If $r = 0$, $K_{4k,n} = K_{4k,4q} = kqK_{4,4}$.

For $r = 1$, $K_{4k,n} = K_{4k,4q+1} = K_{4k,4(q-1)+5} = k(q-1)K_{4,4} \oplus K_{4,5}$.

When $r = 2$, $K_{4k,n} = K_{4k,4q+2} = K_{4k,4(q-1)+6} = k(q-1)K_{4,4} \oplus K_{4,6}$.

When $r = 3$, $K_{4k,n} = K_{4k,4q+3} = kqK_{4,4} \oplus K_{4,3}$.

Then the proof follows from Lemmas 3.8, 3.9, 3.10 and by mathematical induction on k, n . \square

Lemma 3.13. *If $k_1, k_2 \geq 3$ be odd, then $K_{2k_1,2k_2}$ can be decomposed into admissible pairs of P_5 and Y_5 .*

Proof. Since $k_1 \neq 1, k_2 \neq 1$ are odd, $k_1 = 2q_1 + 1$ and $k_2 = 2q_2 + 1$ for $q_1, q_2 \in \mathbb{N}$ and we write, $K_{2k_1,2k_2} = (q_1 - 1)(q_2 - 1)K_{4,4} \oplus (q_1 - 1)K_{4,6} \oplus (q_2 - 1)K_{6,4} \oplus K_{6,6}$.

Then the proof follows from Lemmas 3.9, 3.10, 3.11 and by mathematical induction on k_1, k_2 . \square

Theorem 3.14. *(Sufficient Conditions) If m, n, α and β satisfy the necessary condition given in Theorem 3.3, then $K_{m,n}$ has $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition.*

Proof. Case 1: $m \equiv 0 \pmod{4}$ or $n \equiv 0 \pmod{4}$, w.l.o.g, let $m = 4k$ for $k \in \mathbb{N}$.

Subcase 1.1. $n = 2$.

Lemma 3.6 gives the required decomposition.

Subcase 1.2. $n \geq 3$.

Lemma 3.12 gives the required decomposition.

Case 2: $m \equiv 0 \pmod{2}$ and $n \equiv 0 \pmod{2}$, i.e., $m = 2k_1, n = 2k_2$ for $k_1, k_2 \in \mathbb{N}$.

Subcase 2.1. When one of them is 2, w.l.o.g, let $n = 2$.

When k_1 is even, this falls in Subcase 1.1. If $k_1 \neq 1$ is odd, Lemma 3.7 gives the required decomposition.

Subcase 2.2. $m, n > 2$.

When one of k_1 and k_2 or both of them are even, then the proof follows from Subcase 1.2. If both of them are odd, Lemma 3.13 gives the required decomposition. \square

Theorem 3.15. *(Main Theorem) There exists $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition of $K_{m,n}$ if and only if any one of the following holds:*

1. $m = 2k, k$ is even, $n = 2$ and α is even.
2. $m = 2k, k \geq 3$ is odd, $n = 2$ and α is odd.
3. $m = 4k$ and $n \geq 3$.
4. $m = 2k_1$ and $n = 2k_2$; where $k_1, k_2 \geq 3$ are odd.

Proof. Proof follows from Theorems 3.3 and 3.14. □

4. Conclusion

In this paper, it is proved that the necessary and sufficient condition for the existence of the $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition of the complete graph K_n ($n \geq 8$) is $n \equiv 0$ or $1 \pmod{8}$. Also we have obtained the necessary and sufficient conditions for the $(P_5, Y_5)_{\{\alpha, \beta\}}$ - decomposition of the complete bipartite graph $K_{m,n}$ ($m > 2, n \geq 2$) as $mn = 4(\alpha + \beta)$ whenever

- (i) $m = 2k$, k even; $n = 2$ then α is even.
- (ii) $m = 2k$, $k \geq 3$ odd; $n = 2$ then α is odd.

References

- [1] C. A. Parker. Complete bipartite graph path decompositions. *Ph.D. Dissertation*, Auburn University, Auburn, Alabama, 1998.
- [2] A. Abueida and M. Daven. Multi-designs for graph-pairs of order 4 and 5. *Graphs and Combinatorics*, 19(4):433–447, 2003. <https://doi.org/10.1007/s00373-003-0530-3>.
- [3] S. Caterina De and S. Antonio. Stability number of bull and chair-free graphs. *Discrete Applied Mathematics*, 41(2):121–129, 1993. [https://doi.org/10.1016/0166-218X\(93\)90032-J](https://doi.org/10.1016/0166-218X(93)90032-J).
- [4] S. Gomathi and A. Tamil Elakkiya. Gregarious Y_5 -tree decompositions of tensor product of complete graphs. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 117:185–194, 2023. <https://doi.org/10.61091/jcmcc117-17>.
- [5] C. Huang and A. Rosa. Decomposition of complete graphs into trees. *Ars Combinatoria*, 5:23–63, 1978.
- [6] S. Jeevadoss and A. Muthusamy. Decomposition of complete bipartite graphs into paths and cycles. *Discrete Mathematics*, 331:98–108, 2014. <https://doi.org/10.1016/j.disc.2014.05.009>.
- [7] A. Maria, M. Ryan, and U. Torsten. Twins in graphs. *European Journal of Combinatorics*, 39:188–197, 2014. <https://doi.org/10.1016/j.ejcb.2014.01.007>.
- [8] J. Paulraj Joseph and A. Samuel Issacraj. Fork-decomposition of graphs. *Pre-Conference Proceedings of the International Conference on Discrete Mathematics*, ISBN:978-93-91077-53-2:426–431, 2022.
- [9] A. Samuel Issacraj and J. Paulraj Joseph. Fork-decomposition of some total graphs. *Palestine Journal of Mathematics*, 12(Special Issue II):65–72, 2023.
- [10] A. Samuel Issacraj and J. Paulraj Joseph. Fork-decomposition of the cartesian product of graphs. *Mapana Journal of Sciences*, 22(Special Issue 1):163–178, 2023. <https://doi.org/10.12723/mjs.sp1.13>.
- [11] G. Sethuraman and V. Murugan. Decomposition of complete graphs into arbitrary trees. *Graphs and Combinatorics*, 37(4):1191–1203, 2021. <https://doi.org/10.1007/s00373-021-02299-5>.
- [12] T.-W. Shyu. Decomposition of complete graphs into paths and cycles. *Ars Combinatoria*, 97:257–270, 2010.
- [13] T.-W. Shyu. Decomposition of complete graphs into paths and stars. *Discrete Mathematics*, 310:2164–2169, 2010. <https://doi.org/10.1016/j.disc.2010.04.009>.
- [14] T.-W. Shyu. Decomposition of complete bipartite graphs into paths and stars with same number of edges. *Discrete Mathematics*, 313:865–871, 2013. <https://doi.org/10.1016/j.disc.2012.12.020>.

- [15] M. Truszczyński. Note on the decomposition of $\lambda K_{m,n}(\lambda K_{m,n}^*)$ into paths. *Discrete Mathematics*, 55(1):89–96, 1985. [https://doi.org/10.1016/S0012-365X\(85\)80023-6](https://doi.org/10.1016/S0012-365X(85)80023-6).