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Algorithmic design of a conditional value-at-risk optimization model based on implied volatility for multi-period portfolio adjustment

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ABSTRACT

Based on the definition of volatility and conditional value risk (CVaR), this paper introduces the implied volatility into CVaR model, and further analyzes the partial differential equation of stock portfolio optimization in the form of BS model. In the process of multi-stage investment, in order to reasonably control the investment risk of each stage, the CvaR model based on implied volatility is constructed by using the scenario tree method. With the data of 1166 trading days as the data, 4 stock assets as the data set of this study, the optimization model is applied to the calculation and analysis. The numerical simulation shows that the stock price fluctuation of the four multi-cycle stocks ranges from -23.45% to 41.97%, showing a clustering phenomenon. Among them, the volatility of stocks A and C is more obvious than that of stocks B and D, and the probability density tails of stocks are longer in the cycle, and they all show thick tail characteristics, indicating that the introduction of implied volatility of CVaR model makes the risk control of actual equity asset investment more reasonable.

Keywords: implied volatility, CVaR model, partial differential equations, numerical simulation, probability density

1. Introduction

Regarding the quantitative method of controlling risk, in the past, the main application of measuring financial risk was the variance, but the variance can only represent the fluctuation of the asset price in this period, and the fluctuation can not be represented as a risk, so this method is slowly being

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eliminated [9, 23, 19]. The current methods for quantifying risk indicators mainly include metric value at risk (Va R) and conditional value at risk (CVa R), etc., but both of them are exceptionally difficult to be solved directly through parsing [14, 7].

Risk theory consisting of a large number of mathematical models has been an international research hotspot, and mathematical modeling of risk in finance was one of the earliest studies [18, 17]. In 1963, Baumol proposed a new mathematical model of risk: the value-at-risk (VaR) model. After that, there are some authoritative financial institutions survey shows that VaR model is now widely used by many commercial banks, investment banks, non-financial institutions, institutional investors and regulatory agencies, however, after a long period of continuous exploration by many scholars and practical use of the sector has proved that there are some defects in the VaR model, such as does not satisfy the consistency axiom, VaR does not satisfy the subadditivity and so on [12, 5, 15, 21]. To address the above weaknesses of VaR, developing and improving VaR theory has become an important research topic in recent years.Uryasev and Rockafellar proposed a modified mathematical model of VaR, the conditional value-at-risk CVaR model, in their paper.The CVaR model has the strengths of VaR, while at the same time it has theoretically good properties such as subadditivity, convexity, and computability, etc.[4, 11, 13]. Pownall applies CVaR to the empirical study of the securities market in the Asian financial crisis and compares it with the RiskMetrics methodology, which shows that CVaR is better than the RiskMetrics methodology in capturing the underneath risk that arises when the market risk factors fluctuate drastically under extreme market conditions [3, 2]. The study shows that the CVaR model not only has many theoretical advantages, but also is more effective and practical than VaR in financial risk management [10].

There also exists a class of complex multi-period risky decision-making problems in business, supply chain, power and real estate, etc., i.e., in a time period due to the uncertainty of demand, it needs to make decisions in multiple periods to diversify the risk, and it also needs to find the period that divides the minimum loss, e.g., a production company produces products to be sold in several phases, and different phases, due to seasonal influences, have different prices and demand, therefore, it needs to produce according to the plan of dividing different production cycles to avoid smaller losses, such a problem requires a new mathematical model to solve it, and there are not many studies on such mathematical models [8, 16, 6].

Taking the definition of volatility and conditional value-at-risk (CVaR) as a starting point, the BS model is used to assign implied volatility, after which the lowest risk and highest return of equity assets are taken as an asset portfolio optimization problem, and the scenario tree approach is adopted to algorithmically design the implied volatility-based CvaR model. Four stocks in the multi-period asset type are selected as the research sample, totaling 1166 trading days of multi-period asset data, which constitutes the research dataset. The data set is also analyzed with descriptive statistics to clarify the numerical characteristics of the four types of multi-period stocks. Next, taking A among the four types of multi-period stocks as an example, the optimization model in this paper is used to design an asset portfolio optimization scheme with more accurate and close to the reality.

2. Algorithm design for conditional value-at-risk optimization models

2.1. Volatility theory

2.1.1. Meaning of volatility. Financial markets change all the time, and movements in financial markets can actually be described by changes in asset prices. This change in financial asset prices also creates financial risk, so if risk is to be predicted as accurately as possible, the volatility of prices must be adequately estimated as well as accurately modeled. Volatility is the conditional variance or conditional standard deviation of an asset's return. In statistics, volatility is generally described by the variance or standard deviation of prices [22, 1].

2.1.2. Characterization of volatility. Volatility is not directly observable, but there are some characteristics of volatility that can be seen in the series plot of asset returns.

1) Volatility Aggregation. Volatility aggregation simply means that volatility generally does not follow a large fluctuations in the immediate aftermath of a very small fluctuations, or a small fluctuations in the immediate aftermath of a very large fluctuations. That is to say, a larger fluctuation is often followed by a larger fluctuation, and a smaller fluctuation is often followed by a smaller fluctuation. This suggests that volatility may be larger or smaller over a small time interval, creating aggregation.

2) Leverage. The leverage effect refers to the fact that the volatility of an asset changes differently for the price when the price of the asset rises and falls sharply in two situations. Often the volatility is higher when the price falls sharply than when the price rises sharply.

3) Volatility Persistence. Volatility of the continuity of volatility means that volatility in a short period of time a sudden jump is rare, volatility tends to be in the time period of continuous change.

4) Smoothness. Because volatility does not spread to infinity, that is, volatility is constantly changing within a limited range due to various reasons in the financial market. Therefore, in statistical terms, this phenomenon indicates that volatility is smooth.

2.2. Conditional Value at Risk (CVaR) Theory

2.2.1. Definition of CVaR. Conditional Value at Risk CVaR means the average loss when the portfolio loss exceeds the value of VaR (Value at Risk) at the same confidence level for a given holding period [20]. In mathematical expression it can be expressed as:

$$CVaR_c(x) = E[f(x,r)|f(x,r) > VaR_c(x)], \tag{1}$$

where $f(x,r)$ represents the loss function of the portfolio. The above Eq. (2) can also be expressed as:

$$CVaR_c(x) = VaR_c(x) + E[f(x,r) - VaR_c(x)|f(x,r) > VaR_c(x)]. \tag{2}$$

2.2.2. Factors influencing CVaR. Since CVaR is essentially a conditional mathematical expectation, we know from the above example that the main factors affecting CVaR are the length of the holding period, the confidence level, and the VaR value. Since the confidence level and holding period length are already determined, the VaR value is actually known. Therefore, the main factors affecting the CVaR value are holding period length and confidence level.

1) Confidence level. The selection of confidence level is very important for the calculation results of CVaR. For investment institutions or investors with different risk preferences, different confidence levels will be selected when calculating the CVaR value. For risk averse investment institutions or investors, they will choose a higher confidence level, which will result in a relatively large CVaR

value, but, in general, too high a confidence level VaR and CVaR will tend to converge. At the same time, the probability of occurrence of extreme situations is greatly reduced, resulting in the empirical test of the validity of CVaR can not be carried out. In the actual financial life, banks, insurance companies, asset management companies have a great difference in the degree of preference for risk, and their settings for the confidence level will also be very different. Therefore, in order to get the effective CVaR value, it is necessary to consider the actual situation, make comprehensive measurements, and select the appropriate confidence level.

2) Holding period length. As we know, the longer the holding period, the more volatile this investment will be. This also means that the corresponding CVaR value of the portfolio will be larger, so that if the extreme situation occurs, the greater the loss of the investment organization or investor. Therefore, when making a decision, it is important to consider all factors and choose the appropriate holding period.

2.2.3. Properties of CVaR. According to the above definition of CvaR theory, it can be seen that CVaR theory should satisfy positive chi-squaredness, monotonicity, transfer invariance, and sub-additivity.

1) Positive Chirality. For any positive real number a, at confidence level c:

$$CVaR_c(ax) = aCVaR_c(x). \tag{3}$$

2) Monotonicity. Conditional on confidence level c, portfolios x and y, assuming x the risk of $\geq y$ the risk we have:

$$CVaR_c(x) > CVaR_c(y). \tag{4}$$

3) Transfer invariance. For any positive real number a:

$$CVaR_c(ax) = aCVaR_c(x). \tag{5}$$

4) Subadditivity. Is derived for any portfolios x and y:

$$CVaR_c(x + y) \leq CVaR_c(x) + CVaR_c(y). \tag{6}$$

From the above four properties, it can be seen that CVaR satisfies the consistency risk metric criterion we mentioned earlier.

This is a major advantage of CVaR over VaR, and CVaR was proposed for this one reason.

2.2.4. Advantages of CVaR. CVaR is a risk measure based on VaR, which has been improved by scholars due to the shortcomings of the VaR risk measure. Therefore, CVaR risk measure has advantages that VaR does not have.

Firstly, CVaR is able to respond to the corresponding loss when the extreme situation occurs. According to the definition of CVaR, CVaR value is not a quantile compared with VaR value, but the average value when extreme situations occur. It can provide loss prediction for investment institutions or investors when small probability events occur, enabling investors to better manage risks.

Second, CVaR meets the criteria for a consistent risk measure. Compared to the VaR metric, CVaR satisfies the sub-additivity that VaR does not satisfy, which makes CVaR have better properties when calculating risk.

Thirdly, since in calculating CVaR, we first calculate the VaR value. So when we use CVaR to measure risk, we can get both the maximum loss at the confidence level from the VaR value and the average loss that occurs in extreme cases. This provides investors with a safer risk management precaution.

2.3. Introducing a CvaR optimization model with implied volatility

2.3.1. Implied volatility. Volatility is mainly used to measure the historical volatility of equity asset prices and help investors understand the magnitude of equity asset price movements. Implied volatility, on the other hand, is used for option pricing and risk management, and reflects the market's expectation of future equity asset price fluctuations. Both reflect the volatility of equity asset prices, but one is based on historical data and the other is based on market expectations. Implied volatility can be solved to approximate actual volatility through the Black-Scholes model (BS model for short).The BS model has six assumptions:

(1) The price of the risky stock asset obeys a lognormal distribution.

(2) The return on the risky equity asset and the risk-free rate remain constant over the life of the option.

(3) The market is frictionless, taxes and transaction costs are negligible, and all securities are perfectly divisible.

(4) The option is a European-style option, i.e., the option is only exercisable on the exercise date.

(5) There is no risk-free arbitrage opportunity.

(6) Trading in the security is continuous.

Assume that this is a European call option based on stock $i, i = 1, \cdots, n$ and that the price $S_i(t)$ of stock i obeys the following geometric Brownian distribution:

$$dS_i(t) = S(t)_t(u_i dt + \sigma_i dW(t)). \tag{7}$$

At fixed moment T, if the strike price is K, this stock i has a strike return function of $S_i(T) - K$. According to the BS pricing model, at moment 0, the price of this option is:

$$C_i = S(0)\Phi(d_1) - Ke^{-r_f T}\Phi(d_2). \tag{8}$$

Among them:

$$d_1 = \frac{\ln(S_0/K) + (r_f + \sigma_i^2)T}{\sigma_i\sqrt{T}}. \tag{9}$$

$$d_2 = \frac{\ln(S_0/K) + (r_f - \sigma_i^2)T}{\sigma_i\sqrt{T}}. \tag{10}$$

$\Phi(\cdot)$ is the distribution function of a normal random variable. The option price C_i depends on the volatility of the risky equity asset σ_i and the risk-free interest rate r_f. In general, we can obtain the volatility by looking at the option price C_i at moment 0. σ_i If we substitute volatility σ_i as an unknown into Eq. (10) we obtain the implied volatility σ_i. Due to the nonlinearity of the BS model, we can use dichotomous interpolation to obtain a numerical solution.

2.3.2. Combinatorial optimization problems. The variance of the stock asset portfolio returns is used to measure the corresponding risk, which is defined as the degree of deviation of a random variable from the mean, without distinguishing between positive and negative deviations, i.e., losses and profits. In practice, the CVaR value is generally calculated by constructing an auxiliary function, which allows the CVaR value to be calculated directly without calculating the VaR value. The solution process is a linear programming problem, and CPLEX software is used in this paper for empirical analysis.

We assume that the probability that the expected return $W(x, y)$ of a portfolio does not exceed a specific value a is $\psi(x, a)$, then $\psi(x, a) = \int_{W(x,y)\le a} p(y)dy$. A transformation of the formula yields that the value at risk VaR of a portfolio x, at a certain confidence level β, is:

$$VaR_\beta = \inf\left\{\ a|\psi(x, a) \ge \beta\ \right\}. \tag{11}$$

The corresponding conditional value-at-risk CVaR at a certain confidence level β is:

$$CVaR_\beta = (1-\beta)^{-1} \int_{W(x,y)\ge VaR_\beta} W(x, y)p(y)dy. \tag{12}$$

The theory assumes that the smaller the probability that the portfolio return will fall below a particular value, the better, so the focus of attention is on the probability that the portfolio will fall below the investor's minimum required return.

If the investor's expected return on a portfolio of equity assets is still denoted by $W(x, y)$, and the investor's required minimum return is a, the mathematical meaning of SFP is:

$$\min\left\{\ P(W(x, y) \le a)\ \right\}. \tag{13}$$

So the optimization problem to be solved is to make the probability that $W(x, y)$ is below the investor's minimum required return as small as possible.

2.3.3. Scenario trees. In the multi-period investment process needs to control the investment risk of each stage, for the multi-period stochastic planning problem, the commonly used model is the scenario tree model, which is described in detail as follows:

The scenario tree model assumes that the stochastic parameters we care about in each phase can only take a finite number of values, and all possible values are called scenarios, each of which has a corresponding probability value. Each new scenario is based on the scenarios generated in the previous stage, which ultimately generates a complete scenario tree. The structure and number of scenario trees directly affect the complexity and reliability of the scenario tree, and whether the return scenarios of risky stock assets in multi-period financial investment decision-making can better match the actual is critical.

The number of scenario trees directly affects the overall running time of the algorithm, and binary tree models are generally used to model stock asset returns. In this section, we introduce a linear programming method for single-stage scenario tree generation based on "moment matching", which takes descriptive features into account. "Moment matching" means that the statistical characteristics of the generated scenario tree and the statistical characteristics of the variables are matched as much as possible.

Let X be a random variable, μ be the expectation of a random variable X, and σ be the variance of a random variable X. If $E(X^k), k = 1, 2, \cdots$ exists, it is said to be the kth order moment of origin of X, or the kth order moment for short. If $E\left\{[X-E(X)]^k\right\}, k=2,3,\cdots$ exists, it is said to be the kth order central moment of X.

The third-order central moment $E\left\{[X-E(X)]^3\right\}$ is primarily used to measure whether the distribution of a random variable is skewed, i.e., the extent to which the distribution deviates from symmetry. The skewness is the third-order standardized moment of the sample, defined as $Skew(X) = E\left[(\frac{X-\mu}{\sigma})^3\right]$.

Fourth-order central moments $E\left\{[X-E(X)]^4\right\}$ are used primarily to describe the degree of spikiness in the distribution of a random variable. Kurtosis is the fourth-order standardized moment of the sample, defined as $Kurt(X) = E\left[(\frac{X-\mu}{\sigma})^4\right]$.

If the stock asset's returns are uncorrelated at different stages, then the multi-stage scenario tree generation is simply a step-by-step generation utilizing the single-stage scenario tree generation method. Otherwise, the methodology given below should be adapted. The methodology for generating a single-stage scenario tree is shown below.

Step 1. Determine the distribution interval of stock asset returns using historical data and then divide the interval into a certain number of sub-intervals. In each sub-interval a point is selected as the value that the stock asset returns may take.

Step 2. Check the results for the presence of arbitrage opportunities. If an arbitrage opportunity exists, return to Step 1 and re-divide the distribution interval until there is no arbitrage opportunity.

Step 3. Select data features as matching targets. The probability values corresponding to each outcome are obtained by solving an optimization problem.

To convert the single-stage scenario tree generation problem into an optimal model, the following notations are used. Let S represent the total number of nodes in the scenario tree. The vector of expected returns on risky equity assets is denoted by $\bar{r}$, while Σ represents the covariance matrix of risky equity assets. Higher-order moments are also considered, where M_3 is the vector of third-order central moments, and M_4 is the vector of fourth-order central moments for risky equity assets. The sth outcome vector of the risky stock asset is represented as R^s for $s=1,\ldots,S$, and we define $R=(R^1,R^2,\ldots,R^S)$. Additionally, p^s denotes the probability value corresponding to the sth outcome vector of the risky stock asset, and the probability vector is defined as $p=(p^1,p^2,\ldots,p^S)^T$. Generally speaking, it is sufficient to use the first 4 orders of data features to perform "moment matching" for scenario tree generation. In this paper, we use the absolute deviation degree to measure the "moment matching" in the matching process. The optimization model of "moment matching" is shown below:

$$(MM)\min\sum_{i=1}^{n}\mu_i^0(\bar{r}_i^- + \bar{r}_i^+) + \sum_{i,j=1}^{n}\mu_{ij}^1(\Sigma_{ij}^- + \Sigma_{ij}^+) + \sum_{i=1}^{n}\mu_i^2(\bar{M}_{3i}^- + \bar{M}_{3i}^+) + \sum_{i=1}^{n}\mu_i^3(\bar{M}_{4i}^- + \bar{M}_{4i}^+). \tag{14}$$

The constraints of the model are formulated as follows. The relationship between the portfolio return R_p and the expected return vector $\bar{r}$ is given by $R_p + \bar{r}^- - \bar{r}^+ = \bar{r}$. The covariance constraint is expressed as

$$\sum_{s=1}^{S}(R^s - R_p)(R^s - R_p)^s p^s + \Sigma^- - \Sigma^+ = \Sigma.$$

Similarly, the third-order and fourth-order central moment constraints are given by

$$\sum_{s=1}^{S}(R^s - R_p)^3 p_s + M_3^- - M_3^+ = M_3,$$

$$\sum_{s=1}^{S}(R^s - R_p)^4 p_s + M_4^- - M_4^+ = M_4.$$

Additionally, the probability constraint ensures that the sum of all probabilities equals one,

$$\sum_{s=1}^{S} p^s = 1.$$

The non-negativity constraints are defined for the parameters $\bar{r}^*, \bar{r}, \Sigma_{ij}^*, \Sigma_{ij}^-, M_{3i}^+, M_{3i}^-, M_{4i}^+, M_{4i}^-$ for all $i, j = 1, \ldots, n$, and the probability values satisfy $p^s \geq 0$ for all $s = 1, \ldots, S$. Here, $\mu_i^0, \mu_{ij}^1, \mu_i^2, \mu_i^3$ (where $i, j = 1, \ldots, n$) are given weights. The terms $\bar{r}^+$ and $\bar{r}^-$ represent positive and negative deviations from $\bar{r}$, respectively. As a result, the formulated model is a nonlinear programming problem.

As mentioned earlier, we do not need to assume a random distribution for the distribution; we use moment matching to generate the distribution. For a given number of decision trees N, we generate a discrete scenario tree $\{r_{[j]} \in R^n\}_{j=1}^N$ with corresponding distributional probabilities $\{p_j | \sum\limits_{1}^{N} p_j = 1, p_j \geq 0, \forall j\}$. In this paper, we assume the following notation: $R_s = (r_{[1]}, r_{[2]}, \cdots r_{[N]}), p = (p_1, p_2, \cdots p_N)^T$. In this paper, we assume that R_s is known by setting the randomized return vectors to some number of discrete values, and then solving for the corresponding probability values $p_{[k]} \geq 0, k = 1, \cdots, N$, and $\sum\limits_{i=k}^{N} p_{[k]} = 1$ such that the discrete distributions R_i and p are as close to each other as possible given the data features $R_{BL}, \sum BL, M_3$ and M_4.

To better characterize the problem, the following auxiliary decision variables are introduced in this paper:

$$\begin{cases} R^+ = (R_1^+, \cdots, R_n^+)^T, R^- = (R_1^-, \cdots, R_n^-)^T, \\ \Sigma^+ = \{\Sigma_{ij}^+\}_{i,j=1}^n, \Sigma^- = \{\Sigma_{ij}^-\}_{i,j=1}^n, \\ M_3^+ \in R^n, M_3^- \in R^n, M_4^+ \in R^n, M_4^- \in R^n. \end{cases}$$

Find the appropriate distribution p by solving the following problem (P_{MM}). i.e.,

$$\begin{cases} (P_{MM}) \min \sum\limits_{k=1}^{n} \omega_i^{[1]}(R_i^+ + R_i^-) + \sum\limits_{k=1}^{n} \omega_{ij}^{[2]}(\Sigma_{ij}^+ + \Sigma_{ij}^-), \\ \sum\limits_{k=1}^{n} \omega_i^{[3]}(M_{3i}^+ + M_{3i}^-) + \sum\limits_{k=1}^{n} \omega_{ij}^{[4]}(M_{4i}^+ - M_{4i}^-), \quad s.t \quad R_s p + R^- - R^+ = R_{BL}. \\ \sum\limits_{i=1,j=1}^{n,n} (r_{[i]} - R_s p)(r_{[i]} - R_s p)^T + \Sigma^+ - \Sigma^- = \Sigma_{BL}, \\ \sum\limits_{i=1}^{N}(r_{[i]} - R_s p)^3 p_i + M_3^- - M_3^+ = M_3, \\ \sum\limits_{i=1}^{N}(r_{[i]} - R_s p)^4 p_i + M_4^- - M_4^+ = M_4, \\ \sum\limits_{i=1}^{N} p_i = 1, p_i \geq 0, i = 1, \cdots, N, \qquad R^+, R^-, \Sigma^+, \Sigma^-, M_3^+, M_3^-, M_4^+, M_4^-, \geq 0. \end{cases}$$

3. Analysis of modeling examples

3.1. Sample Selection and Data Sources

The sample is selected from four stocks in the multi-period asset type, A, B, C, and D, in US dollars, with a time span of 2019/03/02 to 2023/10/30, totaling 1,166 trading days of multi-period asset data, which comprise the dataset for this study. The dataset is equally divided into in-sample and out-of-sample data, specifically: the first 500 data points (2019/03/02-2021/03/05) are used for probability distributions and within-sample optimization of the model, while the other 666 data points (2021/03/06-2023/10/30) are used for out-of-sample tests. For the implied volatility-based CVaR model, an additional 261 out-of-sample data points (2020/05/06-2021/05/03) are used on top of the in-sample dataset, and the other 405 out-of-sample data points (2021/05/02-2022/12/30) are used for testing. In order to reflect the level of change in tariffs, the original series is converted to logarithmic returns and the returns are calculated as the difference between the logarithms of the price series. That is, its formula is as follows:

$$R_{i,j} = [\ln(P_{i,j}) - \ln(P_{i,j-1})] \times 100\%, \tag{15}$$

where $R_{i,j}$ is the daily return of the i multi-period asset stock in the j period, and $P_{i,j}$ denotes the price of the i multi-period asset stock in the j period, with selected confidence levels and time window lengths. The confidence levels α are 90%, 95%, and 99%, and the time window length is 480 days for the historical actual series utilized in the methodology of this paper.

Figure 1 shows the trend chart of the four multi-period asset stocks, from which it can be seen that all four multi-period asset stocks have experienced a substantial upward trend since 2019, but from March 2023 A and C stocks have experienced a precipitous decline, and B stocks have been in a sustained downward trend from April 2023 onwards. Meanwhile compared to A and C multi-period asset stocks, B and D stocks have lower valuations, suggesting that B and D stocks have underperformed during this period and reflecting the current risks in the stock market.

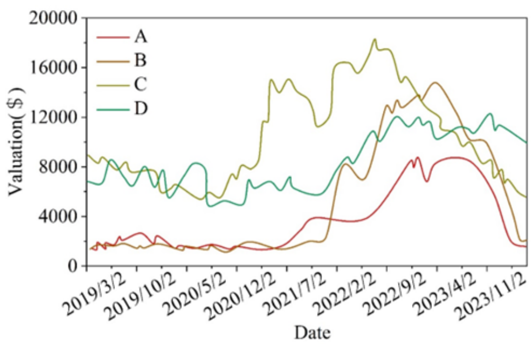
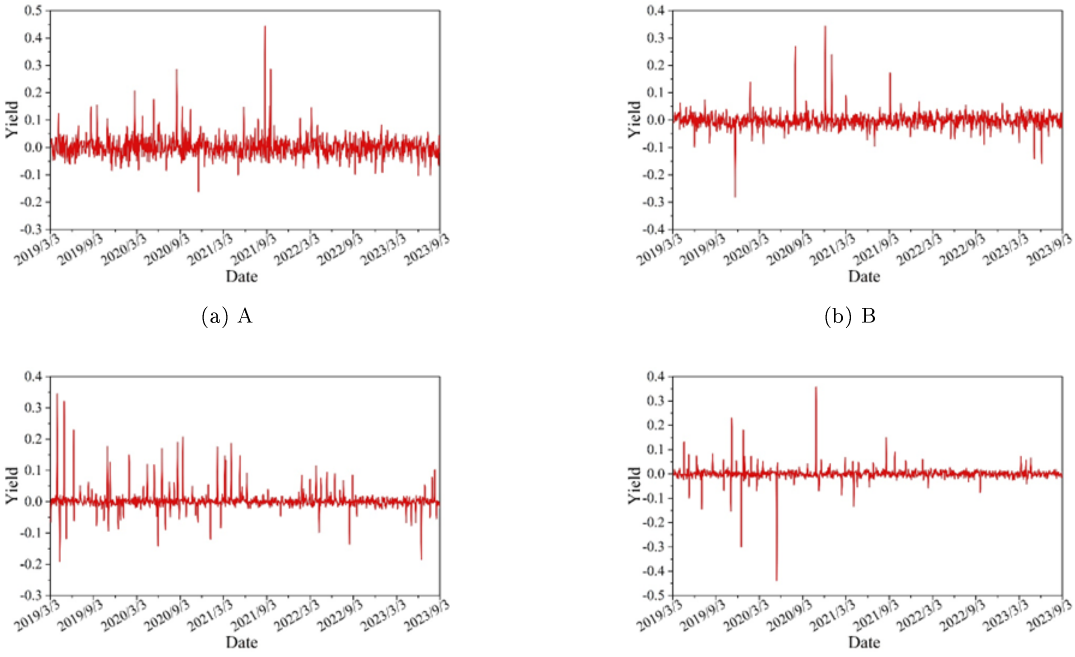


Fig. 1. Stock trend chart for four multi-cycle assets

Figure 2 shows the return series of the four multi-period asset stocks, where 2a to 2d denote A, B, C, and D. From this, it can be seen that the volatility of the return series of the four multi-period asset stocks is stable for most of the time and they all fluctuate around the value of 0, and the volatility range is located in the range of -23.45% to 41.97% for most of the period, and in some The fluctuations continue during the time period and show the phenomenon of aggregation, with the fluctuations of stocks A and C being more volatile than those of stocks B and D. The fluctuations of stocks A and C are more intense than those of stocks B and D.



(a) A (b) B

(c) C (d) D

Fig. 2. Return series of four multi-cycle asset stocks

3.2. Descriptive statistical analysis

Table 1 presents the descriptive statistics of the multi-period stock return series. From the skewness statistics, it can be seen that the skewness values of the four multi-period stock return series are non-zero, indicating that the distribution of the series is skewed with respect to the normal distribution.The skewness value of D is -2.4051, and its logarithmic return distribution shows some left skewness.The skewness values of A, B, and C are positive, and they show skewed distributions, and the distributions are right-skewed, i.e., the distributions have a long tail on the right side. The kurtosis intuitively shows the thickness of the probability density distribution in its tail, and the kurtosis value of the normal distribution is 3. Generally speaking, the larger the kurtosis value, the more extreme data in the data center, which indicates that the thick-tailed characteristic of the probability density distribution is more significant, and the kurtosis value of the four multi-period stock returns is greater than 3, with a significant spiked thick-tailed characteristic. From the Jarque-Bera statistic, the observed values of the four kinds of multi-period stock returns are all larger, and the P-value is 0. The original hypothesis is rejected, which further verifies that the four kinds of multi-period stock return series do not obey the normal distribution. The above data indicate that the return series of the four types of multi-period stock returns are characterized by asymmetric, thick-tailed distribution.

Table 1. Descriptive statistics of multi-cycle stock return series

Stock	Mean	Max	Min	SD	Kurtosis	Skewness	J-B	P
A	0.0007	-0.1707	0.4437	0.0285	71.2542	5.6554	250971.9	0.003
B	-0.0007	-0.2908	0.3505	0.0237	81.2751	2.7026	321554.4	0.001
C	0.0004	-0.1987	0.3507	0.0326	30.1474	2.9152	46077.6	0.008
D	0.0008	-0.4565	0.3582	0.0277	100.2742	-2.4051	490082.4	0.007

Figure 3 shows the relationship between the quantile of the distribution of the four multi-period stock return series and the normal distribution it can be seen that if the data obeys the standard normal distribution, the data should fall on or near the line. However, observation of the above graph reveals that the quantile of the normal distribution of the four multi-period stock return series deviates significantly from the actual quantile, thus showing that it is highly inappropriate to use the tail probabilities of the normal distribution to depict the four multi-period stock return distributions. Figure 4 shows the joint observation of the frequency histogram and the density function plot of the normal distribution of the four multi-period stock return series, from which it can be further seen that the four multi-period stock return series deviate from the normal distribution to a greater extent, and the non-normality is characterized significantly.

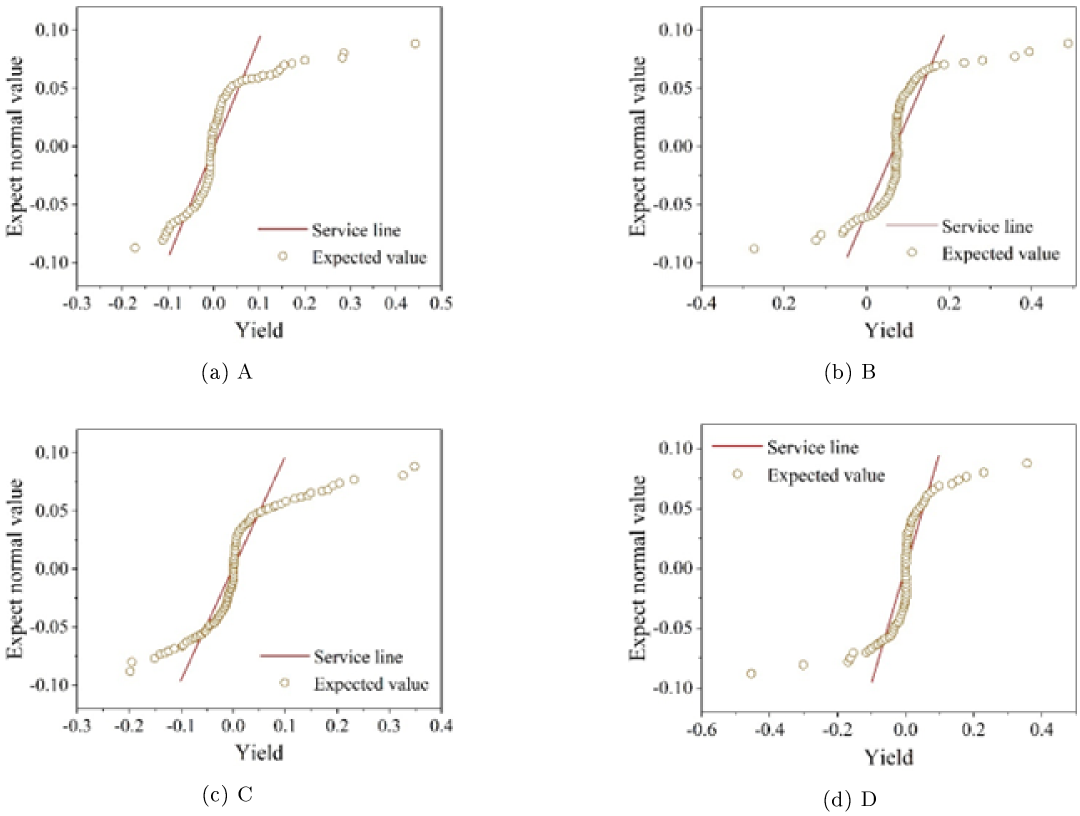


Fig. 3. The relationship between quantile and normal distribution

The partial period probability density distributions of the return series are shown in Figure 5, which shows that for the probability density distributions of the stock returns of the four assets that are in different periods, the probability density distribution plots of each period are all far from normal, and the tails within the periods are all longer, all showing thick-tailed characteristics, which will provide data support for the following research work.

3.3. Optimization model application analysis

The numerical characteristics of the four multi-period stocks can be seen through the above descriptive statistical analysis. This subsection takes A of the four multi-period stocks as an example, and uses the optimization model in this paper to find out the solution to the asset allocation problem with a more accurate and close to the reality, which makes the risk control in the actual stock asset investment more intuitive and specific.

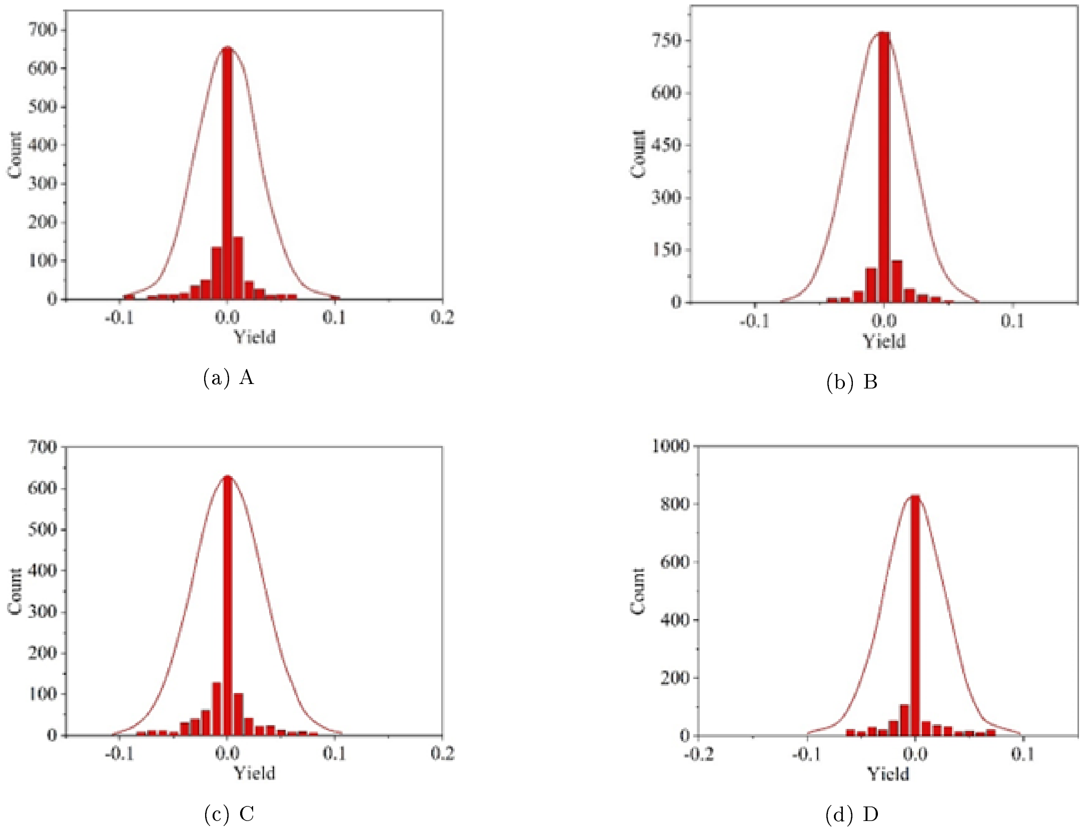


Fig. 4. Frequency histogram and normal distribution density function graph

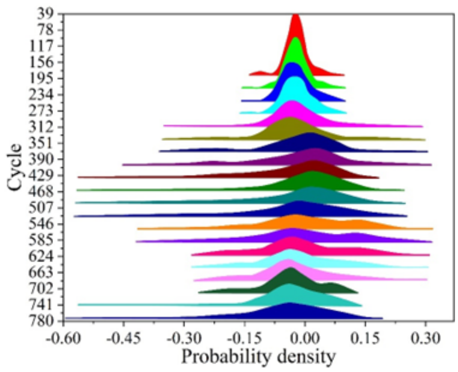


Fig. 5. Partial periodic probability density distribution of return series

When the return target is established, the effective portfolio that minimizes risk can be found as the optimal allocation, thus solving the problem of optimal asset allocation. Figure 6 shows the proportion of equity assets under the same target risk (standard deviation), the horizontal axis is the volatility of the equity portfolio, and the vertical axis is the optimal allocation weight of A equity assets. It is clear that the implied volatility-based CVaR model allocates a lower proportion of A-equity assets for any given volatility target. This is due to the fact that equity assets are much more risky than the other three equity (B, C, and D) classes, especially in terms of tail risk. In real investments, equity A assets are prone to volatility that significantly exceeds historical averages, especially during crises. The scenario tree modeling approach is able to effectively capture the heteroskedasticity of equities and accurately describe the risk of overweighting Equity A allocations resulting in over-expected volatility for multi-period assets. It can be seen that the introduction of implied volatility modeling can more accurately describe the risk of assets, which is an effective improvement to the CvaR modeling approach.

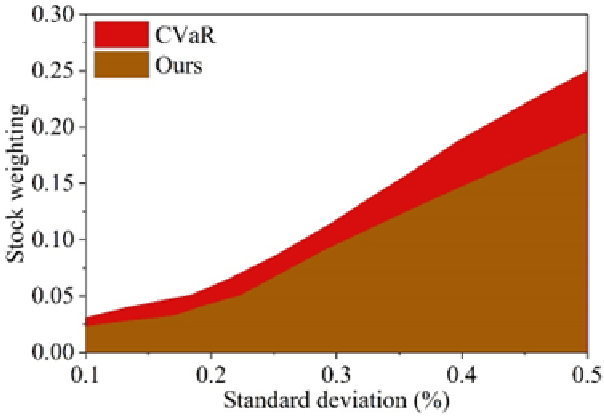


Fig. 6. The proportion of equity assets under the same target risk (standard deviation)

It is assumed that the optimal allocation portfolio corresponds to the portfolio with the maximum expected return, given a specified risk objective for multi-period equity assets. The first 500 data points (from 2019/03/02 to 2021/03/05) are used as in-sample data to calculate the optimal asset allocation weights for the initial period using the proposed model. Subsequently, portfolio adjustments are made every three months (i.e., one quarter) until the end of the sample period (2021/03/05), resulting in a total of 24 adjustments. Each adjustment incorporates all available historical data up to that point and recalculates the optimal asset allocation weights for the following three months using the return forecasting model. The initial net asset value (NAV) is set to 1. Based on the descriptive statistics and model outputs, weekly portfolio NAVs are computed, totaling 500 values. The performance evaluation involves the following metrics:

(a) **Annualized return**: $V^{100/N} - 1$, where V is the final portfolio net value and N is the number of investment periods.

(b) **Annualized volatility**:

$$\left[\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \overline{r})^2\right] \times \sqrt{100},$$

where r_i is the weekly return and $\overline{r} = \frac{1}{N}\sum_{i=1}^{N} r_i$ is the average return.

(c) **Return-to-risk ratio**: Defined as the ratio of annualized return to annualized volatility.

(d) **Maximum drawdown**:

$$\max\left\{(V_i - V_j)/V_i,\ j < i\right\},$$

where V_i and V_j denote the portfolio NAVs at weeks i and j, respectively.

(e) **Return-to-drawdown ratio**: Computed as annualized return divided by maximum drawdown.

It is assumed that the risk policy in the management of multi-period equity investments is to keep the annualized volatility within 1%. Based on this risk objective, the net value of the multi-period equity portfolio is calculated, and the net value of the serial multi-period equity portfolio for the period 2019/03/03-2021/03/03 is shown in Figure 7, which shows that the equity portfolio formulated based on the CVaR model of implied volatility has a consistently higher and smoother net asset value than that of the portfolio constructed based on the CVaR model.

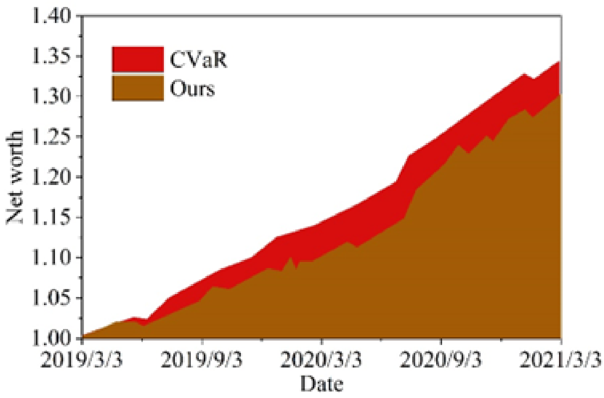


Fig. 7. Multi-cycle stock portfolio net worth series female

The performance indicators of the portfolio are shown in Table 2. Based on the data in the table, it can be seen that the equity asset portfolio constructed by the model in this paper not only dominates in terms of annualized return, but also all the risk indicators are better than those predicted using the empirical estimation method. What is more noteworthy is that the annualized volatility of the constructed portfolio using the CvaR model for asset allocation decision is 1.207%, which breaks through the pre-set risk target. In contrast, with the introduction of implied volatility based on the CvaR model, the portfolio has an annualized volatility of 0.788%, which meets the requirements of the risk policy.

Table 2. Portfolio performance indicators

Index	CVaR	Ours
Annualized return	4.566%	5.087%
Annualized volatility	1.207%	0.788%
Maximum pullback	1.237%	0.404%
Benefit-risk ratio	3.751	6.353
Earnings retracement ratio	3.675	12.459

Assuming that no losses are expected with 95% probability, which is a more commonly used risk management tool relative to volatility, 666 data points (2021/03/06-2023/10/30) are used for the external sample test, which in turn computes the stock portfolio NAV series, which is shown in Figure 8. Although the difference is not very large, the net worth curve of the implied volatility-based CvaR model is more stable.

As shown in Table 3, there is a 95% probability that the risk objective of "no expected loss" is not met for the asset allocation portfolios using the CVaR methodology. In contrast, portfolios that use the implied volatility-based CVaR model for forecasting and decision-making are not expected to lose money, which is consistent with the asset risk management objective. In summary, fitting the return and risk structure of asset returns based on the implied volatility-based CVaR model and constructing optimal allocation portfolios based on it leads to better investment performance and better risk management. It also shows that in-depth study of the distributional characteristics of asset returns and modeling of them can effectively improve the drawbacks of traditional methods in the process of asset allocation, and has strong application value.

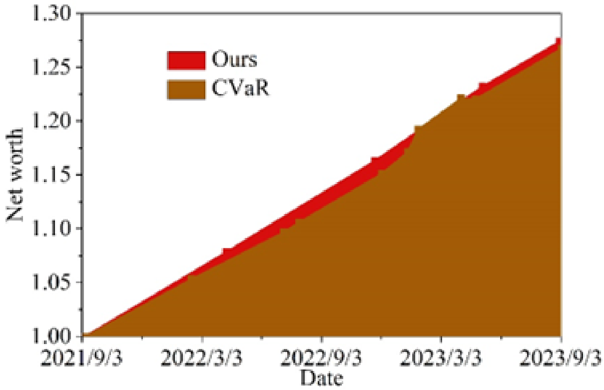


Fig. 8. Net worth series

Table 3. Income-risk characteristics

Index	CVaR	Ours
Annualized return	4.088%	4.086%
Annualized volatility	0.449%	0.235%
Maximum pullback	0.289%	0.018%
Benefit-risk ratio	8.949	17.046
Earnings retracement ratio	13.767	153.351

4. Conclusion

Based on the theory of volatility and conditional value-at-risk (CVaR), the portfolio optimization problem is identified and the implied volatility-based CvaR model is constructed using scenario trees. The research sample is selected and the research data sources are also identified, followed by an arithmetic example analysis of this paper's model. In the actual stock asset investment process, the proportion of resources invested in stock A of this paper's model is relatively small, which effectively avoids the risk of stock price decline caused by stock A, and also verifies the effectiveness of the modeling method using scenario tree. In addition, based on the CvaR method, the risk goal of "expected no loss" cannot be achieved, in contrast, the CVaR model based on implied volatility satisfies the optimal allocation combination of stock assets and will not suffer losses, which effectively overcomes the drawbacks of traditional methods in the process of multi-cycle stock asset allocation, and provides investors with a safer risk management prevention of stock assets.

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