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On edge-disjoint Ramsey numbers of stars

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ABSTRACT

For a graph F and a positive integer t, the edge-disjoint Ramsey number $ER_t(F)$ is the minimum positive integer n such that every red-blue coloring of the edges of the complete graph K_n of order n results in t pairwise edge-disjoint monochromatic copies of a subgraph isomorphic to F. Since $ER_1(F)$ is in fact the Ramsey number of F, this concept extends the standard concept of Ramsey number. We investigate the edge-disjoint Ramsey numbers $ER_t(K_{1,n})$ of the stars $K_{1,n}$ of size n. Formulas are established for $ER_t(K_{1,n})$ for all positive integers n and $t = 2, 3, 4$ and bounds are presented for $ER_t(K_{1,n})$ for all positive integers n and $t \geq 5$. Furthermore, exact values of $ER_t(K_{1,n})$ are determined for $n = 3, 4$ and several integers $t \geq 5$.

Keywords: red-blue coloring, edge-disjoint Ramsey numbers, stars

1. Introduction

Perhaps the best known Ramsey number is $R(K_3, K_3)$, sometimes written as $R(K_3)$. This is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is always a monochromatic subgraph F of K_n where F is isomorphic to K_3. It is well known that $R(K_3) = 6$. This result was established by Greenwood and Gleason [4] in 1955. In fact, this result essentially appeared as Problem A2 in the 1953 William Lowell Putnam Exam (see [4]).

While every red-blue coloring of K_6 always results in a monochromatic triangle K_3 (all whose edges are colored the same), it turns out that every red-blue coloring of K_6 always results in at least two monochromatic triangles. However, it is not true that every red-blue coloring of K_6 always results in two edge-disjoint monochromatic triangles. For example, in the red-blue coloring of K_6 in Figure 1 (where a bold edge represents a red edge and a thin edge represents a blue edge), there do not exist two edge-disjoint monochromatic triangles. On the other hand, every red-blue coloring of K_7 always results in two edge-disjoint monochromatic triangles.

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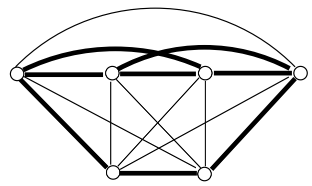

Fig. 1. A red-blue coloring of K_6

Proposition 1.1. [2] *Every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles.*

This brings up the problem of determining for a positive integer t and a graph F without isolated vertices, the minimum positive integer n such that for every red-blue coloring of the complete graph K_n, there always exist t pairwise edge-disjoint monochromatic copies of the graph F. Such an integer n is referred to as the *edge-disjoint Ramsey number* $ER_t(F)$ of F in [2]. If $t = 1$, then $ER_1(F) = R(F)$ is the standard Ramsey number. Therefore, the concept of edge-disjoint Ramsey number extends the standard concept of Ramsey number. It is well-known that Ramsey numbers exist for every graph F (see [6]). This is also the case for edge-disjoint Ramsey numbers.

Observation 1.2. [2] *For every graph F without isolated vertices and every positive integer t, the number $ER_t(F)$ exists and $ER_t(F) \leq ER_{t+1}(F)$.*

By Proposition 1.1, $ER_2(K_3) = 7$. We turn our intention here to investigating the edge-disjoint Ramsey numbers $ER_t(F)$ where $F = K_{1,n}$ is a star of order $n+1$ and size n. We refer to the book [3] for notation and terminology not defined here. In [2] the following result was obtained for $K_{1,2}$.

Theorem 1.3. [2] *For every positive integer t,*

$$ER_t(K_{1,2}) = \lceil 2\sqrt{t} + 1 \rceil.$$

We now investigate the numbers $ER_t(K_{1,n})$ for certain values of $n \geq 3$ and positive integers t. First, we state the Ramsey numbers of stars, obtained by Burr and Roberts [1].

Theorem 1.4. [1] *For $n \geq 3$, $R(K_{1,n}) = \begin{cases} 2n & \text{if } n \text{ is odd,} \\ 2n-1 & \text{if } n \text{ is even.} \end{cases}$*

Proposition 1.5. *For integers $n \geq 2$ and $t \geq 2$,*

$$ER_t(K_{1,n}) \leq ER_{t+1}(K_{1,n}) \leq ER_t(K_{1,n}) + 1.$$

Proof. The lower bound is a consequence of Observation 1.2. To verify the upper bound, let $N = ER_t(K_{1,n}) + 1$ and let there be given a red-blue coloring of $G = K_N$. Next, let $v \in V(G)$. Then $G - v \cong K_{N-1}$. Since $ER_t(K_{1,n}) = N - 1$, there are t pairwise edge-disjoint monochromatic copies $F_1, F_2, \ldots, F_t$ of $K_{1,n}$. By Observation 1.2 and Theorem 1.4, $ER_t(K_{1,n}) = N - 1 \geq R(K_{1,n}) \geq 2n - 1$ and so $\deg_G v = N - 1 \geq 2n - 1$. Hence, there are at least n edges incident with v that have the same color, producing a monochromatic copy F_{t+1} of $K_{1,n}$ that is edge-disjoint from F_i for

$1 \le i \le t$. Therefore, G contains $t+1$ pairwise edge-disjoint monochromatic copies of $K_{1,n}$ and so

$$ER_{t+1}(K_{1,n}) \le N = ER_t(K_{1,n}) + 1.$$

□

2. On the edge-disjoint Ramsey numbers $ER_t(K_{1,3})$

In [5] the following results were obtained.

Proposition 2.1. [5] *$ER_2(K_{1,3}) = 6$ and $ER_3(K_{1,3}) = ER_4(K_{1,3}) = 7$.*

We now determine $ER_t(K_{1,3})$ for $5 \le t \le 7$. First, the following bound for $ER_t(K_{1,n})$ when $t \ge 2$ will be useful.

Proposition 2.2. *For each integer $t \ge 2$ $ER_t(K_{1,3}) \ge \left\lceil \frac{3+\sqrt{9+24t}}{2} \right\rceil$.*

Proof. Let $ER_t(K_{1,3}) = k$. Then $k \ge R(K_{1,3}) = 6$ by Theorem 1.4. Hence, every red-blue coloring of $G = K_k$ results in t pairwise edge-disjoint monochromatic copies of $K_{1,3}$. Consider a red-blue coloring of G such that the red subgraph is $G_r = C_k$ and the blue subgraph is $G_b = G - E(C_k)$. Since $G_r = C_k$ is 2-regular of order $k \ge 6$, there is no red $K_{1,3}$ in G. Thus, G_b contains t pairwise edge-disjoint copies of $K_{1,3}$. Since the size of G_b is $\binom{k}{2} - k$, it follows that $\binom{k}{2} - k \ge 3t$ or $k^2 - 3k - 6t \ge 0$. Hence, $k \ge \frac{3+\sqrt{9+24t}}{2}$ and so $ER_t(K_{1,3}) = k \ge \left\lceil \frac{3+\sqrt{9+24t}}{2} \right\rceil$. □

Proposition 2.3. *$ER_5(K_{1,3}) = 8$.*

Proof. Since $ER_5(K_{1,3}) \le ER_4(K_{1,3}) + 1 = 8$ by Proposition 1.5 and

$$ER_5(K_{1,3}) \ge \left\lceil \frac{3+\sqrt{9+24\cdot 5}}{2} \right\rceil = 8,$$

by Proposition 2.2, it follows that $ER_5(K_{1,3}) = 8$. □

In order to show $ER_6(K_{1,3}) = 8$, we first introduce some additional definitions and notation as well as four lemmas. We omit the proofs of these lemmas since they are straightforward. For a graph G, let $v_1, v_2, \ldots, v_k$ be k vertices of G such that $\deg v_i = d_i$ for $1 \le i \le k$ and $d_1 \le d_2 \le \cdots \le d_k$. Let $S = (s_1, s_2, \ldots, s_k)$ be a sequence of k positive integers. We say that the degree sequence of $v_1, v_2, \ldots, v_k$ is *at least* S if $d_i \ge s_i$ for $1 \le i \le k$. For an integer $n \ge 2$, we write (a^n) for a sequence $(a, a, \ldots, a)$ of n elements a.

Lemma 2.4. *Let H be a graph of order 8. If*
(1) *$\Delta(H) \ge 6$,*
(2) *H contains two vertices x and y such that $\deg x \ge 3$ and $\deg y \ge 4$, or*
(3) *H contains five vertices of degree 3 or more,*
then H contains two pairwise edge-disjoint copies of $K_{1,3}$.

Lemma 2.5. *Let H be a graph of order 8. If H contains*
(1) *two vertices whose degree sequence is at least* $(4, 6)$,
(2) *three vertices whose degree sequence is at least* $(3, 4, 5)$,
(3) *five vertices whose degree sequence is at least* $(3, 4^4)$,
(4) *five vertices whose degree sequence is at least* $(3^4, 6)$, *or*
(5) *six vertices whose degree sequence is at least* $(3^3, 4^3)$,
then H contains three pairwise edge-disjoint copies of $K_{1,3}$.

Lemma 2.6. *Let H be a graph of order 8. If H contains*
(1) *four vertices whose degree sequence is at least* $(3, 4, 5, 6)$,
(2) *five vertices whose degree sequence is at least* $(3, 5^4)$,
(3) *six vertices whose degree sequence is at least* $(3^4, 5, 6)$,
(4) *six vertices whose degree sequence is at least* $(3^2, 4, 5^3)$,
(5) *six vertices whose degree sequence is at least* $(3, 4^4, 6)$,
(6) *six vertices whose degree sequence is at least* $(3^2, 4^3, 7)$, *or*
(7) *seven vertices whose degree sequence is at least* $(3, 4^5, 5)$,
then H contains four pairwise edge-disjoint copies of $K_{1,3}$.

Lemma 2.7. *Let H be a graph of order 8. If H contains*
(1) *seven vertices whose degree sequence is at least* (5^7) *or*
(2) *eight vertices whose degree sequence is at least* $(4^4, 5^4)$,
then H contains five pairwise edge-disjoint copies of $K_{1,3}$.

With the aid of Lemmas 2.4–2.7, we are able to determine the value of $ER_6(K_{1,3})$.

Theorem 2.8. $ER_6(K_{1,3}) = 8$.

Since the proof of Theorem 2.8 is quite lengthy, we will only provide an outline of a proof. Since $ER_6(K_{1,3}) \geq ER_5(K_{1,3}) = 8$ by Proposition 2.3, it remains to show that $ER_6(K_{1,3}) \leq 8$. To establish this inequality, it is required to show that for every red-blue coloring of $G = K_8$, there are six pairwise edge-disjoint monochromatic copies of subgraphs of K_8 that are isomorphic to $K_{1,3}$. So, let there be given a red-blue coloring of G resulting in the red subgraph G_r of order 8 and size m_r and the blue subgraph G_b of order 8 and size m_b. We may assume that $m_r \geq m_b$. Since the size of G is $\binom{8}{2} = 28$, it follows that $m_r \geq 14$. By investigating the possible values of $\delta(G_r)$ and $\Delta_r(G_r)$ as well as the possible degree sequences of G_r, we show that every red-blue coloring of G produces six pairwise edge-disjoint monochromatic copies of subgraphs of K_8 that are isomorphic to $K_{1,3}$.

We begin by investigating the situation where $\delta(G_r) \geq 5$. Here, we show that the red subgraph G_r has six pairwise edge-disjoint copies of $K_{1,3}$ by considering three cases, according to whether $\Delta(G_r)$ is $5, 6$, or 7. If $\Delta(G_r) = 5$, then G_r is a 5-regular graph of order 8. Since the complement $\overline{G_r}$ of G_r is one of C_8, $2C_4$ (the union of two vertex-disjoint copies of a 4-cycle C_4), or $C_3 + C_5$ (the union of the vertex-disjoint cycles C_3 and C_5), there are exactly three possibilities for G_r, namely $\overline{C_8}$, $\overline{2C_4}$, or $\overline{C_3 + C_5}$. In each of these three 5-regular graphs of order 8, there are six pairwise edge-disjoint copies $F_1, F_2, \ldots, F_6$ of $K_{1,3}$. This is shown in Figure 2 where an edge labeled i belongs to F_i for $1 \leq i \leq 6$. Observe that the size of G_r is 20 and $\sum_{i=1}^{6} |E(F_i)| = 18$ and so two edges of G_r do not belong to any F_i for $1 \leq i \leq 6$.

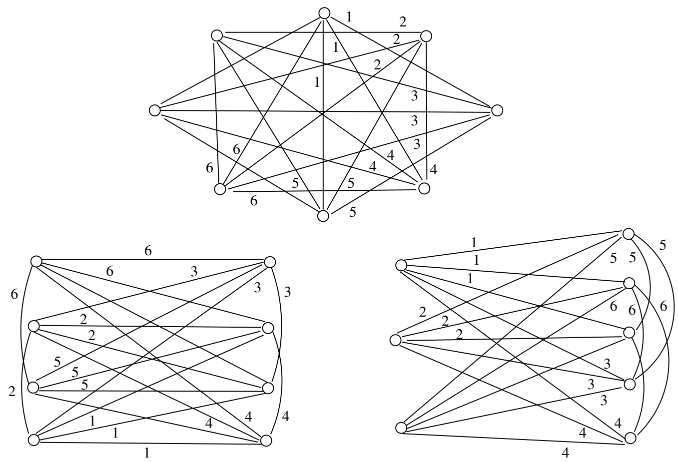

Fig. 2. Six pairwise edge-disjoint copies of $K_{1,3}$ in G_r

If $\Delta(G_r) = 6$ or $\Delta(G_r) = 7$, then there are three possible cases, according to the degree sequences of G_r. Applying Lemmas 2.4–2.7 to the red subgraph G_r, we are able to show that G_r contains six pairwise edge-disjoint copies of $K_{1,3}$ in each of these cases.

For the situation where $\delta(G_r) \le 4$, it follows that $\Delta(G_b) \ge 3$ and so the blue subgraph G_b contains at least one copy of $K_{1,3}$. Here, eight cases are considered according to the degrees of the vertices of G_r. In each case, we show that for a desired number x of pairwise edge-disjoint copies of $K_{1,3}$ in G_r where $3 \le x \le 5$, there are $x-2$ pairwise edge-disjoint copies $F_1, F_2, \ldots, F_{x-2}$ of $K_{1,3}$ in G_r such that there are two vertices u_1 and u_2 in the subgraph $G'_r = G_r - [E(F_1) \cup E(F_2) \cup \cdots \cup E(F_{x-2})]$ for which $\deg_{G'_r} u_1 \ge 3$ and $\deg_{G'_r} u_2 \ge 4$. Consequently, by Lemma 2.4 there are two edge-disjoint copies F_{x-1} and F_x of $K_{1,3}$ in G'_r such that F_{x-1} is centered at u_1 and F_x is centered at u_2. This then implies that G_r has x pairwise edge-disjoint copies of $K_{1,3}$. Furthermore, we show that the blue subgraph G_b has at least $6-x$ pairwise edge-disjoint copies of $K_{1,3}$. Consequently, there are six pairwise edge-disjoint monochromatic copies of $K_{1,3}$ in G.

Therefore, Theorem 2.8 can be verified using an extensive case-by-case analysis together with Lemmas 2.4–2.7. Knowing that $ER_6(K_{1,3}) = 8$ makes it easy to determine $ER_7(K_{1,3})$. First, $ER_7(K_{1,3}) \le ER_6(K_{1,3}) + 1 = 9$ by Proposition 1.5 and Theorem 2.8. Next, $ER_7(K_{1,3}) \ge 9$ by Proposition 2.2. Therefore, we have the following result.

Proposition 2.9. $ER_7(K_{1,3}) = 9$.

3. The edge-disjoint Ramsey numbers $ER_t(K_{1,n})$ for $1 \le t \le 4$

We mentioned that $ER_t(K_{1,3})$ was determined in [5] for $1 \le t \le 4$. Actually, for $1 \le t \le 4$, $ER_t(K_{1,n})$ can be determined for every integer $n \ge 2$.

Theorem 3.1. *For each integer $n \ge 2$, $ER_2(K_{1,n}) = 2n$.*

Proof. We saw that $ER_2(K_{1,2}) = 4$ and $ER_2(K_{1,3}) = 6$. Thus, we may assume that $n \ge 4$. We consider two cases, according to whether n is even or n is odd.

Case 1. *$n \ge 4$ is even.* In this case, $R(K_{1,n}) = 2n-1$. First, we show that $ER_2(K_{1,n}) \le 2n$. Let there be given a red-blue coloring of $G = K_{2n}$. Since $R(K_{1,n}) = 2n-1$, it follows that G contains a monochromatic copy F_1 of $K_{1,n}$. Let $v \in V(G) - V(F_1)$. Since $\deg_G v = 2n-1$, there are at least n

edges incident with v that have the same color, producing a monochromatic copy F_2 of $K_{1,n}$ that is edge-disjoint from F_1. Therefore, $ER_2(K_{1,n}) \leq 2n$.

Next, we show that $ER_2(K_{1,n}) \geq 2n$. Let there be given a red-blue coloring of $H = K_{2n-1}$ such that the blue subgraph G_b is obtained from $H_1 = K_n$ with $V(K_n) = \{u_1, u_2, \dots, u_n\}$ and $H_2 = K_{n-1}$ with $V(K_{n-1}) = \{v_1, v_2, \dots, v_{n-1}\}$ by adding the edges u_iv_i for $1 \leq i \leq n-1$. Thus, $\deg_{G_b} u_i = n$ for $1 \leq i \leq n-1$ and all other vertices of G_b have degree $n-1$. Hence, G_b contains a copy F_i of $K_{1,n}$ whose center is u_i where $1 \leq i \leq n-1$. For each pair $i, j \in \{1, 2, \dots, n-1\}$ and $i \neq j$, both F_i and F_j contain the edge u_iu_j and so are not edge-disjoint. Furthermore, $\deg_{G_b} u_n = \deg_{G_b} v_i = n-1$ for $1 \leq i \leq n-1$. Thus, there is no $K_{1,n}$ centered at u_n or at v_i for $1 \leq i \leq n-1$. Hence, G_b does not contain two edge-disjoint copies of $K_{1,n}$. Every vertex of the red subgraph G_r has degree $n-2$ or $n-1$. Thus, there is no $K_{1,n}$ in G_r. Therefore, this red-blue coloring does not produce two edge-disjoint monochromatic copies of $K_{1,n}$ and so $ER_2(K_{1,n}) \geq 2n$. Hence, $ER_2(K_{1,n}) = 2n$ when $n \geq 4$ is even.

Case 2. $n \geq 5$ is odd. In this case, $R(K_{1,n}) = 2n$. Since $ER_2(K_{1,n}) \geq R(K_{1,n}) = 2n$, it remains to show that $ER_2(K_{1,n}) \leq 2n$. Let there be given a red-blue coloring of $H = K_{2n}$ resulting in a red subgraph H_r and blue subgraph H_b. If $\Delta(H_r) \geq n$ and $\Delta(H_b) \geq n$, then H contains two edge-disjoint monochromatic copies of $K_{1,n}$. Thus, we may assume that $\Delta(H_r) \leq n-1$ and so $\delta(H_b) \geq n$. If $H_b = K_{2n}$, then for every two vertices u and v, there are two edge-disjoint copies of $K_{1,n}$ centered at u and v. If $H_b \neq K_{2n}$, then let x and y be two nonadjacent vertices of H_b. Since $\deg_{H_b} x \geq n$ and $\deg_{H_b} y \geq n$, it follows that H_b contains two edge-disjoint copies of $K_{1,n}$ centered at x and y. Therefore, $ER_2(K_{1,n}) \leq 2n$ and so $ER_2(K_{1,n}) = 2n$ when $n \geq 5$ is even. □

Theorem 3.2. *For each integer $n \geq 2$, $ER_3(K_{1,n}) = 2n + 1$.*

Proof. By Theorem 3.1 and Proposition 1.5, it follows that $ER_3(K_{1,n}) \leq 2n+1$. Thus, it remains to show that $ER_3(K_{1,n}) \geq 2n+1$. Define a red-blue coloring of $G = K_{2n}$ with the red subgraph $G_r = K_n \,\square\, K_2$. Then G_r is an n-regular graph. Thus, two copies of $K_{1,n}$ in G_r are edge-disjoint if and only if their centers are not adjacent. Since the vertex independent number of G_r is $\alpha(G_r) = 2$, there are two edge-disjoint copies of $K_{1,n}$ in G_r but no three pairwise edge-disjoint copies of $K_{1,n}$ in G_r. Furthermore, the blue subgraph G_b is an $(n-1)$-regular graph and so there is no $K_{1,n}$ in G_b. Therefore, this red-blue coloring does not produce three pairwise edge-disjoint copies of $K_{1,n}$ and so $ER_3(K_{1,n}) \geq 2n+1$. □

Next, we show that $ER_4(K_{1,n}) = ER_3(K_{1,n}) = 2n+1$ for each integer $n \geq 2$. Since $ER_4(K_{1,2}) = 5$ and $ER_4(K_{1,3}) = 7$ by Proposition 2.1, we may assume that $n \geq 4$. Applying an extensive case-by-case analysis, the exact value of $ER_4(K_{1,4})$ can be determined, as we state next. Since this result is also a consequence of Proposition 1.5, Theorem 3.2, and a stronger result (Theorem 4.2) in Section 4, we omit its proof.

Theorem 3.3. $ER_4(K_{1,4}) = 9$.

Theorem 3.3 can be extended to $ER_4(K_{1,n})$ for all integers $n \geq 2$.

Theorem 3.4. *For each integer $n \geq 2$, $ER_4(K_{1,n}) = 2n + 1$.*

Proof. Since $ER_4(K_{1,2}) = 5$ and $ER_4(K_{1,3}) = 7$ by Proposition 2.1 and $ER_4(K_{1,4}) = 9$ by Theorem 3.3, we may assume that $n \geq 5$. By Observation 1.2, $ER_4(K_{1,n}) \geq ER_3(K_{1,n}) = 2n + 1$. Hence, it remains to show that $ER_4(K_{1,n}) \leq 2n + 1$. Let there be given a red-blue coloring of $G = K_{2n+1}$. Since $ER_3(K_{1,n}) = 2n + 1$, there are three pairwise edge-disjoint monochromatic copies F_1, F_2, F_3 of $K_{1,n}$. Let

$$H = G[E(F_1) \cup E(F_2) \cup E(F_3)].$$

Then $|V(H)|= n_H \leq 2n + 1$ and $|E(H)|= m_H = 3n$.

$\star$ If $n_H \leq 2n$, then let $v \in V(G) - V(H)$. Since $\deg_G v = 2n$, there are at least n edges incident with v that have the same color and so there is a monochromatic copy F_4 of $K_{1,n}$ centered at v that is edge-disjoint from F_1, F_2, F_3.

$\star$ If there is an end-vertex u in H, then u is incident with $2n - 1$ edges not in H and so there are at least n edges incident with u that have the same color. Thus, there is a monochromatic copy F_4 of $K_{1,n}$ centered at u that is edge-disjoint from F_1, F_2, F_3.

$\star$ Thus, we may assume that $n_H = 2n+1$ and $\delta(H) \geq 2$. Since at least three vertices have degree n in H (namely, the centers of F_1, F_2, F_3), it follows that

$$\sum_{x \in V(H)} \deg_H x \geq 3n + 2(2n - 2) = 7n - 4.$$

Since $n \geq 5$, it follows that $3n = m_H \geq \dfrac{7n - 4}{2} > 3n$, which is impossible. Therefore, $ER_4(K_{1,n}) = 2n + 1$ when $n \geq 5$. $\square$

4. On the edge-disjoint Ramsey numbers $ER_t(K_{1,n})$ for $t \geq 5$

Since the numbers $ER_t(K_{1,n})$ have been determined for all positive integers n and for every integer t with $1 \leq t \leq 4$, it remains to investigate $ER_t(K_{1,n})$ for positive integers n when $t \geq 5$. In Proposition 2.3, we saw that $ER_5(K_{1,3}) = 8$. We now determine $ER_5(K_{1,4})$. First, we state a useful lemma. Since the proof of this lemma is straightforward, we omit its proof.

Lemma 4.1. *Let H be a graph of order 9. If H has*

(1) *two vertices x and y such that $\deg x \geq 4$ and $\deg y \geq 5$ or*

(2) *six vertices of degree 4,*

then H contains two edge-disjoint copies of $K_{1,4}$.

Theorem 4.2. $ER_5(K_{1,4}) = 9$.

Proof. Since $ER_3(K_{1,4}) = 9$ by Theorem 3.2, it follows by Observation 1.2 that $ER_5(K_{1,4}) \geq 9$. Thus, it remains to show that $ER_5(K_{1,4}) \leq 9$. Let there be given a red-blue coloring of $G = K_9$ resulting in a red subgraph G_r of order 9 and size m_r and a blue subgraph G_b of order 9 and size m_b. We may assume that $m_r \geq m_b$. Since the size of G is $\binom{9}{2} = 36$, it follows that $m_r \geq 18$. We consider ten cases depending on the number of vertices of degree 5 or more in G_r.

Case 1. *Every vertex of G_r has degree 5 or more.* Thus, $\Delta(G_b) \leq 3$ and so G_b has no copy of $K_{1,4}$. We show that G_r has five pairwise edge-disjoint copies of $K_{1,4}$. Since $6 \leq \Delta(G_r) \leq 8$, there are three subcases, according to $\Delta(G_r) = 8$, $\Delta(G_r) = 7$ or $\Delta(G_r) = 6$.

Subcase 1.1. $\Delta(G_r) = 8$. Then the degree sequence of G_r is at least $(5^8, 8)$. Let v_1 be a vertex of minimum degree in G_r. Then v_1 is the center of a copy F_1 of $K_{1,4}$. Let $H_1 = G_r - E(F_1)$. Then H_1 contains eight vertices whose degree sequence is at least $(4^4, 5^3, 7)$. Let v_2 be a vertex of minimum degree at least 4 in H_1. Then v_2 is the center of a copy F_2 of $K_{1,4}$. Let $H_2 = H_1 - E(F_2)$. Then H_2 contains seven vertices whose degree sequence is at least

(i) $(3^3, 4, 5^2, 7)$,
(ii) $(3^2, 4^3, 5, 7)$,
(iii) $(3, 4^5, 7)$, or
(iv) $(4^6, 6)$.

Let v be a vertex in H_2 such that $\deg_{H_2} v \geq 7$, let v_3 be a vertex of minimum degree at least 4 in H_2 such that v_3 is adjacent to v, and let F_3 be a copy of $K_{1,4}$ centered at v_3 such that $v \in V(F_3)$. Let $H_3 = H_2 - E(F_3)$. Then H_3 has two vertices x and y such that $\deg_{H_3} x \geq 4$ and $\deg_{H_3} y \geq 5$. By Lemma 4.1, H_3 contains two edge-disjoint copies F_4 and F_5 of $K_{1,4}$ that are edge-disjoint from F_1, F_2, and F_3. Thus, G_r has five pairwise edge-disjoint copies of $K_{1,4}$.

Subcase 1.2. $\Delta(G_r) = 7$. Then the degree sequence of G_r is at least $(5^8, 7)$. Let v_1 be a vertex of minimum degree in G_r that is not adjacent to some vertex of degree 7. Then v_1 is the center of a copy F_1 of $K_{1,4}$. Let $H_1 = G_r - E(F_1)$. Then H_1 contains eight vertices whose degree sequence is at least $(4^4, 5^3, 7)$. Proceeding as in Subcase 1.1, we see that G_r has five pairwise edge-disjoint copies F_1, F_2, F_3, F_4, F_5 of $K_{1,4}$.

Subcase 1.3. $\Delta(G_r) = 6$. Then the degree sequence of G_r is at least $(5^8, 6)$. Let v be a vertex of degree 6 in G_r and let v_1 and v_2 be the two vertices of G_r that are not adjacent to v. Then v_1 is the center of a copy F_1 of $K_{1,4}$ such that $v \notin V(F_2)$. Let $H_1 = G_r - E(F_1)$. Thus, H_1 contains eight vertices whose degree sequence is at least $(4^4, 5^3, 6)$. Then v_2 is the center of a copy F_2 of $K_{1,4}$ in H_1 such that $v \notin V(F_2)$. Let $H_2 = G_r - E(F_2)$.

$\star$ If $\deg_{H_1} v_2 = 4$, then H_2 contains seven vertices whose degree sequence is at least $(3^3, 4, 5^2, 6)$, $(3^2, 4^3, 5, 6)$, or $(3, 4^5, 6)$.

$\star$ If $\deg_{H_1} v_2 \geq 5$, then we can choose F_2 such that one vertex of degree at least 5 that is distinct from v does not belong to F_2. Thus, H_2 contains seven vertices whose degree sequence is at least $(3^4, 5^2, 6)$ or $(3^3, 4^2, 5, 6)$.

Let F_3 be a copy of $K_{1,4}$ in H_2 centered at a vertex of smallest degree at least 4 such that $v \in V(F_3)$. Let $H_3 = H_2 - E(F_3)$. Then there are two vertices x and y such that $\deg_{H_3} x \geq 4$ and $\deg_{H_3} y \geq 5$. By Lemma 4.1, H_3 contains two edge-disjoint copies F_4 and F_5 of $K_{1,4}$ that are edge-disjoint from F_1, F_2, and F_3. Hence, G_r has five pairwise edge-disjoint copies of $K_{1,4}$.

Case 2. *The number of vertices of degree* 5 *or more in* G_r *is* 8. Since G_b contains a vertex of degree 4 or more, it follows that G_b has at least one copy of $K_{1,4}$. Thus, it remains to show that G_r contains four pairwise edge-disjoint copies of $K_{1,4}$. Here, G_r contains eight vertices whose degree sequence is at least (5^8). Let F_1 be a copy of $K_{1,4}$ of G_r centered at a vertex v of smallest degree that is at least 4 and let $H_1 = G_r - E(F_1)$. Thus, H_1 contains seven vertices whose degree sequence is at least $(4^4, 5^3)$. Let F_2 be a copy of $K_{1,4}$ centered at a vertex of smallest degree at least 4. Let $H_2 = H_1 - E(F_1)$. Then either H_2 contains two vertices x and y such that $\deg_{H_2} x \geq 4$ and $\deg_{H_2} y \geq 5$ or H_2 contains six vertices of degree 4 or more. In either situation, it follows by Lemma 4.1 that H_2 has two edge-disjoint copies F_3 and F_4 of $K_{1,4}$ that are edge-disjoint from F_1. Since G_b has a copy of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 3. *The number of vertices of degree* 5 *or more in* G_r *is* 7. Thus, G_r has two vertices of degree at most 4. Hence, G_b has at least one copy of $K_{1,4}$.

$\star$ First, suppose that G_r has two vertices of degree 4. Then the degree sequence of G_r is at least $(4^2, 5^7)$. Let F_1 be a copy of $K_{1,4}$ in G_r centered at a vertex of smallest degree that is at least 4 that is not adjacent to some vertex of degree 5 or more. Let $H_1 = G_r - E(F_1)$. Thus, H_1 contains eight vertices whose degree sequence is at least $(3, 4^3, 5^4)$. Let F_2 be a copy of $K_{1,4}$ in G_r centered at a vertex of smallest degree that is at least 4 and let $H_2 = H_1 - E(F_2)$. Then either H_2 contains two vertices x and y such that $\deg_{H_2} x \geq 4$ and $\deg_{H_2} y \geq 5$ or H_2 contains six vertices of degree 4 or more. In either situation, it follows by Lemma 4.1 that H_2 has two edge-disjoint copies F_3 and F_4 of $K_{1,4}$ that are edge-disjoint from F_1. Since G_b has a copy of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

$\star$ Next, suppose that G_r has at most one vertex of degree 4. Then G_b contains two vertices x and y such that $\deg_{G_b} x \geq 4$ and $\deg_{G_b} y \geq 5$ and so G_b contains two edge-disjoint copies of $K_{1,4}$ by Lemma 4.1. We show that G_r contains three pairwise edge-disjoint copies of $K_{1,3}$. Here, G_r has seven vertices whose degree sequence is at least (5^7). Let F_1 be a copy of $K_{1,4}$ of G_r centered at a vertex of smallest degree that is at least 4 and let $H_1 = G_r - E(F_1)$. Then H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. By Lemma 4.1, it follows that H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ that are edge-disjoint from F_1. Hence, G_r has three pairwise edge-disjoint copies of $K_{1,4}$. Since G_b has two copies of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 4. *The number of vertices of degree* 5 *or more in* G_r *is* 6. Thus, G_r has three vertices of degree at most 4 and G_b has at least three vertices of degree 4 or more. Hence, G_b has at least one copy of $K_{1,4}$. There are six vertices of G_r whose degree sequence is at least (5^6). Let F_1 be a copy of $K_{1,4}$ of G_r centered at a vertex of smallest degree that is at least 4 and let $H_1 = G_r - E(F_1)$. Thus, H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Thus, H_1 has two edge-disjoint copies of $K_{1,4}$ by Lemma 4.1. Hence, G_r contains three pairwise edge-disjoint copies of $K_{1,4}$. Thus, if G_b contains two edge-disjoint copies of $K_{1,4}$, then G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Let T be the set of three vertices in G_r having degree less than 5 and let S be the set of six vertices in G_r having degree 5 or more. If one or more vertices of T have degree less than 4 in G_r, then one or more vertices of T in G_b have degree 5 or more. This would result in G_b containing two edge-disjoint copies of $K_{1,4}$ by Lemma 4.1. Therefore, we may assume that every vertex of T has degree 4 in both G_r and G_b. If two vertices of T are not adjacent in G_b, then these two vertices are the centers of two edge-disjoint copies of $K_{1,4}$ in G_b. Hence, we may assume that every two vertices of T are adjacent in G_b. Therefore, $G_r[T] = \overline{K}_3$. Since every vertex of T has degree 4 in G_r, it follows that every vertex of T is adjacent to four vertices of S. Each of the vertices of T is the center of a copy of $K_{1,4}$ in G_r, resulting in three pairwise edge-disjoint copies of $K_{1,4}$. Furthermore, there are 12 edges joining the sets S and T in G_r. Let $H = G_r[S]$.

$\star$ First, suppose that some vertex of S, say v_1, is adjacent to less than two vertices of T in G_r. Then v_1 is adjacent to at least four vertices in S. Thus, v_1 is the center of a copy F_1 of $K_{1,4}$ in H. Let F_2, F_3, F_4 be the three pairwise edge-disjoint copies of $K_{1,4}$ centered at the vertices of T and let F_5 be a copy of $K_{1,4}$ in G_b. Therefore, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

$\star$ Next, suppose that no vertex of S is adjacent to less than two vertices of T in G_r. Since there are 12 edges joining the sets S and T in G_r, it follows that that every vertex of S is adjacent to exactly two of the three vertices of T.

If some vertex of S, say v_1, has degree greater than 5 in G_r, then v_1 is the center of a copy F_1

of $K_{1,4}$ in H. Again, let F_2, F_3, F_4 be the three pairwise edge-disjoint copies of $K_{1,4}$ centered at the vertices of T and let F_5 be a copy of $K_{1,4}$ in G_b. Therefore, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Thus, we may assume that no vertex of S has degree greater than 5 in G_r. This implies that $H = G_r[S]$ is a 3-regular graph of order 6. There are only two possibilities for H, namely, $K_{3,3}$ and $C_3 \square K_2$. Since every vertex of S is adjacent to exactly two of the three vertices of T, there are four pairwise edge-disjoint copies F_1, F_2, F_3, F_4 of $K_{1,4}$ centered at four vertices of S in each situation. This is shown in Figure 3, where an edge labeled i belongs to F_i for $i = 1, 2, 3, 4$ and each bold edge joins a vertex of S and a vertex of T. Let F_5 be a copy of $K_{1,4}$ in G_b. Therefore, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

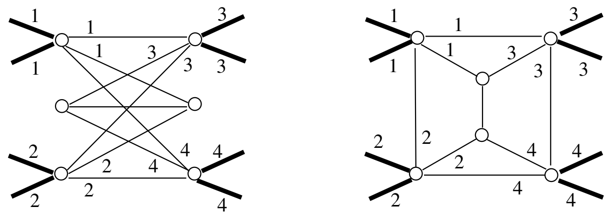

Fig. 3. A step in the proof of Case 4

Case 5. *The number of vertices of degree* 5 *or more in* G_r *is* 5. Again, G_b has at least one copy of $K_{1,4}$. Let T be the set of four vertices in G_r having degree less than 5 and let S be the set of five vertices in G_r having degree 5 or more. Since the number of odd vertices in G_r is even, either G_r has a vertex of degree 3 or less or G_r has a vertex degree 6 or more. We consider these two subcases.

Suncase 5.1. G_r *has a vertex of degree* 3 *or less.* Then G_b has two edge-disjoint copies F_4 and F_5 of $K_{1,4}$ by Lemma 4.1. We show that G_r has three pairwise edge-disjoint copies of $K_{1,4}$. Let $u \in S$. Since $\deg_{G_r} u \geq 5$, it follows that u is adjacent to a vertex $v \in T$. Let F_1 be a copy of $K_{1,4}$ centered at u such that $uv \in E(F_1)$. Let $H_1 = G_r - E(F_1)$. Thus, H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Thus, H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. Hence, G_r contains three pairwise edge-disjoint copies of $K_{1,4}$. Therefore, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Subcase 5.2. G_r *has a vertex of degree* 6 *or more.* Then G_r has five vertices whose degree sequence is at least $(5^4, 6)$. Let F_1 be a copy of $K_{1,4}$ centered at a vertex of smallest degree that is at least 4 and let $H_1 = G_r - E(F_1)$. Thus, H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$ and so H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. Hence, G_r contains three pairwise edge-disjoint copies of $K_{1,4}$. If G_r has a vertex of degree less than 4, then G_b has two edge-disjoint copies of $K_{1,4}$ by Lemma 4.1 and so G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$. Thus, we may assume that every vertex of T has degree 4 in G_r and in G_b. If there are two vertices of T that are not adjacent in G_b, then G_b has two edge-disjoint copies of $K_{1,4}$ centered at these two vertices. Thus, we may assume that $G_b[T] = K_4$ and so $G_r[T] = \overline{K}_4$. Therefore, there are four pairwise edge-disjoint copies of $K_{1,4}$ in G_r centered at the four vertices of T. Since G_b has a copy of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 6. *The number of vertices of degree* 5 *or more in* G_r *is* 4. Let T be the set of five vertices in G_r having degree less than 5 and let S be the set of four vertices in G_r having degree 5 or more. Next, let $u \in S$. Since $\deg_{G_r} u \geq 5$, it follows that u is adjacent to a vertex $v \in T$. Let F_1 be a copy of $K_{1,4}$ centered at u such that $uv \in E(F_1)$ and let $H_1 = G_r - E(F_1)$. Thus, H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Thus, H_1 has two edge-disjoint copies

F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. If $G_b[T] = K_5$, then $G_r[T] = \overline{K}_5$ and so there are five pairwise edge-disjoint copies of $K_{1,4}$ in G_r centered at the vertices of T. Thus, we may assume that there are two vertices in T that are not adjacent in G_b. Then G_b has two edge-disjoint copies F_4 and F_5 of $K_{1,4}$ centered at these two vertices. Therefore, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 7. *The number of vertices of degree* 5 *or more in* G_r *is* 3. Let u and v be two vertices of degree 5 or more in G_r, let F_1 be a copy of $K_{1,4}$ centered at u such that $uv \notin E(F_1)$, and let $H_1 = G_r - E(F_1)$. Thus, H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Hence, H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. Since G_b has six vertices of degree 4, it follows by Lemma 4.1 that G_b has two edge-disjoint copies F_4 and F_5 of $K_{1,4}$. Therefore, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 8. *The number of vertices of degree* 5 *or more in* G_r *is* 2. By Lemma 4.1, both G_r and G_b have at least two edge-disjoint copies of $K_{1,4}$. We consider four subcases.

Subcase 8.1. G_r *has at least two vertices of degree* 3 *or less.* Then G_b has seven vertices whose degree sequence is at least $(4^5, 5^2)$. Let $u, v \in V(G_b)$ such that $\deg_{G_b} u \geq 5$ and $\deg_{G_b} v \geq 5$. Then there is a copy F_1 of $K_{1,4}$ in G_b centered at u such that $uv \notin E(F_1)$. Let $H_1 = G_b - E(F_1)$. Then H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Thus, H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. Hence, G_b has three pairwise edge-disjoint copies of $K_{1,4}$. Since G_r has at least two edge-disjoint copies of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Subcase 8.2. G_r *has a vertex of degree* 2 *or less.* Then G_b has seven vertices whose degree sequence is at least $(4^6, 6)$. Let F_1 be a copy of $K_{1,4}$ cantered at a vertex of smallest degree at least 4 and let $H_1 = G_b - E(F_1)$. Then H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Thus, H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. Hence, G_b has three pairwise edge-disjoint copies of $K_{1,4}$. Since G_r has at least two edge-disjoint copies of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Subcase 8.3. G_r *has exactly one vertex of degree* 3 *and no other vertex of degree less than* 4. Then G_b has seven vertices with degree sequence at least $(4^6, 5)$. Therefore, G_b has a vertex u of degree 4 that is not adjacent to the vertex v of degree 5 in G_b. Let F_1 be a copy of $K_{1,4}$ and let $H_1 = G_b - E(F_1)$. Then H_1 contains two vertices x and y such that $\deg_{H_1} x \geq 4$ and $\deg_{H_1} y \geq 5$. Thus, H_1 has two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ by Lemma 4.1. Since G_r has at least two edge-disjoint copies of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Subcase 8.4. G_r *has no vertex of degree* 3 *or less.* Then the degree sequence of G_r is at least $(4^7, 5^2)$. Then G_r has three pairwise edge-disjoint copies of $K_{1,4}$ (by the argument for G_b in Subcase 8.3). Since G_b has at least two edge-disjoint copies of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 9. *The number of vertices of degree* 5 *or more in* G_r *is* 1. Again, both G_r and G_b have at least two edge-disjoint copies of $K_{1,4}$. If G_r has seven vertices with degree sequence at least $(4^6, 5)$, then G_r has three pairwise edge-disjoint copies of $K_{1,4}$ (by the argument for G_b in Subcase 8.3). If G_r does not have seven such vertices, then G_r has at most six vertices of degree 4 or more and so G_r has at least three vertices of degree 3 or less. Hence, G_b has eight vertices with degree sequence at least $(4^5, 5^3)$. Then G_b has three pairwise edge-disjoint copies of $K_{1,4}$ (by the argument for G_b in Subcase 8.1). In either situation, G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

Case 10. *The number of vertices of degree* 5 *or more in* G_r *is* 0. Since $m_r \geq 18$, it follows that

G_r is a 4-regular graph of order 9 and so G_b is also a 4-regular graph of order 9. By Lemma 4.1, both G_r and G_b contain two edge-disjoint copies of $K_{1,4}$. Let F_1 be a copy of $K_{1,4}$ in G_b and let $H_1 = G_b - E(F_1)$. Then H_1 contains eight vertices whose degree sequence is $(3^4, 4^4)$. Let T be the set of four vertices of degree 4 in H_1.

$\star$ If there are two vertices of T of degree 4 that are not adjacent in G_b, then these two vertices are centers of two edge-disjoint copies F_2 and F_3 of $K_{1,4}$ in H_1. Since G_r has two edge-disjoint copies of $K_{1,4}$ in G_r, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$.

$\star$ If every two vertices in T are adjacent in G_b, then $G_b[T] = K_4$ and so $G_r[T] = \overline{K}_4$. Thus, there are four pairwise edge-disjoint copies of $K_{1,4}$ in G_r. Since G_b has at least one copy of $K_{1,4}$, it follows that G has five pairwise edge-disjoint monochromatic copies of $K_{1,4}$. $\square$

As we mentioned earlier, Theorem 3.3 is a consequence of Proposition 1.5 as well as Theorems 3.2 and 4.2. Knowing that $ER_5(K_{1,4}) = 9$ now makes it easy to determine $ER_6(K_{1,4})$.

Theorem 4.3. $ER_6(K_{1,4}) = 10$.

Proof. Since $ER_6(K_{1,4}) \le ER_5(K_{1,4}) + 1 = 10$ by Theorem 4.2 and Observation 1.2, it remains to show that $ER_6(K_{1,4}) \ge 10$. Consider a red-blue coloring of $G = K_9$ such that the degree sequence of the red subgraph G_r is $(5^8, 6)$ and so the degree sequence of the blue subgraph G_b is $(2, 3^8)$. Since $\Delta(G_b) = 3$, there is no copy of $K_{1,4}$ in G_b. Furthermore, the size of G_r is 23 and the size of six pairwise edge-disjoint copies of $K_{1,4}$ is 24. Thus, G_r does not contain six pairwise edge-disjoint copies of $K_{1,4}$. Hence, this red-blue coloring of G does not produce six pairwise edge-disjoint monochromatic copies of $K_{1,4}$. Therefore, $ER_6(K_{1,4}) \ge 10$ and so $ER_6(K_{1,4}) = 10$. $\square$

We close this section by presenting lower bounds for $ER_t(K_{1,n})$ where $t \ge 5$ and $n \ge 3$.

Theorem 4.4. *For integers t and n where $t \ge 5$ and $n \ge 3$, let $ER_t(K_{1,n}) = k$.*

(1) *If k is even or n is odd, then* $ER_t(K_{1,n}) \ge \left\lceil \dfrac{n + \sqrt{n^2 + 8nt}}{2} \right\rceil$.

(2) *If k is odd and n is even, then* $ER_t(K_{1,n}) \ge \left\lceil \dfrac{n + \sqrt{n^2 + 8nt - 4}}{2} \right\rceil$.

Proof. Let $ER_t(K_{1,n}) = k$. Then every red-blue coloring of $G = K_k$ results in t pairwise edge-disjoint monochromatic copies of $K_{1,n}$. We consider two cases, according to whether either (1) k is even or n is odd or (2) k is odd and n is even.

Case 1. *k is even or n is odd.* Then there is an $(n-1)$-regular graph F of order k. Consider a red-blue coloring of G such that the red subgraph is $G_r = F$ and the blue subgraph is $G_b = G - E(F)$. Since G_r is $(n-1)$-regular, there is no red $K_{1,n}$ in G. Thus, G_b contains t pairwise edge-disjoint blue copies of $K_{1,n}$. Let m_b be the size of G_b. Since the size of F is $k(n-1)/2$ and the size of $K_{1,n}$ is n, it follows that

$$m_b = \binom{k}{2} - \frac{k(n-1)}{2} \ge nt.$$

Thus, $k(k-1) - k(n-1) \ge 2nt$ or $k^2 - nk - 2nt \ge 0$. Hence, $k \ge \dfrac{n + \sqrt{n^2 + 8nt}}{2}$. Since k is an integer, $k \geq \left\lceil \frac{n+\sqrt{n^2+8nt}}{2} \right\rceil$.

Case 2. *k is odd and n is even.* Then there is a graph H of order k such that exactly $k-1$ vertices of H have degree $n-1$ and one vertex of H has degree $n-2$. Consider a red-blue coloring of G such that the red subgraph is $G_r = H$ and the blue subgraph is $G_b = G - E(H)$. Since $\Delta(G_r) = n-1$ there is no red $K_{1,n}$ in G. Thus, G_b contains t pairwise edge-disjoint blue copies of $K_{1,n}$. Let m_b be the size of G_b. Since the size of H is $\frac{1}{2}[(k-1)(n-1)+(n-2)]$ and the size of $K_{1,n}$ is n, it follows that

$$m_b = \binom{k}{2} - \frac{(k-1)(n-1)+(n-2)}{2} \geq nt.$$

Then $k(k-1) - [(k-1)(n-1)+(n-2)] \geq 2nt$ or $k^2 - nk + (1-2nt) \geq 0$. Hence, $k \geq \frac{n+\sqrt{n^2+8nt-4}}{2}$. Since k is an integer, $k \geq \left\lceil \frac{n+\sqrt{n^2+8nt-4}}{2} \right\rceil$. □

From the results obtained thus far, we see that the lower bound in (1) of Theorem 4.4 is the exact value for $ER_t(K_{1,3})$ for $t = 5, 6, 7$ and the lower bound in (2) of Theorem 4.4 is the exact value for $ER_t(K_{1,4})$ for $t = 5, 6$. On the other hand, strict inequality holds for $ER_t(K_{1,5})$ for $t = 5$. Therefore, finding improved bounds for $ER_t(K_{1,n})$ for integers $t \geq 5$ and $n \geq 3$ would be valuable.

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