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Research on the strategy of rational utilization of agroforestry resources and maximization of ecological benefits based on the variable score method

Dan Wu[1, ✉]

[1] *People's Government of Dingan County, Dingan, Hainan, 571200, China*

ABSTRACT

This paper puts forward countermeasures to maximize the ecological benefits of agroforestry resources from the perspective of sustainable development of urban agroforestry resources. Taking the maximization of ecological benefits as the goal, the optimal allocation of agricultural and forestry resources is carried out. Based on the results of the optimal allocation of water resources, the planting structure of crops in the irrigation area is adjusted with the water allocation of irrigated crops as the constraint. The optimization model under the constraint of eco-efficiency objective was constructed based on the variational method and optimal control model, and the model was solved by the method of Pontryagin's great value. After the model adjustment in this paper, the planting structure of crops in the irrigation area of city A was obviously optimized, and the planting area of potatoes accounted for the largest share of the planting area of all the crops in the irrigation area, which was about 40.61%, and the ecological benefits of potato crops were higher, which got the priority of the model, and at the same time, the model also reduced the planting area of the crops with low ecological benefits, and this reasonable allocation adjustment method satisfied the goal of maximizing ecological benefits.

Keywords: variational method, optimal control model, planting structure, agroforestry resources

1. Introduction

The significance of sustainable utilization of agricultural and forestry specialty resources lies in changing the weak status of agriculture, improving the competitive quality of agricultural and forestry specialty products, prompting agriculture to break through the traditional scope of agricultural production, expanding to the processing industry, business, science and technology, forming a production, processing, science and technology, services, sales of the industrial chain, and gradually moving towards the characteristics of the agriculture [1-3]. This new form of production and

✉ Corresponding author.
E-mail address: wd0333@126.com (D. Wu).
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management, the realization of special resources in a wider range and higher level of optimization of the allocation and recombination of factors of production, and become an important form of operation in the new period to increase the income of farmers, the structural adjustment of the agricultural economy and the industrialization of rural areas [4-6].

China's sustainable utilization of agricultural and forestry specialty resources has entered a rapid development stage after active exploration [7]. However, most of the utilization of speciality resources is low, the scale of the leading enterprises is small, the competitiveness is not strong, the leading role is not obvious, industrialized management of intermediary organizations management is imperfect, the development is not sound, behavior is not standardized, the operation of the internal benefit distribution mechanism is not sound, in the farmers and the processing, marketing enterprises have not yet formed the true meaning of the community of interest, management system is not adapted to the support policy is not in place, talent, Technology, lack of funds is still a constraint on the sustainable use of agricultural and forestry specialties resources development of the “bottleneck” [8-10]. Modern agriculture and forestry specialty resources can not rely solely on the internal strength of agriculture itself to obtain sustainable and stable development, but need to rely on a complete set of adaptive support system to promote. It is of great significance to build a support system for the sustainable utilization of agricultural and forestry specialties and enhance the level of utilization of agricultural and forestry specialties for the construction of modern agriculture, promote the strategic adjustment of rural economic structure, increase the income of farmers and promote the coordinated development of urban and rural economy [11-14].

The sustainable utilization of agricultural and forestry resources has a specific meaning, that is, it is market-oriented, based on agricultural and forestry characteristic products and commodity production bases, and relying on leading enterprises, forming an industrial chain and interest community that organically combines agriculture, industry and commerce or agribusiness business, and forming a "one-stop" development pattern of regional layout, specialized production, large-scale operation, enterprise management, and socialized services [15-16]. The overall technical options for the sustainable use of agricultural and forestry specialties resources include: the positioning of leading characteristic industries, the development and growth of leading enterprises, the construction of market platforms, scientific and technological innovations, the construction of capital support systems, organizational system innovations, and the improvement of the benefit linkage mechanism, etc., which constitute the support system for agricultural and forestry specialties resources [17-18].

Participants in the sustainable use of agricultural and forestry speciality resources mainly include farmers (forest) households, processing enterprises, intermediary organizations, marketing enterprises, scientific and technological institutions and the government. Each participant plays different roles, produces different effects and creates their own interests [19-20]. Correct understanding of the different roles in the positioning of agricultural industrialization management, in order to play its own advantages and functions, to create maximum benefits. In the sustainable utilization of agricultural and forestry specialties, farmers and dragon-head enterprises are the main business and interest bodies [21-22].

At present, China's economy has made long-lasting progress and development, and the rapid development of the economy has caused greater pollution and damage to the agroforestry environment, and the environmental problems are becoming more and more worrying [23]. For the pollution of the agroforestry environment, only by strengthening the environmental protection of agroforestry can the ecological environment be avoided from being further damaged [24]. For the protection of the agroforestry environment, active and effective measures should be taken to ensure that the agroforestry environment can be effectively developed and utilized to realize the real sustainable development of resources [25-26].

Agroforestry resources are natural economic resources endowed by nature, which can not only purify the atmosphere, consolidate the soil, but also promote the development of biodiversity, and at the same time can provide a boost to human economic development, so it is very necessary to study the sustainable development and utilization of agroforestry resources. Literature [27] explains that forest resources play a positive role in controlling air pollution and soil erosion, and provide abundant biomass resources for human survival and economic development, and finally discusses the sustainable development and utilization of forest resources. Literature [28] revealed that wood burning is still the main source of energy in the Geche region of southwestern Ethiopia, which has led to excessive deforestation and exploitation of forest resources, and concluded that this situation can be improved by adopting energy efficient technologies, such as the planting of fast-growing trees to serve as fuelwood energy. Literature [29] based on multi-stage random sampling method, village household survey, while combining interviews, literature and other methods, reveals that the Manasbal range of forest zoning in Sindh province realizes the deep utilization of traditional forestry resources, while the residents of the region have a deep understanding of the botanical resources, and proposes to combine the development of diversified livelihoods with the region's forestry resources to enhance the incomes of the residents, and to promote the development of the local economy. The proposal is to develop diversified livelihoods by combining regional forestry resources to improve the income of the residents and promote local economic development.

In addition to the sustainable utilization and development of agroforestry resources, agroforestry ecological environment management and protection is also a key area of academic concern, and the focus of the research mainly includes biomass waste management, ecological environment prevention and control, and land degradation and other issues. Literature [30] reviewed the biodegradable, polluting and utilizable characteristics of biomass waste generated from agroforestry resources, as well as the sources of biomass waste, and focused on analyzing the potential value of biomass waste and means of utilization. Literature [31] analyzed the cutting-edge research, practical performance and technological barriers of biomass waste utilization means based on the perspective of technical routes such as direct combustion/gasification for power generation, methane, green hydrogen, biodiesel fuels, and methanol preparation, which made a positive contribution to the efficient utilization of biomass waste. Literature [32] used literature review methods such as search, evaluation, synthesis and analysis to analyze the research related to agroforestry ecosystem protection and agroforestry ecosystem services (AFES), focusing on summarizing the results of agroforestry ecosystem services (AFES) research and the future development trend, and at the same time summarizing and discussing around the topics of ecological disease prevention and control, land degradation, and so on.

This paper discusses the method of rational allocation of agroforestry resources, and constructs the model of maximizing the ecological benefit of agroforestry resources based on the variational method. The model takes water resources as the primary consideration, and adjusts the crop planting structure in the irrigation area under the limit of total water resources. Taking the maximization of ecological benefits as the goal, the variational method and optimal control theory are used to obtain the results of agroforestry resource allocation. In the solution process of the model for maximizing the ecological benefit of agroforestry resources, the ecological benefit coefficient is optimized, and the allocation of agroforestry resources is indirectly adjusted according to the optimization of the weights of the ecological benefit coefficient, and the resources are preferentially allocated to the parts with higher ecological benefit coefficients, so as to achieve the maximization of the overall ecological benefit. The validity of the model was verified by designing simulation experiments.

2. Multi-objective optimal allocation of agroforestry resources

2.1. Model for optimal allocation of water resources

According to the water use areas and subjects firstly, City A is divided into 12 sub-districts according to the administrative division, which is indicated by k. According to the living, production and ecology according to the different subjects of water use, divided into 7 water use sectors: living, ecology, planting, forestry, livestock, industry, construction, and tertiary industry, expressed in n. Then the water use of a sector in a certain region can be expressed as w_n^k, with a total of 84 variables in 10,000 m^3 units.

1) Objective function. Based on the current water resources situation in City A, combined with its development plan, this model is selected to maximize the ecological benefits as the objective, then the objective function is:

$$F(W) = opt\left[F_1(W), F_2(W), F_3(W)\right] \tag{1}$$

where: W is the decision variable, $F(W)$ is the overall objective of the multi-objective optimal allocation. $F_1(W)$, $F_2(W)$ and $F_3(W)$ are the expressions of the social, economic and ecological objective functions resulting from the allocation of water resources, respectively.

a. Social objective: The objective is to minimize the total water deficit in the region, to allow more water resources to be allocated to as many sectors as possible, and to minimize the pressure on the water supply.

$$F_l(W) = \min\left\{\sum_{n-1}^{N} \varphi_n \sum_{k-1}^{K}\left(D_n^k - w_n^k\right)\right\} \tag{2}$$

where: φ_n represents the water equity coefficient for water use sector n, and D_n^k represents the water demand for water use sector n.

b. Economic objective: The objective is to maximize the total regional production value and to maximize the ecological benefits of water resources.

$$F_2(X) = \max\left\{\sum_{k=1}^{K}\sum_{n=1}^{N} w_n^k \rho_n^k\right\} \tag{3}$$

where: ρ_n^k represents the coefficient of water production value of the nrd water sector in region k, in yuan$/m^3$.

c. Ecological objectives: ecological to wastewater sewage pollution of the environment to minimize the target, selected representative COD emissions as an indicator, the unit is t.

$$F_3(X) = \min\left\{0.01\sum_{n=1}^{N}\left(\alpha_n \mu_n \sum_{k=1}^{K} w_n^k\right)\right\} \tag{4}$$

where: 0.01 denotes the unit coefficient, α_n denotes the effluent discharge coefficient of the nnd water sector, μ_i denotes the COD content of the effluent discharged by the n water sector, mg/L and the COD content of the effluent discharged by the 4th water sector.

2) Constraints

a. Water demand constraints:

$$D_{\min,n}^k \le w_n^k \le D_{\max,n}^k \tag{5}$$

where $D_{\min,n}^k$ and $D_{\max,n}^k$ represent the upper and lower bounds of sector n water demand in the region k in 10,000 m^3.

b. Available water supply constraints:

$$\sum_{n=1}^{N} w_n^k \leq T^k \tag{6}$$

where: T^k indicates the total amount of water available for regional k supply, 10,000 m^3.

c. Ecological environmental protection constraints:

$$0.01\sum_{n=1}^{N}\left(\alpha_n \mu_n \sum_{k=1}^{K} w_n^k\right) \leq Q \tag{7}$$

where: Q denotes the maximum allowable COD emission, t; α_n denotes the emission factor; and μ_n denotes the emission concentration, mg / L.

d. non-negative constraints:

$$w_i^k \geq 0 \tag{8}$$

1) Objective function weight coefficient. The objective function weight coefficient λ_p indicates the importance of objective p in the multi-objective function in the total objective, this paper uses the binary comparison weighting method [33] to determine the weight of the objective function. For the three objective functions in the decision-making system [34] are ranked according to the importance from the highest to the lowest:

$$P = \{p_1, p_2, p_3\} \tag{9}$$

where: p_1 denotes economic objectives, p_2 denotes ecological objectives, and p_3 denotes social objectives.

Weighting formula:

$$\lambda_{\mathrm{p}} = \frac{\omega_i}{\sum_{i=1}^{3} \omega_i} \tag{10}$$

Normalized for ω_1, ω_2, and ω_3:

$$\omega_i = \frac{1-\beta_{li}}{\beta_{li}} \tag{11}$$

where: β_{li} indicates the fuzzy scale value of objective p_1 relative to objective p_i, and the fuzzy scale value needs to be quantified when comparing the importance of the objective function.

2) Water use efficiency coefficient. It is difficult to determine the coefficients of domestic water use and ecological efficiency, and the coefficient is 10 yuan/m^3 in combination with the situation in the region, and the coefficients of water use efficiency in plantation, forestry, animal husbandry, fishery and livestock, industry, construction, and tertiary industry are determined with reference to the relevant studies on water resources planning in City A, and the water use sectors and their output value in City A.

3) Equity coefficient of water use. According to the importance of each water use sector in the region, priority will be given to meeting the needs of domestic water use and ecological environment water use, and then production water use will be considered. In descending order of importance, production water is used by the tertiary industry, the secondary industry (construction and industry), and the primary industry (planting, forestry, animal husbandry, fishery and livestock industry). After clarifying

the importance of each user, the equity coefficient of water use is determined according to the following formula:

$$\varphi_n = \frac{1+N-m_n}{\sum_{n=1}^{N} 1+N+m_n} \tag{12}$$

where: m_n denotes the water use sequence number for water sector n, and N denotes the maximum value of the water sector sequence number.

4) Ecological correlation coefficient. The ecological correlation coefficient mainly adopts COD emission index [35] to measure the ecological environment in the region. According to the emission standard planned for City A, the discharge coefficient of different water-using sectors in City A mainly refers to the "Shaanxi Province Yellow River Basin Sewage Comprehensive Discharge Standard" to determine the COD content of sewage from different water-using sectors.

5) Upper and lower limits of water demand. The water demand prediction result of the water supply and demand prediction is taken as the upper limit value of water demand. The lower limit value of water demand is based on the nature of water consumption in each sector, and is determined in the form of a percentage of the upper limit value. In each sector, priority is given to meeting domestic water demand, so the upper and lower limits for domestic water use are the same, and ecological water use is 90% of the upper limit. Other sectors are set at 60%, but if there are individual sub-regions that cannot satisfy the lower limit value of water allocated to each sector at the same time, the sectors can be synchronized to lower the value to an appropriate range.

6) Ecological environmental protection. According to the COD emissions published in the National Economic and Social Development Statistics Bulletin of City A in the past years, the ecological environmental protection constraint in the model is formulated as COD emissions not exceeding 30,000t.

7) Available water supply. Based on the prediction results, the water supply available in several areas of the city can be estimated at 50% frequency.

2.2. Crop structure adjustment in irrigation areas

1) Decision variables. The 12 sub-districts are divided according to the 12 counties and districts in the administrative division of city A, which can be denoted by i and the unit is hm^2. The study object is the 4 crops of maize, potato, cereal and apple, which can be denoted by j. In summary, the area planted with i sub-districts j crops can be expressed as z_{ij}, unit hm^2.

2) Objective function. Maximize the benefit of increased production:

$$\max f(z) = \sum_{i=1}^{I}\sum_{j=1}^{J} z_{ij} p_j G_j \tag{13}$$

where: x_{ij} denotes the acreage of the j crop in sub-area i, hm^2; p_j denotes the average unit price of the j crop, yuan $/kg$; and G_j denotes the incremental yield of the j crop, kg/hm^2.

3) Constraints

a. Area area constraint:

$$\sum_{j=1}^{J} z_{ij} \leq AG_i \tag{14}$$

where: AG_i denotes the area of irrigation area in sub-area i, hm^2.

b. Yield constraints:

$$\sum_{i=1}^{I} z_{ij} G_j \geq SG_j \tag{15}$$

where: SG_j denotes the lower yield limit of j crop (maize, grain, potato, apple), kg.

c. Water quantity constraint:

$$\sum_{j=1}^{4} E_{ij} z_{ij} \leq G_i' \tag{16}$$

where: E_{ij} denotes the gross irrigation quota for crop j in the i area, m^3/hm^2 and G_i' denotes the available water supply in the area i in million m^3.

d. Non-negative constraints:

$$z_{ij} \geq 0 \tag{17}$$

3. Strategies for maximizing eco-efficiency

3.1. Variational approach

With the emergence of dynamic optimization problems in economic and management disciplines, the variational theory is gradually applied to the field of economic management, and the optimal control problem of dynamic problems has promoted the variational theory and gradually gained wide application.

1) Generalized function. A generalized function differs from the ordinary notion of a composite function as a mapping of a function to a real number by assuming that for every function $x(t) \in \mathrm{F}$ on the set of functions F, and noting that $\mathrm{J}\ (x(t))$ is a generalized function defined on F, J denotes the mapping of a function $x(t)$ to a real number, and F is called the set of permissible functions of J.

The basic form of a generalized function is:

$$\mathrm{J}\ (x(t)) = \int_{t_0}^{t_1} F\left(t, x(t), x'(t)\right) dt \tag{18}$$

$x'(t)$ denotes the derivative of the function $x(t)$, and the generalized function J maps $x(t)$ to the function F integrating values in the interval $[t_0, t_1]$. The product function F is a curve about t, $x(t)$ and $x'(t)$, the main purpose of the variational method is to seek to satisfy $\mathrm{J}\ (x(t))$ the maximum or minimum of $x(t)$, that is, $\mathrm{J}\ (x(t))$ the extreme value of the problem, at this time $\mathrm{J}\ (x(t))$ is called the target generalized function.

2) The variational method. The extreme value problem of $\mathrm{J}\ (x(t))$ needs to be given initial values or constraints, so the basic model of variational method can be expressed as:

$$\begin{gathered} MaxorMin\mathrm{J}\ (x(t)) = \int_{t_0}^{t_1} F\left(t, x(t), x'(t)\right) dt \\ s.t. x\left(t_0\right) = x_0 \\ x\left(t_1\right) = x_1 \end{gathered} \tag{19}$$

where x_0 and x_1 are given constants. At this point the product function $\mathrm{J}\ (x(t))$ is a generalized function of a single state variable, the initial point t_0 and the final point t_1 are given, no other constraints are given, and the generalized function takes the form of the basic, or simplest, problem of the calculus of variations. In general, the generalized function must be productible, and this condition is always assumed to hold in research.

It can be solved according to Euler's equation with the necessary conditions for its extremal path, i.e., Euler's equation is expressed as:

$$F_x - \frac{d}{dt} F_{x'} = 0 t \in \left[t_0, t_1\right] \tag{20}$$

Expanding the derivative $dF_{x'} / dt$ gives the detailed form of the equation:

$$F_x = F_{x't} + F_{x'x} x' + F_{x'x'} x'' t \in \left[t_0, t_1\right] \tag{21}$$

The general solution of a second-order nonlinear differential equation usually contains two arbitrary constant terms to be determined, which can be determined from the values of the functions at the initial and final points in the basic problem to obtain a definite solution to the basic problem, i.e., an extremal path.

In addition to the basic problem, there are more cases of variational methods. The basic problem is a typical fixed endpoint problem, while the basic problem of the initial point and the end point of the value is fixed, when the end point of the value is not fixed, that is, the free end point case, the extreme value path in addition to the need to satisfy the Euler equation needs to satisfy the initial conditions and the transversal conditions. When x_1 is a free value, the transversality condition for the free final value case is:

$$At\ t_1, F_{x'} = 0 \tag{22}$$

When the endpoints are fixed and the endpoints satisfy an equation constraint, the model can be represented as:

$$\begin{gathered} MaxorMin\mathrm{J}\ (x(t)) = \int_{t_0}^{t_1} F\left(t, x(t), x'(t)\right) dt \\ s.t. x\left(t_0\right) = x_0 \\ Q\left(t_1, x_1\right) = 0 \end{gathered} \tag{23}$$

The extremal paths satisfy the Euler equations, the initial conditions, in addition to the transversality condition becomes:

$$At\ t_1, F - F_{x'}\left(x' + Q_t / Q_x\right) = 0 \tag{24}$$

The extreme value problem can be extended to the case of multiple functions, assuming there are two functions $x(t)$ and $y(t)$, the basic problem can be formulated as:

$$\begin{gathered} MaxorMin\mathrm{J}\ (x(t)) = \int_{t_0}^{t_1} F\left(t, x(t), y(t), x'(t), y'(t)\right) dt \\ s.t. x\left(t_0\right) = x_0, y\left(t_0\right) = y_0 \end{gathered} \tag{25}$$

Similarly, the terminal moment and the end value of the function can be not fixed and the solutions of the function all satisfy the Euler equation, i.e.:

$$F_x - \frac{d}{dt}F_{x'} = 0, F_y - \frac{d}{dt}F_{y'} = 0, t \in [t_0, t_1] \tag{26}$$

When both the terminal moment and the end value are fixed, the optimal solution can be obtained by determining any constant term through the initial point, the end point and the corresponding value. And when the end point is fixed and the end value is not fixed, the solution can be based on a similar transversality condition. When the endpoints are fixed and the endpoints satisfy Equation Constraint $Q(t_1, x_1, y_1) = 0$, the cross-section condition becomes if both $x(t)$ and $y(t)$ endpoints are not fixed:

$$At\ t_1, F - F_{x'}(x' + Q_t / Q_x) - F_{y'}(y' + Q_t / Q_y) = 0 \tag{27}$$

3.2. *Optimal control model*

Optimal control theory is an extension of the variational approach to solving dynamic optimization problems, and the maximum principle of optimal control can solve all variational problems. The basic problem of the optimal control model can be formulated as:

$$\begin{gathered} Max\mathrm{J} = \int_{t_0}^{t_1} F(t, x(t), u(t))dt \\ s.t. x'(t) = f(t, x(t), u(t)) \\ x(t_0) = x_0 \\ x(t_1) Unconstrained \end{gathered} \tag{28}$$

where $x(t)$ denotes the state variable, $u(t)$ denotes the control variable, and J denotes the effectiveness generalized function. The difference with the variational approach is that optimal control theory is about determining the optimal control path for control variable $u(t)$ so that the optimal state path for state variable $x(t)$ can be solved.

Define the Hamiltonian function:

$$H(t, x, u, \lambda) = F(t, x, u) + \lambda(t) f(t, x, u) \tag{29}$$

where $\lambda(t)$ is an auxiliary variable. After introducing the Hamiltonian function, the conditional representation of the optimal solution of the model is given as:

$$x'(t) = \frac{\partial H}{\partial \lambda} \tag{30}$$

$$\lambda'(t) = -\frac{\partial H}{\partial x} \tag{31}$$

$$\frac{\partial H}{\partial u} = 0 \tag{32}$$

$$\lambda(t_1) = 0 \tag{33}$$

Eq. (30) is the state equation of the optimal control model, (31) is the covariance equation, (32) is the extreme value condition, and (33) is the cross-truncation condition, and the above solution results can be obtained according to the theory of variational method, and the specific process will not be repeated. If the terminal point satisfies the function $Q(t_1, x_1) = 0$, the cross-section condition becomes:

$$At\ t_1, \lambda(t_1) = p\partial Q / \partial x(t_1) \tag{34}$$

When there are multiple state variables and multiple control variables, assuming that there is n state variable $x_1(t),\cdots,x_n(t)$ and m control variables $u_1(t),\cdots,u_m(t)$, the optimal control model's can be formulated as:

$$\begin{gathered}MaxJ \ = \int_{t_0}^{t_1} F\left(t, x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)\right)dt \\ s.t. x_i'(t) = f_i\left(t, x_1(t),\cdots,x_n(t),u_1(t),\cdots,u_m(t)\right) \\ x_i\left(t_0\right) = x_{i0} \\ 1 \le i \le n\end{gathered} \tag{35}$$

The state variables in model (35) can be either fixed terminal values or free terminal values. The Hamiltonian function is constructed as:

$$\begin{aligned}H = {} & F\left(t, x_1,\cdots,x_n,u_1,\cdots,u_m\right) \\ & +\sum_{i=1}^{n}\lambda_i(t) f_i\left(t, x_1,\cdots,x_n,u_1,\cdots,u_m\right)\end{aligned} \tag{36}$$

Then the state equation of the optimal solution can be expressed as:

$$x_i'(t) = \frac{\partial H}{\partial \lambda_i} 1 \le i \le n \tag{37}$$

The covariance equation can be expressed as:

$$\lambda_i'(t) = -\frac{\partial H}{\partial x_i} 1 \le i \le n \tag{38}$$

The extreme value condition can be expressed as:

$$\frac{\partial H}{\partial u_j} = 0 1 \le j \le m \tag{39}$$

The crossover condition is satisfied when the final value of $x_i\left(t_1\right)$ is not fixed: $\lambda_i\left(t_1\right) = 0$. The crossover condition is satisfied when t_1 satisfies the function $Q\left(t_1, x_1,\cdots,x_n\right) = 0$:

$$\lambda_i\left(t_1\right) = p\partial Q / \partial x_i 1 \le i \le n \tag{40}$$

where p is some constant.

3.3. Variational method based eco-efficiency maximization solution

Introduce the Lagrange multiplier vector functions $\lambda_1(t)$ and $\lambda_2(t)$ and construct the Hamiltonian function as:

$$\begin{aligned}H = {} & \frac{\left(\exp\left(\left(\theta Y_i(t) - u_i(t) - r_i(t)\right)/L_i(t)\right)\right)^{1-\tau} - 2 + \tau}{1-\tau} L_i(t) e^{-\rho t} \\ & + \lambda_1(t)\left(r_i(t) - \delta_{R_i} R_i(t)\right) + \lambda_2(t)\left(u_i(t) - \delta_{K_i} K_i(t)\right)\end{aligned} \tag{41}$$

According to the optimal control model solution method, the covariance equation of the model can be expressed as:

$$\begin{cases} \lambda_1'(t) = -\dfrac{\partial H}{\partial R_i(t)} \\ \lambda_2'(t) = -\dfrac{\partial H}{\partial K_i(t)} \end{cases} \tag{42}$$

The governing equations of the model are:

$$\begin{cases} \dfrac{\partial H}{\partial r_i(t)} = 0 \\ \dfrac{\partial H}{\partial u_i(t)} = 0 \end{cases} \tag{43}$$

This is obtained according to equation (42):

$$\begin{aligned} \lambda_1'(t) &= -\frac{\partial H}{\partial R_i(t)} \\ &= \lambda_1(t)\delta_{R_i} - \theta\beta Y_i(t) R_i(t)^{-1} e^{(1-\tau)(\theta X_i(t)-u_i(t)-r_i(t))L_i(t)-\rho t} \end{aligned} \tag{44}$$

$$\begin{aligned} \lambda_2'(t) &= -\frac{\partial H}{\partial K_i(t)} \\ &= \lambda_2(t)\delta_{K_i} - \theta\alpha Y_i(t) K_i(t)^{-1} e^{(1-\tau)(\theta Y_i(t)-u_i(t)-r_i(t))/L_i(t)-\rho t} \end{aligned} \tag{45}$$

This is obtained according to equation (43):

$$\frac{\partial H}{\partial r_i(t)} = \lambda_1(t) - e^{(1-\tau)(\theta Y_i(t)-u_i(t)-r_i(t))/L_i(t)-\rho t} = 0 \tag{46}$$

$$\frac{\partial H}{\partial u_i(t)} = \lambda_2(t) - e^{(1-\tau)(\theta Y_i(t)-u_i(t)-r_i(t))/L_i(t)-\rho^2} = 0 \tag{47}$$

Available:

$$\lambda_1(t) = \lambda_2(t) = e^{(1-r)(\theta r_i(t)-u_i(t)-r_i(t))/L_i(t)-\rho t} \tag{48}$$

Substituting (48) into (47) and (46) gives:

$$\begin{cases} \delta_{R_i} - \theta\beta Y_i(t) R_i(t)^{-1} = \delta_{K_i} - \theta\alpha Y_i(t) K_i(t)^{-1} \\ (1-\tau) d\left(\left(\theta Y_i(t) - u_i(t) - r_i(t)\right)/L_i(t)\right)/dt - \rho \\ = \delta_{R_i} - \theta\beta Y_i(t) R_i(t)^{-1} \end{cases} \tag{49}$$

When eco-efficiency is taken as a constraint, the cross-cutting condition for the optimal control model is:

$$\begin{cases} \lambda_1(t_1) = p\partial EE_i(t)/\partial R_i(t)\big|_{t=t_1} \\ \lambda_2(t_1) = p\partial EE_i(t)/\partial K_i(t)\big|_{t=t_1} \end{cases} \tag{50}$$

As:

$$\frac{\partial EE_i(t)}{\partial R_i(t)} = -\Phi\beta EE_i(t) R_i(t)^{-1} \tag{51}$$

$$\frac{\partial EE_i(t)}{\partial K_i(t)} = -\Phi\alpha EE_i(t)K_i(t)^{-1} \tag{52}$$

Neither $\lambda_1(t)$ nor $\lambda_2(t)$ can be zero, hence the cross-cutting condition:

$$\alpha R_i(t_1) = \beta K_i(t_1) \tag{53}$$

When industry output is taken as a constraint, the cross-section condition for the optimal control model is:

$$\begin{cases} \lambda_1(t_1) = p\partial Y_i(t)/\partial R_i(t)\big|_{t=t_1} \\ \lambda_2(t_1) = p\partial Y_i(t)/\partial K_i(t)\big|_{t=t_1} \end{cases} \tag{54}$$

The same can be obtained for $\alpha R_i(t_1) = \beta K_i(t_1)$, so both constraints derive the same transversality condition.

The Bellman equation cannot obtain the functional relationship between the variables in the solution, and the final result of the optimal solution is not different, in this paper, we use Pontryagin's great value principle method to solve the model.

4. Simulation experiment and analysis of eco-efficiency maximization strategy

The pareto frontier surfaces obtained by the variational method and the NSGA-III algorithm are shown in Fig. 1. Where f2 and f3 represent the two objective functions of the algorithm respectively, and f1 represents the number of iterations of the algorithm. Compared with the NSGA-III algorithm, the pareto frontier surface generated by the variational method is closer to the centroid, which indicates that the results of the simulation test of the variational method applied to the model of optimal allocation of water resources are more excellent in terms of convergence and diversity. The computational time of the variational method is about 0.958s and that of the NSGA-III algorithm is about 0.979s.

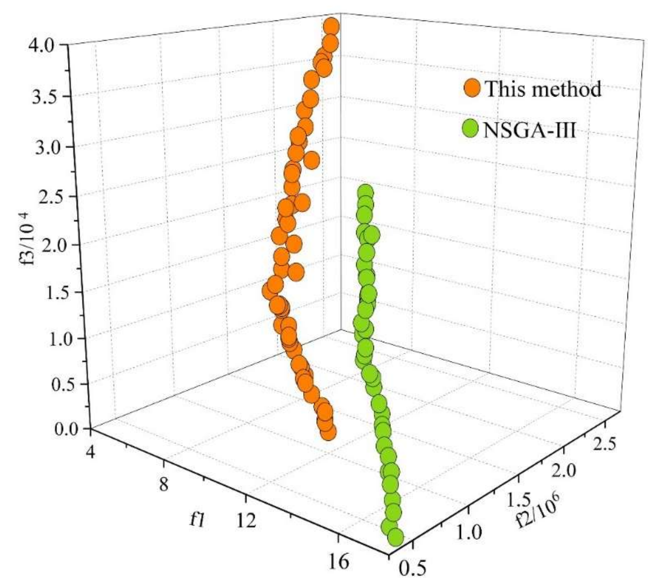

Fig. 1. The pareto front side of the two algorithms

The relationship between social and ecological benefits is shown in Fig. 2. Compared with the NSGA-III algorithm, the COD content value of the solution generated by the variational method is smaller in the case of the same water shortage rate, and the COD content value of the solution generated by the variational method when the water shortage rate is all 10% is $0.71*10^4$mg, which is much smaller than the COD content value of the solution generated by the NSGA-III algorithm. It shows that the variational method is more conducive to finding a better solution set and its convergence speed is faster.

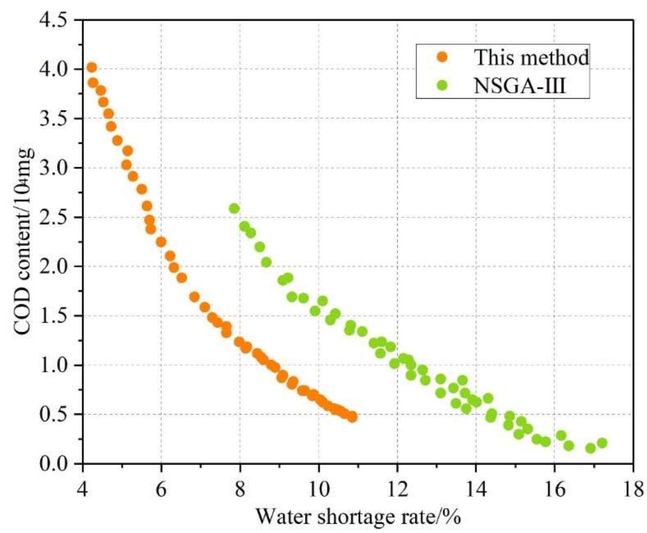

Fig. 2. The relationship between social benefit and ecological benefit

The relationship between economic and ecological benefits is shown in Figure 3. In the case of the same economic benefits, the COD content produced by the variational method is lower than that produced by the NSGA-III algorithm, and when the economic benefits are $1.0*10^6$ million yuan, the COD content produced by the variational method is $1.25*10^4$ mg, and the COD content produced by the NSGA-III algorithm is $2.0*10^4$ mg, which is higher than that produced by the variational method, which indicates that the variational method is more powerful for the better ability to optimize the economic and ecological benefit objectives designed in this paper.

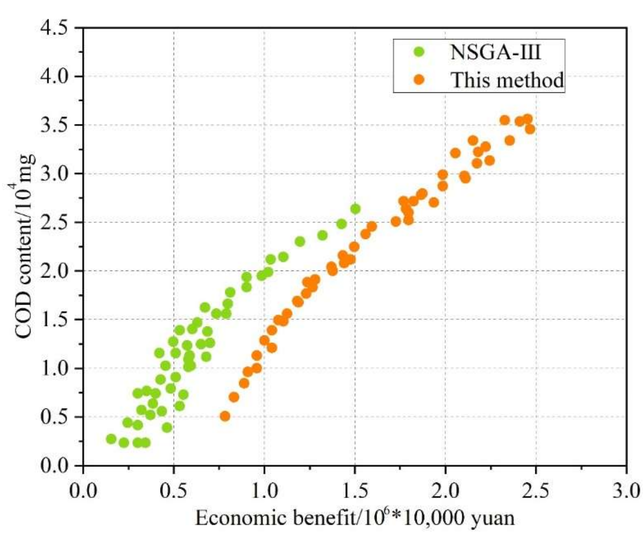

Fig. 3. Economic benefits and ecological benefits

The relationship between social and economic benefits is shown in Fig. 4. Compared with the NSGA-III algorithm, the variational method finds a better Pareto solution set in the case of the same water shortage rate both 10%. And except for some solution sets, the variational method can find better Pareto optimal solution sets in most cases.

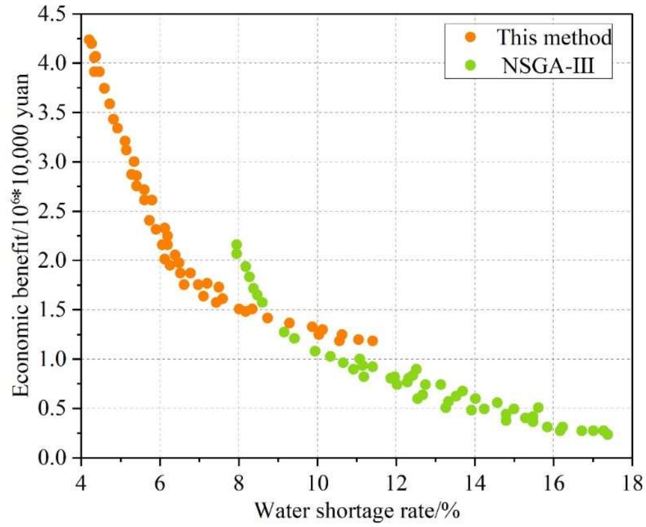

Fig. 4. The relationship between social and economic benefits

The social benefits (water shortage rate), economic benefits, and ecological benefits (COD content) under the water allocation scenarios are shown in Table 1. Scenario 1 is the preferred social benefit scenario, which has the lowest water shortage rate (highest social benefit). Scenario 2 is the preferred economic benefit scenario, which has the highest economic benefit value. Scenario 3 is the preferred eco-efficiency scenario, which has the lowest COD content (highest eco-efficiency). Scenario 4 is the mean value scenario, which has the middle social, economic and ecological benefits. According to the actual situation of maximizing the demand for ecological benefits in this paper, Scheme 3 is selected as the water resource allocation scheme.

Table 1. Comparison of the benefits of four water allocation schemes

Water allocation plan	Water shortage rate/%	Economic benefit/10000 yuan	Cod content/mg
1	4.145	2770079.687	398451.366
2	4.256	2849854.957	408751.665
3	5.168	2094565.595	295871.166
4	4.215	2795436.589	397851.254

5. Optimization analysis of planting crop structure

The optimization scheme of crop planting area in irrigation district of A city in 2028 is shown in Table 2. After optimization, the total crop planting area of A city is 114,254hm^2, and the crop planting area of each district and county is smaller than the constraint upper limit of the total crop planting area. Figure 5 shows the crop planting area of each district and county, and the crop planting structure of each district and county in A city shows obvious diversity and regional characteristics. In many districts and counties, the planting area of potato crops occupies a large proportion, especially in county e, the potato crop covers an area of 17,452hm^2, accounting for 37.61% of the city's potato crops, this is because county e has a large effective irrigated area, which can be allocated to the planting area of potato crops will be more, and secondly due to the fact that county e is in the sandy and windy area, the irrigation quota of the potato crops is relatively small compared to other crops, and the average amount of irrigation water per mu is also smaller than other crops. Secondly, because county e is in the sandy and windy area, the irrigation quota for potato crops is relatively small compared with other crops, the average amount of irrigation water per mu will be less, which can save more irrigation water resources, and the high yield per unit area of potato crops can get more ecological benefits.

Corn is also an important crop, especially in county f, which covers an area of 6,032hm^2 under corn cultivation. Maize, as a traditional food crop, is widely grown in the region due to its greater ecological benefits per unit area and more stable yields. Wheat is planted in almost zero percent of the area because its yield is not only low but also its ecological benefits are limited compared to other food crops such as corn and potatoes. Soybeans are grown on a relatively small area in city A, but are more prominent in certain districts and counties such as counties b and d. Oil crops are also grown more prominently in county a and other districts and counties. Vegetables have the largest planting area in county a, at 11,150hm^2, indicating that the district has better conditions and market demand for vegetable cultivation. Melon crops, on the other hand, are grown more in Suide County. As a whole, the crop structure in City A maintains a certain balance between food crops (e.g., corn, potatoes) and cash crops (e.g., oilseeds, vegetables). This structure helps to ensure food safety while improving agro-ecological efficiency.

At the same time, in order to achieve rationalization and diversification of crop production in City A for the future, in order to cope with the changing market demand and to improve the overall efficiency and sustainability of agricultural production, there is a need to determine the ratio constraints on the cultivation of various crops. Currently, the ratio of vegetable cultivation area to total cultivated area in City A is about 19.38%. According to the future development plan, in order to ensure an adequate and diversified supply of vegetables for the population, it has been determined that the proportion of the area under vegetable cultivation in the planning year should not be less than 5%, a policy that is intended to respond to increasing market demand and also reflects the government's support for the trend towards healthy diets. Food security has always been a priority in local and national policies, and in order to ensure the security of the food supply, City A stipulates that the proportion of the area planted with food crops (including wheat, corn, potatoes, and soybeans) shall not be less than 40%. This measure is intended to ensure self-sufficiency and a stable long-term supply of basic foodstuffs. In order to maintain the diversity and balance of crop cultivation and to avoid over-concentration on a single crop at the expense of other crops, the area planted with maize should not exceed 40 per cent of the total planted area, the area planted with oilseeds should account for more than 3.5 per cent of the total planted area, the proportion of melons planted should not be less than 1 per cent, and the proportion of the area planted with other types of crops should be more than 7 per cent of the total planted area.

Table 2. Optimization Plan for Planting Structure in A City

District	Soybean	Potato	Corn	Wheat	Vegetable	Melons	Oil material	Other	Tot
a	18	8557	5124	7	11150	1318	1957	5389	33520
b	736	6068	532	3	3495	128	2702	866	14530
c	11	4267	798	0	1224	229	238	767	7534
d	924	5597	2773	2	1172	204	128	42	10842
e	30	17452	5107	0	1873	799	467	241	25969
f	16	1270	6032	2	433	285	2	247	8287
g	291	1626	216	4	973	16	1134	790	5050
h	478	629	661	8	649	335	434	384	3578
i	421	88	246	2	178	258	7	19	1219
j	4	13	107	1	106	8	17	51	307
k	59	6	42	3	83	0	103	103	399
l	539	831	161	3	808	7	645	25	3019
A	3537	46404	21799	35	22144	3587	7834	8924	114254

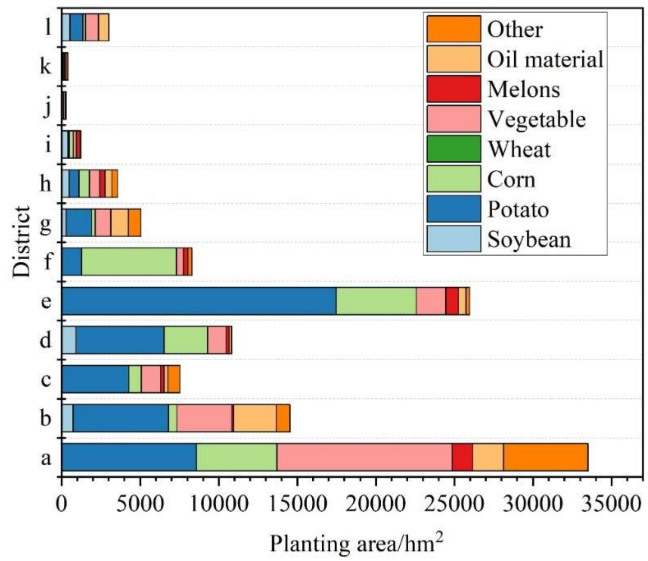

Fig. 5. Crop planting area in each district and county

6. Conclusion

In this paper, the method of rational utilization of agroforestry resources is explored, and the optimal allocation model for maximizing ecological benefits is constructed based on the variational method, and compared with the NSGA-III algorithm in simulation experiments.

The experimental results show that the pareto front surface obtained by the variational method is closer to the spatial center point, and its computational time is about 0.958s, which is 0.021s less than that of the NSGA-III algorithm, indicating that the variational method is more excellent in terms of convergence and computational efficiency in the simulation experiments of optimal resource allocation. The variational method is better at preferring the three objective functions of the multi-objective model design for agricultural and forestry water resources, and the optimized allocation model for maximizing ecological benefits using the variational method can provide a reference for the efficient use of resources and the maximization of ecological benefits in the city.

The planting structure of crops in city A was adjusted according to the allocated water volume of irrigated crops, and the planting area of potato was the largest after adjustment, about 46,403hm^2, which was due to the fact that the model set ecological benefit as the most important objective. Due to the high ecological value of the potato crop, it was prioritized and its planting area was the largest, accounting for 40.61% of the total planting area. This configuration is in line with the direction of this paper to pursue the maximum ecological benefits.

References

[1] Grebner, D. L., Bettinger, P., Siry, J. P., & Boston, K. (2021). Introduction to forestry and natural resources. *Academic press*.

[2] Usman, M., & Makhdum, M. S. A. (2021). What abates ecological footprint in BRICS-T region? Exploring the influence of renewable energy, non-renewable energy, agriculture, forest area and financial development. *Renewable Energy*, 179, 12-28.

[3] Suleiman, M. S., Wasonga, V. O., Mbau, J. S., Suleiman, A., & Elhadi, Y. A. (2017). Non-timber forest products and their contribution to households income around Falgore Game Reserve in Kano, Nigeria. *Ecological Processes*, 6, 1-14.

[4] Kumar Sarangi, P., Subudhi, S., Bhatia, L., Saha, K., Mudgil, D., Prasad Shadangi, K., ... & Arya, R. K. (2023). Utilization of agricultural waste biomass and recycling toward circular bioeconomy. *Environmental Science and Pollution Research*, 30(4), 8526-8539.

[5] Pimentel, D. (2019). Handbook of energy utilization in agriculture. CRC press.

[6] Antar, M., Lyu, D., Nazari, M., Shah, A., Zhou, X., & Smith, D. L. (2021). Biomass for a sustainable bioeconomy: An overview of world biomass production and utilization. *Renewable and Sustainable Energy Reviews*, 139, 110691.

[7] Alatzas, S., Moustakas, K., Malamis, D., & Vakalis, S. (2019). Biomass potential from agricultural waste for energetic utilization in Greece. *Energies*, 12(6), 1095.

[8] Olorunfemi, T. O., Olorunfemi, O. D., & Oladele, O. I. (2020). Determinants of the involvement of extension agents in disseminating climate smart agricultural initiatives: Implication for scaling up. *Journal of the Saudi Society of Agricultural Sciences*, 19(4), 285-292.

[9] Oswalt, S. N., Smith, W. B., Miles, P. D., & Pugh, S. A. (2019). Forest resources of the United States, 2017. General Technical Report-US Department of Agriculture, Forest Service. *Forest Service*.

[10] Gaurav, N., Sivasankari, S., Kiran, G. S., Ninawe, A., & Selvin, J. (2017). Utilization of bioresources for sustainable biofuels: a review. *Renewable and sustainable energy reviews*, 73, 205-214.

[11] Suzuki, K., Tsuji, N., Shirai, Y., Hassan, M. A., & Osaki, M. (2017). Evaluation of biomass energy potential towards achieving sustainability in biomass energy utilization in Sabah, Malaysia. *Biomass and Bioenergy*, 97, 149-154.

[12] Liu, Y., Sun, D., Wang, H., Wang, X., Yu, G., & Zhao, X. (2020). An evaluation of China's agricultural green production: 1978–2017. *Journal of Cleaner Production*, 243, 118483.

[13] Tun, M. M., Juchelkova, D., Win, M. M., Thu, A. M., & Puchor, T. (2019). Biomass energy: An overview of biomass sources, energy potential, and management in Southeast Asian countries. *Resources*, 8(2), 81.

[14] Xie, M., Irfan, M., Razzaq, A., & Dagar, V. (2022). Forest and mineral volatility and economic performance: evidence from frequency domain causality approach for global data. *Resources Policy*, 76, 102685.

[15] Kumar, A., Adamopoulos, S., Jones, D., & Amiandamhen, S. O. (2021). Forest biomass availability and utilization potential in Sweden: A review. *Waste and Biomass Valorization*, 12, 65-80.

[16] Kangas, A., Astrup, R., Breidenbach, J., Fridman, J., Gobakken, T., Korhonen, K. T., ... & Olsson, H. (2018). Remote sensing and forest inventories in Nordic countries–roadmap for the future. *Scandinavian Journal of Forest Research*, 33(4), 397-412.

[17] Shaibur, M. R., Husain, H., & Arpon, S. H. (2021). Utilization of cow dung residues of biogas plant for sustainable development of a rural community. *Current Research in Environmental Sustainability*, 3, 100026.

[18] Kallio, A. M. I., Solberg, B., Käär, L., & Päivinen, R. (2018). Economic impacts of setting reference levels for the forest carbon sinks in the EU on the European forest sector. *Forest Policy and Economics*, 92, 193-201.

[19] Harbi, J., Erbaugh, J. T., Sidiq, M., Haasler, B., & Nurrochmat, D. R. (2018). Making a bridge between livelihoods and forest conservation: Lessons from non timber forest products' utilization in South Sumatera, Indonesia. *Forest policy and economics*, 94, 1-10.

[20] Li, L., Liu, J., Long, H., de Jong, W., & Youn, Y. C. (2017). Economic globalization, trade and forest transition-the case of nine Asian countries. *Forest Policy and Economics*, 76, 7-13.

[21] Heinonen, T., Pukkala, T., Mehtätalo, L., Asikainen, A., Kangas, J., & Peltola, H. (2017). Scenario analyses for the effects of harvesting intensity on development of forest resources, timber supply, carbon balance and biodiversity of Finnish forestry. *Forest Policy and Economics*, 80, 80-98.

[22] Abd El-Sayed, E. S., El-Sakhawy, M., & El-Sakhawy, M. A. M. (2020). Non-wood fibers as raw material for pulp and paper industry. *Nordic Pulp & Paper Research Journal*, 35(2), 215-230.

[23] Siraj, M., Zhang, K., Xiao, W., Bilal, A., Gemechu, S., Geda, K., ... & Xiaodan, L. (2018). Does participatory forest management save the remnant forest in Ethiopia?. *Proceedings of the National Academy of Sciences, India Section B: Biological Sciences*, 88, 1-14.

[24] Naqvi, S. R., Jamshaid, S., Naqvi, M., Farooq, W., Niazi, M. B. K., Aman, Z., ... & Afzal, W. (2018). Potential of biomass for bioenergy in Pakistan based on present case and future perspectives. *Renewable and Sustainable Energy Reviews*, 81, 1247-1258.

[25] Sgroi, F. (2020). Forest resources and sustainable tourism, a combination for the resilience of the landscape and development of mountain areas. *Science of the Total environment*, 736, 139539.

[26] Hannam, K. D., Venier, L., Allen, D., Deschamps, C., Hope, E., Jull, M., ... & Hazlett, P. W. (2018). Wood ash as a soil amendment in Canadian forests: what are the barriers to utilization?. *Canadian Journal of Forest Research*, 48(4), 442-450.

[27] Malik, A. Y. (2024). Forest Resources Status, Utilization, and Conservation. *In Sustainable Forest Resources Management* (pp. 93-108). Apple Academic Press.

[28] Feyisa, B. N., Feyssa, D. H., & Jiru, D. B. (2017). Fuel wood utilization impacts on forest resources of Gechi District, South Western Ethiopia. *Journal of Ecology and the Natural Environment*, 9(8), 140-150.

[29] Pala, N. A., Banday, M., Islam, M. A., Rashid, M., Malik, Z. A., & Bussmann, R. W. (2021). Ethnobotanical utilization of forest resources in Sindh Forest of Kashmir Himalaya, India. *Ethnobotany Research and Applications*, 21, 1-18.

[30] Guo, J., Zhang, Y., Fang, J., Ma, Z., Li, C., Yan, M., ... & Bian, M. (2024). Reduction and reuse of forestry and agricultural bio-waste through innovative green utilization approaches: A review. *Forests*, 15(8), 1372.

[31] XU, L., WU, Y., ZHANG, Y., & HU, S. (2023). Progress of comprehensive utilization technology of agricultural and forestry wastes in Guangdong under the background of “carbon peaking and carbon neutrality”. *Chemical Industry and Engineering Progress*, 42(11), 5648.

[32] Xiao, J., & Xiong, K. (2022). A review of agroforestry ecosystem services and its enlightenment on the ecosystem improvement of rocky desertification control. *Science of The Total Environment*, 852, 158538.

[33] Kaihui Ma,Junyi Deng,Qinghui Wu,Peng Hu,Shuxing Qiu,Mao Chen & Jian Xu. (2025). Comparison of transient shrinkage behaviours between binary/ternary- FeO-rich oxides in CO/H2. International Journal of Hydrogen Energy299-306.

[34] Christoph Wilhelm,Anke Steckelberg & Felix G Rebitschek. (2025). Benefits and harms associated with the use of AI-related algorithmic decision-making systems by healthcare professionals: a systematic review.. The Lancet regional health. Europe101145.

[35] Shuai Shao,Zhenbing Yang,Le Xu & Lili Yang. (2024). Environmental protection experience of secretaries and effectiveness of environmental governance: Evidence from COD discharge in China. Journal of Economic Behavior and Organization237-253.