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Research on Computational Simulation and Capital Structure Adjustment in Corporate Investment and Financing Decisions under ESG Rating Mechanisms

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ABSTRACT

The problems of debt value and optimal capital structure of enterprises are the main issues in corporate finance research. Under the ESG rating mechanism, the article first utilizes the real option theory to study the optimal capital structure and investment and financing decision-making methods of enterprises. Then it puts forward model assumptions and combines the jump diffusion model for the construction of enterprise project investment and financing decision-making model and the dynamic planning adjustment of capital structure. Finally, through specific numerical experiments, the influence process of each variable in the model on the enterprise investment and financing decision is analyzed, and the agency problem is analyzed. Through the experiment, it can be obtained that when the residual value after stopping production, the risk-free interest rate, the variable production cost and the tax rate are set to γ =1, r =0.1, ξ =0.1, θ =0.2, respectively, with the increase of the frequency of the jump, the investment price of the positive-jump model gradually decreases, and the investment price of the negative-jump model gradually increases, which can be obtained that the reasonable simulation estimation of the relevant parameters has an important impact on the enterprise's investment strategy, so the enterprise should make a more accurate assessment of the parameters, otherwise they will lose part of the benefits or lose good investment opportunities.

Keywords: real options theory, financing decision model, investment and financing decision, ESG rating mechanism, jump diffusion model

1. Introduction

With the flourishing development of ESG and its entry into the public eye, it has become an important issue of concern to global investors, enterprises and financial institutions, and this concept has become the foundation and core handhold for realizing the goal of "dual-carbon" at the macro level, and provides a new perspective for evaluating the high-quality development of enterprises and the

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fulfillment of their social responsibility at the micro level [1-3]. ESG evaluates the sustainability of an enterprise's operation and its role in social values from the three important dimensions of environmental protection, social responsibility fulfillment and scientific corporate governance, advocates the unity of economic value and social value, represents a healthier and better way of development, a more responsible corporate image, and a more scientific mode of corporate governance, which is a perfect match for many major development concepts in China at present. This is perfectly in line with many major development concepts in China, and brings new opportunities for high-quality development of enterprises [4-7].

On the basis of the continuous improvement of the new development concept, ESG provides value leadership and internal driving force for enterprises to realize the transformation of their operation and management modes and reshape the competitive pattern of the market, as well as providing a new comprehensive matrix index system for stakeholders, therefore, enterprises integrate ESG concepts into their own strategic decision-making and voluntarily cooperate with third-party rating agencies in the capital market to publish their ESG ratings [8-11].

Green investment and financing is an important part of China's low-carbon economy and sustainable development, and an important driving force for the realization of the “dual-carbon” goal [12]. Green investment behavior has both market and environmental considerations, and transforming environmental responsibility and strategies into corporate actions is the key to maximizing corporate value and social value.ESG ratings are closely related to green investment, and by practicing ESG responsibility, companies can obtain more policy preferences and financing support, so that they can proactively use their funds for climate change, resource utilization, green construction and other environmental governance activities, with positive impacts. Environmental governance activities, with positive environmental and social externalities, to promote its performance improvement, long-term business development is of great practical significance [13-16]. It can be seen that ESG ratings convey positive signals to the capital market that enterprises have adopted an environmentally friendly development model, break down the communication barriers between internal and external information of enterprises, and provide enterprises with the opportunity to win more support from social capital and provide a source of motivation for green investment [17-19].

ESG performance has gradually evolved into an important non-financial information to measure the quality and potential of enterprise development, which can express the potential sustainable development ability of enterprises to investors, and have an important impact on enterprise investment and financing decision-making [20-21], and the enterprise entity investment is an important direction to study the enterprise investment decision-making, can ESG performance enhance the level of enterprise entity investment? If ESG performance has an impact on corporate entity investment, what is its role mechanism? These issues deserve in-depth research [22-23].

The article firstly briefly introduces the preparatory knowledge of model building, and then applies the analysis method of real options to theoretically analyze the established mathematical model. Subsequently, the model assumptions and the detailed theoretical analysis and solution process are given, including the construction of the investment and financing decision model of the new project and the expansion project, and the analysis of the agency cost is carried out at the same time. Finally, numerical experiments are carried out, and sensitivity analysis experiments are conducted on the relevant parameters of the model. Comparative analysis of the commissioning price under the jump diffusion model and the geometric Brownian motion model, and analyze the agency cost and so on.

2. Method

2.1. Preparatory knowledge

2.1.1. Real options. Real options is the financial options concept of investment in the reality of the right to choose, it gives the investment decision maker in the project investment decision-making possesses in the future in accordance with the changes in the market to adopt a variety of flexible strategies.

1) Bifurcation Tree Method. The binomial tree method is a common approach to discrete-time option pricing. Under the single-period binomial tree model, suppose that there is a firm whose current stock price is S_0. After a period of time t the stock price could be one of two things: the stock price rises to S_u with probability p, and the stock price declines to S_d with probability $q(p+q=1)$. If the firm is used as the underlying asset, there exists a call option with an exercise price of K and an exercise time of t, which is currently priced at V_0. Let the risk-free interest rate be r (which is a constant) The risk-free rate is 10 (a constant). The option price after time t may be $V_u = \max\left(0, S_u - K\right)$ with probability p, or $V_d = \max\left(0, S_d - K\right)$ with probability q. If one hedges by buying a share of Δ in order to hedge the risk, one has:

$$V_u - \Delta S_u = \left(V_0 - S_0\right)(1+r) \tag{1}$$

$$V_d - \Delta S_d = \left(V_0 - S_0\right)(1+r) \tag{2}$$

Solving the equation associatively gives:

$$\Delta = \frac{V_u - V_d}{S_u - S_d} \tag{3}$$

$$V_0 = \frac{1}{1+r}\left(pV_u + qV_d\right) \tag{4}$$

If $p = q = \frac{1}{2}$, the market is risk neutral and the value of the option is the discounted value of its expected value at the risk-free rate r.

2) The Black-Scholes model. Assuming that no dividends are paid on the underlying asset and there are no taxes, transaction costs, or arbitrage opportunities, the risk-free rate r is a constant and the borrowing and lending rates are the same; the price of the underlying asset obeys geometric Brownian motion:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t \tag{5}$$

where α is the expected return, σ is the volatility, and W_t is the standard geometric Brownian motion.

The value of a European call option based on the underlying asset V is satisfied:

$$\frac{dV}{dt} + \frac{1}{2}\sigma^2 S^2 \frac{d^2V}{dS^2} + rS\frac{dV}{dS} - rV = 0 \tag{6}$$

If the effective time of the option is $[0,T]$ and the knockout price of the option is K, the option value V satisfies the above equation within $[0,T]$ and $V(T) = (S-K)^+$, and the pricing formula for a European call option can be obtained by using the function transformation:

$$V(S,t) = SN\left(d_1\right) - Ke^{-r(T-t)}N\left(d_2\right) \tag{7}$$

$$d_1 = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)\right] \tag{8}$$

$$d_2 = d_1 - \sigma\sqrt{T-t} \tag{9}$$

$N(\cdot)$ represents the probability distribution function of the standard normal distribution, and the same can be obtained from the pricing formula of the European put option.

3) Dynamic programming method. Suppose an asset price is expressed as S_t at any moment t, the asset price at a future moment is $S_{t+1}, S_{t+2}, \cdots$, the control variable at each stage is β, the option price is not executed when $\beta = 0$, the option price is executed when $\beta = 1$, the control variable at moment t is β_t, and if the decision β_t is chosen at moment t, the return at this time can be obtained as $\pi(x_t, \beta_t)$, and the discount rate between each stage is $\frac{1}{1+R}$, and the choice of the control variable $\{\beta_t\}$ to maximize the expected return yields the final return $\Omega_T(S_T)$ and the optimal option return from moment t is $F_t(S_t)$.

The decision maker chooses the control variable to be β_t at moment t, when the immediate cash flow is $\pi(x_t, \beta_t)$, and the asset price is S_{t+1} at the next moment $t+1$, when the optimal decision outcome is chosen to be $F_{t+1}(S_{t+1})$, and

$$F_{t+1}(S_{t+1}) = \pi(x_t, \beta_t) + \frac{1}{1+R} E_t\left[F_{t+1}(S_{t+1})\right] \tag{10}$$

In the decision making process is that choice β maximizes the expected return, then the choice at this point can be obtained (Bellman's equation):

$$F_t(S_t) = \max_{\beta_t}\left\{\pi(x_t, \beta_t) + \frac{1}{1+R} E_t\left[F_{t+1}(S_{t+1})\right]\right\} \tag{11}$$

For a multi-stage decision problem, working backwards from the last moment T, set the gain at the last moment to $\Omega_T(S_T)$, then:

$$F_{T-1}(S_{T-1}) = \max_{\beta_{T-1}}\left\{\pi(x_{T-1}, \beta_{T-1}) + \frac{1}{1+R} E_{T-1}\left[\Omega_T(S_T)\right]\right\} \tag{12}$$

Taking this backwards gives the expected returns and decision choices at the optimal decision at each stage, and the decision choices at any moment t are:

$$F_t(S_t) = \max_{\beta_1}\left\{\Omega(S_t), \pi(x_t, \beta_t) + \frac{1}{1+R} E_t\left[F_{t+1}(S_{t+1})\right]\right\} \tag{13}$$

2.1.2. Theories of corporate capital structure:

1) Capital structure theory based on the transfer of corporate control. Under the ESG rating mechanism, the market value of the enterprise $V(t)$ is regarded as the underlying asset, and due to market changes and operational uncertainty, it is assumed that the market value of the enterprise obeys the geometric Brownian motion, and the opening value of the enterprise is $V_0 = E + B$, where E is the amount of equity financing, and B is the amount of bond financing [24]. Assume that the debt principal and interest repayment at the end of the period $t = T$ is stipulated in the financing agreement as $B \times e^{kT}$, and R is the interest rate on the debt. If at the end of the period the firm is unable to repay its debt as it falls due, the firm will file for bankruptcy, viewing the shareholders' ownership of the firm as a future right to purchase the market value of the firm (option), and the debt principal and interest repayments at maturity as the exercise price of the purchase option.

Derive the amount of equity financing at the beginning of the period:

$$E = V_0 N(d_1) - Be^{(R-r)T} N(d_2) \tag{14}$$

Among them:

$$d_1 = \frac{\ln\left(\frac{V_0}{B}\right) + \left(r - R + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} \tag{15}$$

$$d_2 = d_1 - \sigma\sqrt{T} \tag{16}$$

Then the corresponding opening bond financing amount B is:

$$B = V_0\left[1 - N(d_1)\right] + Be^{(R-r)T} N(d_2) \tag{17}$$

It can be set $\omega = \frac{B}{V_0}$ period at the beginning of the enterprise gearing ratio that is the capital structure of the enterprise, then the above formula becomes:

$$\begin{aligned} \omega = 1 - N&\left[\frac{\ln\left(\frac{1}{\omega}\right) + \left(r - R + \frac{\sigma^2}{2}\right)\times T}{\sigma\sqrt{T}}\right] + \omega \times e^{(R-r)T} \\ \times N&\left[\frac{\ln\left(\frac{1}{\omega}\right) + \left(r - R + \frac{\sigma^2}{2}\right)\times T}{\sigma\sqrt{T}}\right] \end{aligned} \tag{18}$$

Then the solution of the above equation can be obtained by calculation.

2) The theory of optimal capital structure of enterprises. The ESG rating mechanism, which divides the market into n segments, and the firm's pre-tax income in each period is K_j, without loss of generality, can be made $K_{n\geq}K_{n-1} \geq \cdots \geq K_2 \geq K_1$. Let L be the amount of money the firm pays to its creditors in each stage period, C_j be the cost of bankruptcy in stage period j, Q_j be the present value of a dollar in stage period j, and S_j be the tax rate in stage period j. Then the return to creditors in each stage period can be set to Y_j, and Y_j can be expressed as:

$$Y_j = \begin{cases} L, L \leq K_j \\ K_j - C_j, L > K_j \end{cases} \tag{19}$$

Then the market value of the firm's debt can be set to $B(L)$ and:

$$B(L) = \sum_{j=1}^{n} Y_j Q_j = \begin{cases} L\sum_{j=1}^{n} Q_j, 0 \leq L \leq K_1 \\ \sum_{j=1}^{k-1}\left(K_j - C_j\right)Q_j + L\sum_{j+k}^{n} Q_j, K_{k-1} < L \leq K_k, k = 2, \cdots n \\ \sum_{j=1}^{n}\left(K_j - C_j\right)Q_j, L > K_n \end{cases} \tag{20}$$

A shareholder's return in each phase period may be set to Z_j and:

$$Z_j = \begin{cases} K_j\left(1-S_j\right)+S_jL-L, L \le K_j \\ 0, L > K_j \end{cases} \tag{21}$$

A firm's return on equity can be set to $T(L)$ and:

$$\begin{aligned} T(L) &= \sum_{j=1}^{n} Z_j Q_j \\ &= \begin{cases} \sum_{j=1}^{n}\left[K_j\left(1-S_j\right)+S_jL-L\right] 0 \le L \le K_1 \\ \sum_{j=1}^{n}\left[K_j\left(1-S_j\right)+S_jL-L\chi_j, K_{k-1} < L \le K_k, k = 2, \cdots n \right. \\ 0, K_k < L \end{cases} \end{aligned} \tag{22}$$

In summary, then the value of the firm's assets can be expressed as $V(L) = B(L) + T(L)$.

Since the enterprise value of the non-leveraged firm is $V(0)$, the asset value of the firm can be expressed as:

$$V\left(L\right) = \begin{cases} V(0) + SB(L), 0 \le L \le K_1 \\ V(0) + SB(L) - (1-S)\sum_{j=1}^{k-1} C_j Q_j, K_j < L \le K_k, k = 2, \cdots n \\ V(0) + SB(L) - (1-S)\sum_{j=1}^{n} C_j Q_j, L \le K_n \end{cases} \tag{23}$$

For securities that can be expressed in terms of enterprise value and time, this can be expressed as:

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (rV - C)F_V - rF + F_t + C_y = 0 \tag{24}$$

where σ^2 denotes the firm's return per unit of time (covariance), r denotes the risk-free rate, C denotes the firm's expenditure per unit of time, C_y denotes the security's expenditure per unit of time, V denotes the value of the firm, and F denotes the value of the bond [25].

2.1.3. Corporate investment and financing decision-making methods under ESG rating mechanism. The traditional theory of investment decision-making is dominated by the dynamic analysis method represented by the discounted cash flow method (DCF method), which forecasts the future cash flows and risks of a business and selects the most appropriate discount rate. The value of any asset is equal to the sum of the present values of its expected future generated cash flows, i.e.:

$$V = \sum_{t=1}^{n} \frac{F_t}{(1+R)^t} \tag{25}$$

where V is the value of the asset, n is the life of the asset, F_t is the cash flow in year t, and R is the discount rate (which incorporates the risk of the projected cash flows).

The net present value method (NPV method) is to use the total present value of the amount of net cash benefits and the amount of net cash investment to calculate the net present value, and then based on the size of the net present value to assess the investment program, if $NPV > 0$, you can invest in the program; $NPV < 0$, may not invest in the program.

The formula for calculating the net present value NPV of an investment project is:

$$NPV = \sum_{t=1}^{n} \frac{C_t}{(1+R)^t} - C_0 \tag{26}$$

where: C_0 is the initial investment amount, C_t is the NPV flow in year t, R is the discount rate, and n is the period of the investment project. The ROA-DCF four-quadrant analysis is proposed to determine which method to use for the analysis of investment decisions, and the ROA-DCF four-quadrant analysis is shown in Figure 1.

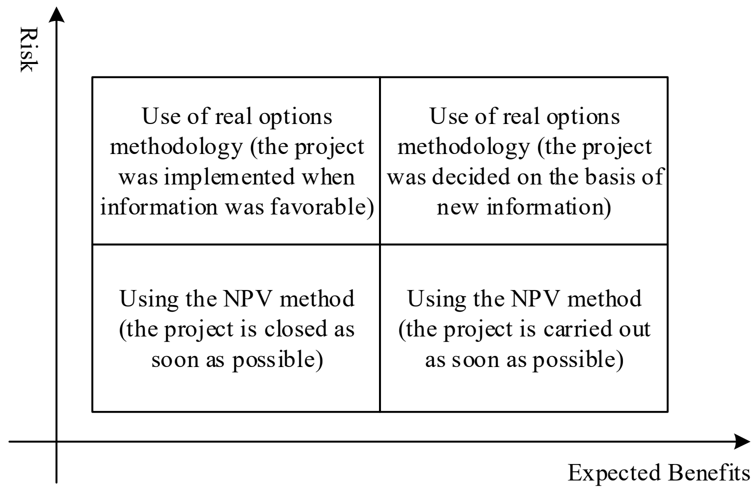

Fig. 1. ROA-DCF four-quadrant analysis

2.2. Model construction and theoretical analysis

2.2.1. Basic assumptions and fundamental theorems. $(\Omega, \mathrm{F}, (\mathrm{F})_{t..0}, \mathrm{P})$ is a complete domain flow space and $W = \{W_i; t..0\}$ is a standard Brownian motion. The price of the firm's product at moment t is P_t, $P_0 = p$. Suppose $\{P_t\}$ is a stochastic process satisfying the following stochastic differential equation:

$$dP_t = a(b - P_t)dt + \sigma dW_t \tag{27}$$

where a, b, σ are all constants, the firm's cost of producing a unit of product is ξ, and the firm's revenue from each unit of product is $P_t - \xi$. Once the price of the product has fallen to a particular level, the firm will liquidate all of its assets, and the investment is irreversible, and at the same time, we assume that the firm does not understart, and that if the firm decides to shut down its production, it will have a salvage value of γ for a single plant, and we will discuss the problem of investment and financing decisions in the following two cases to discuss the firm's investment and financing decisions: (H_1) At the beginning of the period, the firm does not have a plant, and it will decide whether or not to build a new plant for production based on the price trend of the product. At the beginning of period (H_2), the firm has already built a plant, and it will decide whether to build another plant to expand its scale according to the price trend of the product.

Lemma 1 Let $\{P_t\}$ satisfy equation $dP_t = a(b - P_t)dt + \sigma dW_t$, then P_t can be expressed in the following form:

$$P_t = b + e^{-at}\left(p - b + \int_0^t \sigma e^{as} dW_s\right) \tag{28}$$

PROOF: Let $Y_t = P_t e^{at}$, which follows from Ito's formula:

$$\begin{aligned} dY_t &= aP_t e^{at}dt + e^{at}dP_t \\ &= aP_t e^{at}dt + e^{at}\left(a\left(b-P_t\right)dt + \sigma dW_t\right) \\ &= abe^{at}dt + \sigma e^{at}dW_t \end{aligned} \tag{29}$$

There are Ito points for both sides at the same time:

$$Y_t - Y_0 = be^{at} - b + \int_0^t \sigma e^{at}dW_t \tag{30}$$

Then:

$$P_t = b + e^{-at}\left(p - b + \int_0^t \sigma e^{at}dW_t\right) \tag{31}$$

Lemma 2 Assume that $X = \{X(t), 0 \le t \le T\}$ is a simple stochastic process with for $\int_0^T X(t)dW_t$:

$$E\int_0^T X(t)dW_t = 0 \tag{32}$$

Proof: by the Cauchy-Schwarz inequality, there is:

$$E\left|\xi_i\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)\right|,, \sqrt{E\left(\xi_i^2\right)E\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)^2} < \infty \tag{33}$$

And:

$$E\left|\sum_{i=0}^{n-1}\xi_i\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)\right|,, \sum_{i=0}^{n-1}E\left|\xi_i\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)\right| < \infty \tag{34}$$

This shows that the expectation of this random integral exists, and also that there is, according to Brownian motion and the fact that ξ_i is F_{t_i} measurable:

$$E\left(\xi_i\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)|\,\mathrm{F}_{t_i}\right) = \xi_i E\left(\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)|\,\mathrm{F}_{t_i}\right) = 0 \tag{35}$$

Then there is:

$$E\left(\xi_i\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)\right) = 0 \tag{36}$$

$E\int_0^T X(t)dW_t = 0$ by $E\left(\xi_i\left(W\left(t_{i+1}\right)-W\left(t_i\right)\right)\right) = 0$.

Lemma 3 Let the value function $V(s)$ be satisfied:

$$V(s) = E_s\left(e^{-r(\eta\wedge\tau)}R(\tau,\eta)\right) \tag{37}$$

where η, τ is the point in time when the issuer and the holder execute the option contract, and for the holder, the payoff function is $R(\tau,\eta) = X_\tau I_{\tau\le\eta} + Y_\eta I_{\eta<\tau}, I_{\tau\le\eta}$ and $I_{\eta<\tau}$ are indicative functions of $\tau \le \eta$ and $\eta < \tau$. Let $\gamma = \dfrac{\gamma}{\sigma^2} + \dfrac{1}{2}$, if $\delta < \delta^* = \dfrac{K}{2\gamma}\left(\dfrac{2\gamma-1}{2\gamma}\right)^{(2\gamma-1)}$, the permanent American Israeli put option is priced as:

$$I^{P}(s)=\begin{cases} K-s & s\in\left(0,k^{*}\right] \\ \left(K-k^{*}\right)\left(\dfrac{s}{k^{*}}\right)^{-(\gamma-1)}\dfrac{\left(\dfrac{s}{K}\right)^{\gamma}-\left(\dfrac{s}{K}\right)^{-\gamma}}{\left(\dfrac{k^{*}}{K}\right)^{\gamma}-\left(\dfrac{k^{*}}{K}\right)^{-\gamma}} & \\ +\delta\left(\dfrac{s}{K}\right)^{-(\gamma-1)}\dfrac{\left(\dfrac{s}{k^{*}}\right)^{\gamma}-\left(\dfrac{s}{k^{*}}\right)^{-\gamma}}{\left(\dfrac{k^{*}}{K}\right)^{\gamma}-\left(\dfrac{k^{*}}{K}\right)^{-\gamma}} & s\in\left(k^{*},K\right) \\ \delta\left(\dfrac{s}{K}\right)^{-(2\gamma-1)} & s\in[K,\infty) \end{cases} \tag{38}$$

In the above equation, $\frac{k^{*}}{K}$ is the solution of the following equation:

$$y^{2\gamma}+2\gamma-1=2\gamma\left(1+\frac{\delta}{K}\right)y \tag{39}$$

Also, the optimal stopping time strategies for the holder as well as the seller can be obtained, respectively:

$$\begin{cases} \tau^{*}=\inf\left\{t..0:S_{t}\le k_{*}\right\} \\ \eta^{*}=\inf\left\{t..0:S_{t}=K\right\} \end{cases} \tag{40}$$

PROOF: From the expression for $R(\tau,\eta)$, $V(s)$ can be rewritten as:

$$V(s)=\begin{cases} K-s & s\in(0,k] \\ (K-k)E_{s}\left(e^{-r\tau}1_{(r,,\,\eta)}\right)+E_{s}\left(\delta e^{-m\eta}1_{(r>\eta)}\right) & s\in(k,K) \\ \delta E_{s}\left(e^{-r\eta}\right) & s\in[K,\infty) \end{cases} \tag{41}$$

2.2.2. Decision-making model for investment and financing of new projects:

1) The ESG rating mechanism in which all the capital needed by the firm to finance the construction of the plant is obtained from equity financing. Define the shareholder value function (the sum of cash flows received by shareholders) as:

$$V^{U}(p,c)=\sup_{\tau\in S}\mathrm{B}^{P}\left[\int_{0}^{\tau}e^{-rs}\left(P_{s}-\xi\right)ds+e^{-r\tau}\gamma\right] \tag{42}$$

A solution to the optimal stopping time problem $V^{U}(p,c)=\sup_{\tau\in S}\mathrm{E}^{P}\left[\int_{0}^{\tau}e^{-rs}\left(P_{s}-\xi\right)ds+e^{-r\tau}\gamma\right]$ exists and is of the form:

$$\begin{cases} V^{U}(p,c)=\dfrac{rp+ab}{ra+r^{2}}-\dfrac{\xi}{r}+\dfrac{1}{\kappa(a+r)} \\ P_{A}=\dfrac{(a+r)\xi}{r}-\dfrac{1}{k}-\dfrac{ab}{r}+\gamma(a+r) \end{cases} \tag{43}$$

Here, P_A is the optimal stopping price.

Proof: let:

$$J^U(p,c) = \mathrm{E}^P\left[\int_0^{\tau} e^{-rs}\left(P_s - \xi\right)ds + e^{-r\tau}\gamma\right] \tag{44}$$

Using the conclusions of Lemma 1 and Lemma 2, we transform $J^U(p,c)$:

$$\begin{aligned}
J^U(p,c) &= \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(P_s - \xi\right)ds - \int_\tau^{+\infty} e^{-rs}\left(P_s - \xi\right)ds\right] + \mathrm{E}^P\left[e^{-r\tau}\gamma\right] \\
&= \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(b + e^{-as}\left(p - b + \int_0^s \sigma e^{ar} dW_t\right)\right)ds\right] - \frac{\xi}{r} \\
&\quad -\mathrm{E}^P\left[\int_0^{+\infty} e^{-r(t+\tau)}\left(P_{t+\tau} - \xi\right)dt\right] + \mathrm{E}^P\left[e^{-r\tau}\gamma\right] \\
&= \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(b + e^{-as}(p-b)\right)ds\right] - \frac{\xi}{r} \\
&\quad -\mathrm{E}^p\left[\mathrm{E}_\tau \int_0^{+\infty} e^{-r(t+\tau)}\left(P_{r+\tau} - \xi\right)dt\right] + \mathrm{E}^P\left[e^{-r\tau}\gamma\right] \\
&= \frac{rp + ab}{ra + r^2} - \frac{\xi}{r} \\
&\quad -\mathrm{E}^P\left[e^{-r\tau}\mathrm{E}_\tau\left[\int_0^{+\infty} e^{-rr}\left(b + e^{-at}\left(P_\tau - b + \int_\tau^{t+\tau}\sigma e^{a(s-\tau)} dW_s\right) - \xi\right)dt\right]\right] \\
&\quad +\mathrm{E}^P\left[e^{-r\tau}\gamma\right] \\
&= \frac{rp + ab}{ra + r^2} - \frac{\xi}{r} + \mathrm{E}^P\left[-e^{-rr}\left(\frac{rP_\tau + ab}{ra + r^2} - \frac{\xi}{r} + \gamma\right)\right]
\end{aligned} \tag{45}$$

Also, let $\tilde{V}^U(p,c) = \mathrm{E}^P\left[-e^{-r\pi}\left(\frac{rP_\tau + ab}{ra + r^2} - \frac{\xi}{r} + \gamma\right)\right]$, then degree $\tilde{V}^U(p,c)$ satisfies the equation:

$$\frac{1}{2}\sigma^2 \frac{d^2V}{dp^2} - rV + a(b-p)\frac{dV}{dp} = 0 \tag{46}$$

It can be solved:

$$\tilde{V}^U(p,c) = C_1\int_0^{+\infty} \eta(t) e^{-\kappa(b-p)} dt + C_2\int_0^{+\infty} \eta(t) e^{\kappa(b-p)} dt \tag{47}$$

where $\kappa = \frac{\sqrt{2a}}{\sigma}$, $\lambda = \frac{r}{a}$, $\eta(t) = t^{\lambda-1}\exp\left(-\frac{t^2}{2}\right)$. We have attainment conditions:

$$\begin{cases}
\tilde{V}^U\left(P_A, c\right) = \gamma - \dfrac{rP_A + ab}{ra + r^2} + \dfrac{\xi}{r} \\
\dfrac{d\tilde{V}^U\left(P_A, c\right)}{dp} = -\dfrac{r}{ra + r^2}
\end{cases} \tag{48}$$

where P_A is the optimal critical value for stopping operations, which can be solved by the above two equations and the relationship between $V^U(p,c)$ and $\tilde{V}^U\left(P_A,c\right)$:

$$\begin{cases} V^U(p,c) = \dfrac{rp+ab}{ra+r^2} - \dfrac{\xi}{r} + \dfrac{1}{\kappa(a+r)} \\ P_A = \dfrac{(a+r)\xi}{r} - \dfrac{1}{\kappa} - \dfrac{ab}{r} + \gamma(a+r) \end{cases} \tag{49}$$

2) In the case of the ESG rating mechanism, the firm's capital for the preparation of the plant is all obtained from debt financing, in which case the shareholder value function is expressed as:

$$V^D(p,c) = \sup_{\tau\in S} \mathrm{E}^P\left[\int_0^\tau e^{-rs}\left(P_s - \xi - (1-\theta)c\right)ds\right] \tag{50}$$

where θ is the tax rate and P_B is the bankruptcy price.

A solution to the optimal stopping time problem $V^D(p,c) = \sup_{\tau\in S} E^P\left[\int_0^\tau e^{-rs}\left(P_s - \xi - (1-\theta)c\right)ds\right]$ exists and is of the form:

$$\begin{cases} V^D(p,c) = \dfrac{rp+ab}{ra+r^2} - \dfrac{(\xi+(1-\theta)c)}{r} + \dfrac{1}{\kappa(a+r)} \\ P_B(c) = \dfrac{(a+r)(\xi+(1-\theta)c)}{r} - \dfrac{1}{\kappa} - \dfrac{ab}{r} \end{cases} \tag{51}$$

Here, P_B is the bankruptcy price of the business.

Proof: order:

$$J^D(p,c) = \mathrm{E}^P\left[\int_0^\tau e^{-rs}\left(P_s - \xi - (1-\theta)c\right)ds\right] \tag{52}$$

Again, for $J^D(p,c)$, we have:

$$\begin{aligned} J^D(p,c) &= \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(P_s - \xi - (1-\theta)c\right)ds - \int_\tau^{+\infty} e^{-rs}\left(P_s - \xi - (1-\theta)c\right)ds\right] \\ &= \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(b + e^{-as}\left(p - b + \int_0^s \sigma e^{at} dW_t\right)\right)ds\right] - \frac{(\xi+(1-\theta)c)}{r} \\ &\quad -\mathrm{E}^P\left[\int_0^{+\infty} e^{-r(t+\tau)}\left(P_{t+\tau} - \xi - (1-\theta)c\right)dt\right] \\ &= \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(b + e^{-as}(p-b)\right)ds\right] - \frac{(\xi+(1-\theta)c)}{r} \\ &\quad -\mathrm{E}^p\left[\mathrm{E}_\tau \int_0^{+\infty} e^{-r(t+\tau)}\left(P_{t+\tau} - \xi - (1-\theta)c\right)dt\right] \\ &= \frac{rp+ab}{ra+r^2} - \frac{(\xi+(1-\theta)c)}{r} \\ &\quad -\mathrm{E}^P\left[e^{-r\tau}\mathrm{E}_\tau\left[\int_0^{+\infty} e^{-rt}\left(b + e^{-at}\left(P_\tau - b + \int_\tau^{t+\tau} \sigma e^{a(s-\tau)} dW_s\right)\right.\right.\right. \\ &\quad \left.\left.\left. -\xi - (1-\theta)c\right)dt\right]\right] \\ &= \frac{rp+ab}{ra+r^2} - \frac{(\xi+(1-\theta)c)}{r} \\ &\quad +\mathrm{E}^P\left[-e^{-r\tau}\left(\frac{rP_\tau + ab}{ra+r^2} - \frac{(\xi+(1-\theta)c)}{r}\right)\right] \end{aligned} \tag{53}$$

Let $\hat{V}^D(p,c)=\mathrm{E}^P\left[-e^{-r\tau}\left(\frac{rP_\tau+ab}{ra+r^2}-\frac{(\xi+(1-\theta)c)}{r}\right)\right]$, according to the edge-value condition

$$\begin{cases}\tilde{V}^D\left(P_B(c),c\right)=-\dfrac{rP_B(c)+ab}{ra+r^2}+\dfrac{(\xi+(1-\theta)c)}{r}\\ \dfrac{d\bar{V}^D\left(P_B(c),c\right)}{dp}=-\dfrac{r}{ra+r^2}\end{cases} \tag{54}$$

It can be solved:

$$\begin{cases}V^D(p,c)=\dfrac{rp+ab}{ra+r^2}-\dfrac{(\xi+(1-\theta)c)}{r}+\dfrac{1}{\kappa(a+r)}\\ P_B(c)=\dfrac{(a+r)(\xi+(1-\theta)c)}{r}-\dfrac{1}{\kappa}-\dfrac{ab}{r}\end{cases} \tag{55}$$

3) Under the ESG rating mechanism, a firm's capital for the construction of a plant is obtained partly from equity financing and partly from debt financing.

Let the equity value of the enterprise be $E(p,c)$, and the debt value be $D(p,c)$, then the overall value of the enterprise

$$\begin{aligned}V^L(p,c)&=D(p,c)+E(p,c)\\ &=\frac{c}{r}+\frac{\left(\gamma-\frac{c}{r}\right)\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt}{\int_0^{+\infty}\eta(t)e^{\kappa(b-P_D)}dt}\\ &\quad+\frac{rp+ab}{ra+r^2}-\frac{\xi}{r}+\frac{1}{\kappa(a+r)}\end{aligned} \tag{56}$$

Proof: similar to the proof of the theorem, for $E(p,c)$ and $D(p,c)$, there are

$$\begin{cases}\tilde{E}\left(P_D,c\right)=-\dfrac{rP_D+ab}{ra+r^2}+\dfrac{\xi}{r}\\ \dfrac{d\tilde{E}\left(P_D,c\right)}{dp}=-\dfrac{r}{ra+r^2}\end{cases} \tag{57}$$

where P_D is the moment of bankruptcy, is solved:

$$\begin{cases}E(p,c)=\dfrac{rp+ab}{ra+r^2}-\dfrac{\xi}{r}+\dfrac{1}{\kappa(a+r)}\\ P_D=\dfrac{(a+r)\xi}{r}-\dfrac{1}{k}-\dfrac{ab}{r}\end{cases} \tag{58}$$

At the same time:

$$D(p,c)=\mathrm{E}^P\left[\int_0^\tau e^{-rs}cds+e^{-r\tau}\gamma\right]=\frac{c}{r}+\left(\gamma-\frac{c}{r}\right)\mathrm{E}^P\left[e^{-r\tau}\right] \tag{59}$$

According to the edge conditions satisfied by $D(p,c)$:

$$\begin{cases}D\left(P_D,c\right)=\gamma\\ D(p,c)=\dfrac{c}{r}\end{cases} \tag{60}$$

where $D(p,c)=\frac{c}{r}$ holds when p is sufficiently large. From this, it can be solved:

$$D(p,c)=\frac{c}{r}+\frac{\left(\gamma-\frac{c}{r}\right)\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt}{\int_0^{+\infty}\eta(t)e^{\kappa(b-P_o)}dt} \tag{61}$$

So:

$$\begin{aligned}V^L(p,c)&=D(p,c)+E(p,c)\\&=\frac{c}{r}+\frac{\left(\gamma-\frac{c}{r}\right)\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt}{\int_0^{+\infty}\eta(t)e^{\kappa(b-P_D)}dt}\\&+\frac{rp+ab}{ra+r^2}-\frac{\xi}{r}+\frac{1}{\kappa(a+r)}\end{aligned} \tag{62}$$

2.2.3. Investment and financing decision model for expansion projects:

1) Under the ESG rating mechanism, assuming that the firm's investment in preparing for the construction of a new plant is fully financed by debt financing, and starting with the scenario in which the firm already has two plants in production, the payoffs are $2P_t-2\xi$, the firm issues a perpetual debt when it invests in the new plant $\frac{c}{r}$, the contract stipulates that the firm is required to pay back the debt every period c, and the manager runs the firm with the objective of maximizing shareholder value [26]. After the new plant is in operation, the manager selects a bankruptcy level P_b at which the firm declares bankruptcy once the price P_t of the product falls below P_b or P_b. Assuming that debt is risky, e.g., creditors may ultimately fail to collect the value of loans totaling $\frac{c}{r}$, shareholders will get nothing when the firm goes bankrupt, based on these assumptions above, the shareholder value function at this point is:

$$V_2(p,c)=\sup_{r\in S}\mathrm{E}^p\left[\int_0^{\tau}e^{-rs}\left(2P_s-2\xi-(1-\theta)c\right)ds\right] \tag{63}$$

Order:

$$J(p,c)=\mathrm{E}^P\left[\int_0^{\tau}e^{-rs}\left(2P_s-2\xi-(1-\theta)c\right)ds\right] \tag{64}$$

So, for $J(p,c)$, there is:

$$
\begin{aligned}
&J(p,c)=\mathrm{E}^{P}\left[\int_0^{+\infty}e^{-rs}\left(2P_s-2\xi-(1-\theta)c\right)ds\right.\\
&\left.-\int_\tau^{+\infty}e^{-rs}\left(2P_s-2\xi-(1-\theta)c\right)ds\right]\\
&=\mathrm{E}^{P}\left[\int_0^{+\infty}2e^{-rs}\left(b+e^{-as}\left(p-b+\int_0^s\sigma e^{at}dW_t\right)\right)ds\right]-\frac{(2\xi+(1-\theta)c)}{r}\\
&-\mathrm{E}^{P}\left[\int_0^{+\infty}e^{-r(t+\tau)}\left(2P_{r+\tau}-2\xi-(1-\theta)c\right)dt\right]\\
&=\mathrm{E}^{P}\left[\int_0^{+\infty}2e^{-rs}\left(b+e^{-as}(p-b)\right)ds\right]-\frac{(2\xi+(1-\theta)c)}{r}\\
&-\mathrm{E}^{p}\left[\mathrm{E}_\tau\int_0^{+\infty}e^{-r(t+\tau)}\left(2P_{r+\tau}-2\xi-(1-\theta)c\right)dt\right]\\
&=2\frac{rp+ab}{ra+r^2}-\frac{(2\xi+(1-\theta)c)}{r}\\
&-\mathrm{E}^{P}\left[e^{-r\tau}\mathrm{E}_\tau\left[\int_0^{+\infty}e^{-rt}\left(2\left(b+e^{-at}\left(P_\tau-b+\int_\tau^{t+\tau}\sigma e^{a(s-\tau)}dW_s\right)\right)\right.\right.\right.\\
&-2\xi-(1-\theta)c)dt]]=\frac{rp+ab}{ra+r^2}-\frac{(2\xi+(1-\theta)c)}{r}\\
&+\mathrm{E}^{P}\left[-e^{-r\tau}\left(2\frac{rP_\tau+ab}{ra+r^2}-\frac{(2\xi+(1-\theta)c)}{r}\right)\right]
\end{aligned}
\tag{65}
$$

When $P_t\le P_b(c)$, the firm is in bankruptcy, and when $P_t\ge P_b(c)$, the firm receives an undetermined benefit, let:

$$
\tilde{V}(p,c)=\mathrm{E}^{P}\left[-e^{-rr}\left(2\frac{rP_\tau+ab}{ra+r^2}-\frac{(2\xi+(1-\theta)c)}{r}\right)\right]
\tag{66}
$$

In the continuous interval, $\tilde{V}(p,c)$ should satisfy the equation:

$$
\frac{1}{2}\sigma^2\frac{d^2V}{dp^2}-rV+a(b-p)\frac{dV}{dp}=0
\tag{67}
$$

It can be solved:

$$
\tilde{V}(p,c)=C_3\int_0^{+\infty}\eta(t)e^{-\alpha(b-p)}dt+C_4\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt
\tag{68}
$$

where $\kappa=\frac{\sqrt{2a}}{\sigma}$, $\lambda=\frac{r}{a}$, $\eta(t)=t^{\lambda-1}\exp\left(-\frac{t^2}{2}\right)$. Then according to the margin condition

$$
\begin{cases}
\tilde{V}\left(P_b(c),c\right)=-2\frac{rP_b(c)+ab}{ra+r^2}+\frac{(2\xi+(1-\theta)c)}{r}\\
\frac{d\tilde{V}\left(P_b(c),c\right)}{dp}=-2\frac{r}{ra+r^2}
\end{cases}
\tag{69}
$$

And the relationship between $V_2(p,c)$ and $\bar{V}(p,c)$ can be determined for $V_2(p,c)$ and $P_b(c)$, respectively:

$$\begin{cases} V_2(p,c) = 2\dfrac{rp+ab}{ra+r^2} - \dfrac{(2\xi+(1-\theta)c)}{r} + \dfrac{2}{\kappa(a+r)} \\ P_b(c) = \dfrac{(a+r)(2\xi+(1-\theta)c)}{2r} - \dfrac{1}{\kappa} - \dfrac{ab}{r} \end{cases} \tag{70}$$

At the beginning of the period there is only one plant but at the same time there is an expansion option, the firm can decide the implementation strategy of the expansion option according to the change of the product price, at this time, the shareholder value function is:

$$\begin{aligned} V_1(p,c) = \sup_{\tau_m,\tau_d \in \mathrm{S}} \mathrm{E}^P \Big[& \int_0^{\tau_m \wedge \tau_d} e^{-rs}\left(P_s - \xi\right) ds \\ & + e^{-r\tau_m}\gamma 1_{|\tau_m < \tau d|} + e^{-r\tau_d} V_2\left(P_{\tau_d}, c\right) 1_{|\tau_m > \tau_d|} \Big] \end{aligned} \tag{71}$$

Here, τ_m is the moment of shutdown of the original plant without investment, τ_d is the moment of the decision to start production, and there exists $\left(P_m, P_d\right)$ such that the generalized form of $\tau_m = \inf\left\{t; P_t \le P_m\right\}$, $\tau_d = \inf\left\{t; P_t \ge P_d\right\}$, and $V_1(p,c)$ is:

$$V_1(p,c) = \begin{cases} \gamma & P \le P_m(c) \\ C_5 \int_0^{+\infty} \eta(t) e^{-\kappa(b-p)} dt & \\ + C_6 \int_0^{+\infty} \eta(t) e^{\kappa(b-p)} dt + \dfrac{rp+ab}{ra+r^2} - \dfrac{\xi}{r} & P_m(c) < P < P_d(c) \\ V_2(p,c) & P_d(c) \le P \end{cases} \tag{72}$$

where C_5 and C_6 are to be determined by the edge-value condition:

$$\begin{cases} \dfrac{dV_1\left(P_m(c),c\right)}{dp} = 0 \\ C_5 \int_0^{+\infty} \eta(t) e^{-\kappa\left(b-P_m\right)} dt + C_6 \int_0^{+\infty} \eta(t) e^{\kappa\left(b-P_m\right)} dt \\ + \dfrac{rP_m(c)+ab}{ra+r^2} - \dfrac{\xi}{r} = \gamma \\ \dfrac{dV_1\left(P_d(c),c\right)}{dp} = \dfrac{dV_2\left(P_d(c),c\right)}{dp} \\ C_5 \int_0^{+\infty} \eta(t) e^{-\kappa\left(b-P_d\right)} dt + C_6 \int_0^{+\infty} \eta(t) e^{\kappa\left(b-P_d\right)} dt \\ + \dfrac{rP_d(c)+ab}{ra+r^2} - \dfrac{\xi}{r} = V_2\left(P_d(c),c\right) \end{cases} \tag{73}$$

Can be concluded:

If $a \ne -r$, then the solution of $V_1(p,c)$ is:

$$V_1(p,c) = \begin{cases} \gamma & P \le P_m(c) \\ C_5 \int_0^{+\infty} \eta(t) e^{-x(b-p)} dt & \\ + C_6 \int_0^{+\infty} \eta(t) e^{\kappa(b-p)} dt + \dfrac{rp+ab}{ra+r^2} - \dfrac{\xi}{r} & P_m(c) < P < P_d(c) \\ V_2(p,c) & P_d(c) \le P \end{cases} \tag{74}$$

where $P_m(c)$, $P_d(c)$, C_5, and C_6 are satisfied by the implicit equations:

$$\begin{cases} \kappa C_5 \int_0^{+\infty} \eta(t) e^{-\kappa(b-P_m)} dt \\ -\kappa C_6 \int_0^{+\infty} \eta(t) e^{\kappa(b-P_m)} dt + \dfrac{1}{r+a} = 0, \\ C_5 \int_0^{+\infty} \eta(t) e^{-\kappa(b-P_m)} dt + C_6 \int_0^{+\infty} \eta(t) e^{\kappa(b-P_m)} dt \\ + \dfrac{rP_m(c)+ab}{ra+r^2} - \dfrac{\xi}{r} = \gamma \\ \kappa C_5 \int_0^{+\infty} \eta(t) e^{-\kappa(b-P_d)} dt \\ -\kappa C_6 \int_0^{+\infty} \eta(t) e^{\kappa(b-P_d)} dt + \dfrac{1}{r+a} = \dfrac{2}{r+a} \\ C_5 \int_0^{+\infty} \eta(t) e^{-\kappa(b-P_d)} dt + C_6 \int_0^{+\infty} \eta(t) e^{\kappa(b-P_d)} dt \\ + \dfrac{rP_d(c)+ab}{ra+r^2} - \dfrac{\xi}{r} = 2\dfrac{rP_d(c)+ab}{ra+r^2} \\ -\dfrac{(2\xi+(1-\theta)c)}{r} + \dfrac{2}{\kappa(a+r)} \end{cases} \tag{75}$$

2) In the following, we consider the case where the firm decides to go into production and the capital is raised entirely by equity financing, at which point the shareholder value function is:

$$U_2(p,c) = \sup_{r\in S} \mathrm{E}^P\left[\int_0^{\tau} e^{-rs}\left(2P_s - 2\xi\right) ds + e^{-r\tau} 2\gamma\right] \tag{76}$$

Similar to the solution for $V_2(p,c)$, for $\bar{J}(p,c)$, there is:

$$\begin{aligned} &\bar{J}(p,c) = \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs}\left(2P_s-2\xi\right)ds - \int_{\tau}^{+\infty} e^{-rs}\left(2P_s-2\xi\right)ds\right] \\ &+\mathrm{E}^P\left[e^{-r\tau}2\gamma\right] = \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs} 2\left(b + e^{-as}\left(p - b + \int_0^s \sigma e^{at} dW_t\right)\right) ds\right] \\ &-\frac{2\xi}{r} - \mathrm{E}^P\left[\int_0^{+\infty} e^{-r(t+\tau)}\left(2P_{t+\tau}-2\xi\right)dt\right] \\ &+\mathrm{E}^P\left[e^{-r\tau}2\gamma\right] = \mathrm{E}^P\left[\int_0^{+\infty} e^{-rs} 2\left(b + e^{-as}(p-b)\right)ds\right] - \frac{2\xi}{r} \\ &-\mathrm{E}^P\left[\mathrm{E}_{\tau}\int_0^{+\infty} e^{-r(t+\tau)}\left(2P_{t+\tau}-2\xi\right)dt\right] + \mathrm{E}^P\left[e^{-r\tau}2\gamma\right] \\ &= 2\frac{rp+ab}{ra+r^2} - \frac{2\xi}{r} \\ &-\mathrm{E}^P\left[e^{-r\tau}\mathrm{E}_{\tau}\left[\int_0^{+\infty} 2e^{-rt}\left(b + e^{-at}\left(P_{\tau} - b + \int_{\tau}^{t+\tau}\sigma e^{a(s-\tau)}dW_s\right) - \xi\right)dt\right]\right] \\ &+\mathrm{E}^P\left[e^{-r\tau}2\gamma\right] \\ &= 2\frac{rp+ab}{ra+r^2} - \frac{2\xi}{r} + \mathrm{E}^P\left[-2e^{-r\tau}\left(\frac{rP_{\tau}+ab}{ra+r^2} - \frac{\xi}{r} + \gamma\right)\right] \end{aligned} \tag{77}$$

According to the margin conditions:

$$\begin{cases} \tilde{U}_2\left(P_c,c\right)=2\gamma-2\dfrac{rP_c+ab}{ra+r^2}+\dfrac{2\xi}{r} \\ \dfrac{d\tilde{U}_2\left(P_c,c\right)}{dp}=-2\dfrac{r}{ra+r^2} \end{cases} \tag{78}$$

And the relationship between $U_2(p,c)$ and $\tilde{U}_2\left(P_c,c\right)$ can be solved:

$$\begin{cases} U_2(p,c)=2\dfrac{rp+ab}{ra+r^2}-\dfrac{2\xi}{r}+\dfrac{2}{\kappa(a+r)} \\ P_c=\dfrac{(a+r)\xi}{r}-\dfrac{1}{\kappa}-\dfrac{ab}{r}+\gamma(a+r) \end{cases} \tag{79}$$

Accordingly, we return to the case where the firm has only one plant at the beginning of the period and also has an expansion option. At this point, the shareholder value function is:

$$U_1(p,c)=\sup_{\tau_l,\tau_e\in S}\mathrm{E}^P\left[\int_0^{\tau_l\wedge\tau_e}e^{-rs}\left(P_s-\xi\right)ds+e^{-r\tau_l}\gamma 1_{\{\tau_l<\tau_e\}}+e^{-r\tau_e}U_2\left(P_{\tau_e},c\right)1_{\{\tau_j>\tau_e\}}\right] \tag{80}$$

Here, τ_l is the moment when the original plant is shut down without investment, and τ_e is the moment when the decision is made to start production. And there exists $\left(P_l,P_e\right)$ such that $\tau_l=\inf\left\{t;P_t\le P_l\right\},\tau_e=\inf\left\{t;P_t\ge P_e\right\}$. It can be seen that the solution of $U_1(p,c)$ satisfies the form:

$$U_1(p,c)=\begin{cases} \gamma & P\le P_l(c) \\ C_7\int_0^{+\infty}\eta(t)e^{-\kappa(b-p)}dt & \\ +C_8\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt+\dfrac{rp+ab}{ra+r^2}-\dfrac{\xi}{r} & P_l(c)<P<P_e(c) \\ U_2(p,c)-I & P_e(c)\le P \end{cases} \tag{81}$$

According to its edge-value condition:

$$\begin{cases} \dfrac{dU_1\left(P_l(c),c\right)}{dp}=0 \\ C_7\int_0^{+\infty}\eta(t)e^{-\kappa\left(b-P_l\right)}dt+C_8\int_0^{+\infty}\eta(t)e^{\kappa\left(b-P_l\right)}dt+\dfrac{rP_l(c)+ab}{ra+r^2}-\dfrac{\xi}{r}=\gamma \\ \dfrac{dU_1\left(P_e(c),c\right)}{dp}=\dfrac{dU_2\left(P_e(c),c\right)}{dp} \\ C_7\int_0^{+\infty}\eta(t)e^{-\kappa\left(b-P_l\right)}dt+C_8\int_0^{+\infty}\eta(t)e^{\kappa\left(b-P_l\right)}dt \\ +\dfrac{rP_e(c)+ab}{ra+r^2}-\dfrac{\xi}{r}=U_2\left(P_e(c),c\right)-I \end{cases} \tag{82}$$

The following conclusion can be drawn:

If $a\ne -r$, then the solution of $U_1(p,c)$ is:

$$U_1(p,c)=\begin{cases}\gamma & P\le P_l(c)\\ C_7\int_0^{+\infty}\eta(t)e^{-\kappa(b-p)}dt & \\ +C_8\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt+\dfrac{rp+ab}{ra+r^2}-\dfrac{\xi}{r} & P_l(c)<P<P_e(c)\\ U_2(p,c)-I & P_e(c)\le P\end{cases} \tag{83}$$

where the implicit equation satisfied by P_l,P_e,C_7,C_8 is:

$$\begin{cases}\kappa C_7\int_0^{+\infty}\eta(t)e^{-\kappa(b-P_l)}dt-\kappa C_8\int_0^{+\infty}\eta(t)e^{\kappa(b-P_l)}dt\\ +\dfrac{1}{r+a}=0\\ C_7\int_0^{+\infty}\eta(t)e^{-\kappa(b-P_l)}dt+C_8\int_0^{+\infty}\eta(t)e^{\kappa(b-P_l)}dt\\ +\dfrac{rP_1+ab}{ra+r^2}-\dfrac{\xi}{r}=\gamma\\ \kappa C_7\int_0^{+\infty}\eta(t)e^{-\kappa(b-P_e)}dt-\kappa C_8\int_0^{+\infty}\eta(t)e^{\kappa(b-P_e)}dt\\ +\dfrac{1}{r+a}=\dfrac{2}{r+a}\\ C_7\int_0^{+\infty}\eta(t)e^{-\kappa(b-P_e)}dt+C_8\int_0^{+\infty}\eta(t)e^{\kappa(b-P_e)}dt\\ +\dfrac{rP_e+ab}{ra+r^2}-\dfrac{\xi}{r}\\ =2\dfrac{rP_e+ab}{ra+r^2}-\dfrac{2\xi}{r}\\ +\dfrac{2}{\kappa(a+r)}-I\end{cases} \tag{84}$$

2.2.4. Agency cost analysis. We will compute the agency costs incurred in getting investment decisions under the ESG rating mechanism by carving out the value of the expansion option held by the firm [27]. Let $F(p,c)$ be the value of the expansion option and P_d be the optimal investment timing. When the firm executes the option by maximizing the value of equity, at $P=P_d^2$, it is known from the continuity condition and the optimality condition:

$$\begin{cases}F_2\left(P_d^2,c\right)=E\left(P_d^2,c\right)-\left(I-\dfrac{c}{r}\right)\\ \dfrac{dF_2\left(P_d^2,c\right)}{dp}=\dfrac{dE\left(P_d^2,c\right)}{dp}\end{cases} \tag{85}$$

Easy to solve for:

$$\begin{cases}F_2(p,c)=\dfrac{\left(E\left(P_d^2,c\right)-\left(I-\dfrac{c}{r}\right)\right)\left(\int_0^{+\infty}\eta(t)e^{\kappa(b-p)}dt\right)}{\int_0^{+\infty}\eta(t)e^{\kappa\left(b-P_d^2\right)}dt}\\ P_d^2=\dfrac{(a+r)\xi}{r}-\dfrac{2}{\kappa}-\dfrac{ab}{r}\end{cases} \tag{86}$$

Under the ESG rating mechanism, when the firm executes the option to maximize firm value, at $P = P_d^1$, it is known from the continuity condition and the optimality condition:

$$\begin{cases} F_1\left(P_d^1, c\right) = V^L\left(P_d^1, c\right) - \left(I - \dfrac{c}{r}\right) \\ \dfrac{dF_1\left(P_d^1, c\right)}{dp} = \dfrac{dV^L\left(P_d^1, c\right)}{dp} \end{cases} \tag{87}$$

where $V^L(p,c)$ is the value of the firm and:

$$V^L(p,c) = D(p,c) + E(p,c) \tag{88}$$

Available:

$$F_1(p,c) = \frac{\left(V^L\left(P_d^1, c\right) - \left(I - \dfrac{c}{r}\right)\right)\left(\int_0^{+\infty} \eta(t) e^{\kappa(b-p)} dt\right)}{\int_0^{+\infty} \eta(t) e^{\kappa\left(b - P_d^1\right)} dt} \tag{89}$$

where P_d^1 is a solution that satisfies the following equation:

$$-\kappa\left(V^L\left(P_d^1, c\right) - \left(I - \frac{c}{r}\right)\right) = \frac{1}{a+r} - \kappa \frac{\left(\gamma - \dfrac{c}{r}\right)\int_0^{+\infty} \eta(t) e^{\kappa\left(b - P_d^l\right)} dt}{\int_0^{+\infty} \eta(t) e^{\alpha(b - P_D)} dt} \tag{90}$$

Thus, agency costs $AC = \dfrac{F_1 - F_2}{F_2}$.

3. Results and discussion

3.1. Numerical results

In this section we give the numerical results of the model and then analyze the impact of jumps on the value of corporate debt, the value of the firm and the optimal capital structure under the ESG rating mechanism. The tax rate τ is 0.35, the risk-free rate r is 0.06, the volatility σ, 0.25, the bankruptcy cost a is 0.5, and V we take as $100. In our calculations we choose λ to be a value between 0 and 0.1. First we look at the effect of jumps on the value of debt. The effect of Poisson jump strength on the value of debt is shown in Figure 2. From the figure we can see that as the increase of debt dividend on the one hand will increase the debt value of the enterprise, at the same time, the increase of debt dividend will increase the chances of enterprise bankruptcy. As a result, the debt value of the enterprise with the increase of debt dividend shows the trend of appearing to increase and then decrease. And when the jump intensity increases, the value of debt first decreases and then gradually increases in C the same situation. Meanwhile, from the figure, it can be seen that the increase of jump intensity, on the one hand, will delay the debt value to reach the maximum, while on the other hand, it will make the maximum value of debt value become larger.

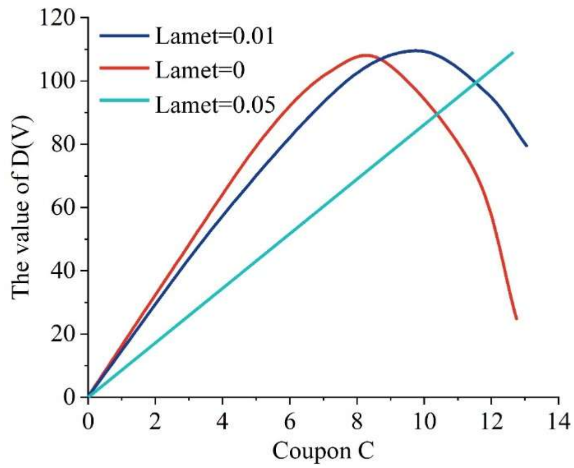

Fig. 2. The effect of poisson jump intensity on debt value

The effect of Poisson jump strength on the option value is shown in Figure 3. From the figure we can see that the option value decreases as the debt dividend increases and then rebounds when it reaches the minimum. At the same debt dividend, the option value increases as the jump intensity increases.

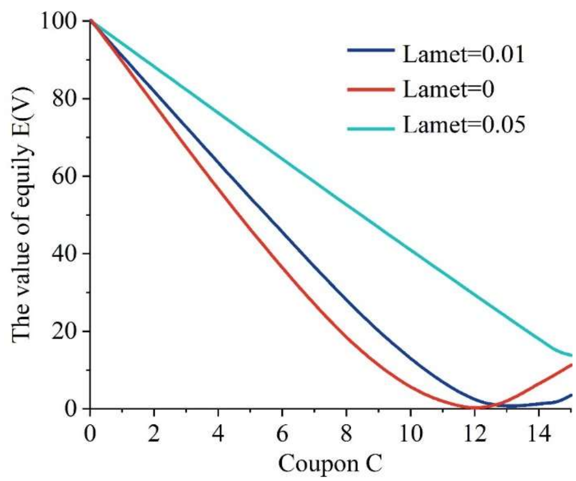

Fig. 3. The effect of poisson jump intensity on the value of options

The effect of Poisson jump intensity on the total value of the firm is shown in Figure 4. From the figure, we can see that the total value of the company will change with the change of jump intensity, showing a trend of increasing and then decreasing.

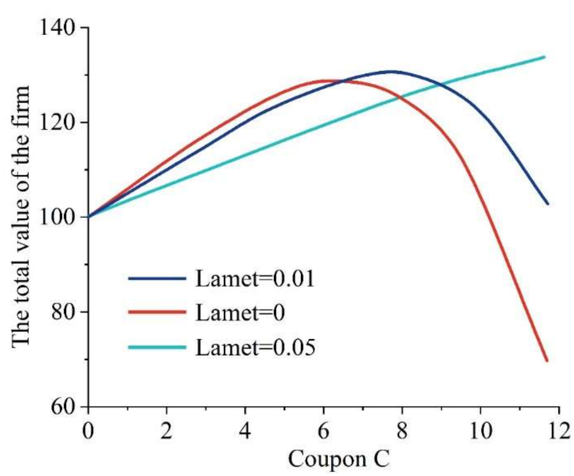

Fig. 4. The effect of poisson jump strength on the value of the company

3.2. Numerical Experiments and Proxy Problem Analysis

Numerical experiments are first performed to analyze how the jump-diffusion model and the geometric Brownian motion are invested with the same parameters under the ESG rating mechanism. The results of the numerical experiments are obtained given the parameters $\gamma=1$, $r=0.1$, $\lambda=0.1$, $\theta=0$, and $\xi=0.1$. The results of the numerical experiment are shown in Figure 5. With the increase of volatility, the positive jump investment price gradually increases but the magnitude is not obvious, the geometric Brownian motion and negative jump model investment price gradually increases and the latter is significantly larger than the former. Elevated volatility means that the investment risk increases, the investment price is bound to rise when the product price obeyed the jump expansion time dispersion model, positive jump offset part of the volatility risk, negative jump obviously part of the increase in the volatility risk, so it leads to different investment strategies. This shows that under different laws of motion enterprise investment strategy is different, especially in the negative jump model if the positive jump or Brownian motion model strategy will bring unnecessary losses to the enterprise, but if the positive jump model under the use of the other two investment strategy will lose some good investment opportunities.

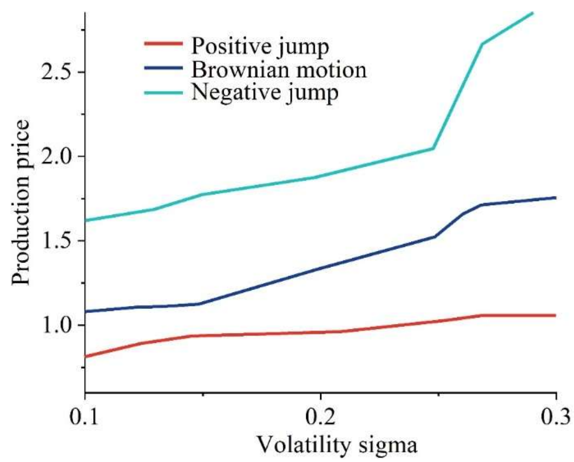

Fig. 5. Numerical results

Next, the impact of parameter changes on investment decisions is analyzed. The relevant parameters $\gamma=1$, $r=0.1$, $\lambda=0.1$, $\xi=0.1$, constant $\theta=0.2$. The impact of positive jump frequency and negative jump frequency on the investment price is shown in Fig. 6 and Fig. 7.Under the ESG rating mechanism, with the increase of the jump frequency, the investment price of the positive jump model decreases gradually, and the investment price of the negative jump model increases gradually, because the increase of the jump frequency is equivalent to the decrease of the bankruptcy risk of the positive jump model or the increase of the bankruptcy risk of the negative jump model, so the investment strategy under the jump diffusion model will change as shown. Because the increase of hopping frequency is equivalent to decreasing the bankruptcy risk of positive hopping model or increasing the bankruptcy risk of negative hopping model, the investment strategies under the hopping diffusion model will change as shown in the figure. This shows that under the same law of motion, the reasonable simulation estimation of the relevant parameters has a great influence on the enterprise's investment strategy, and if the parameters are not estimated correctly, the enterprise will lose part of the benefits or lose good investment opportunities.

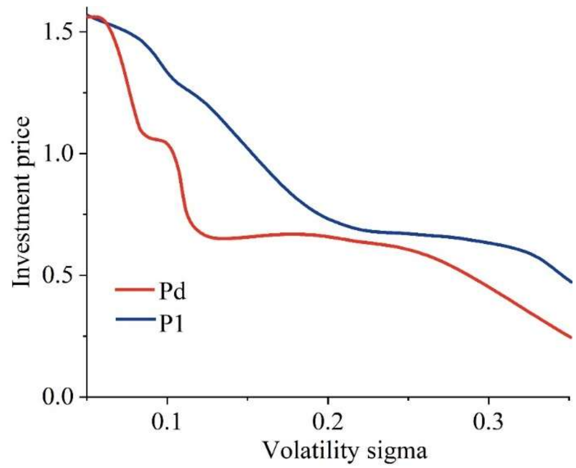

Fig. 6. The rate of jump is affected by the investment price

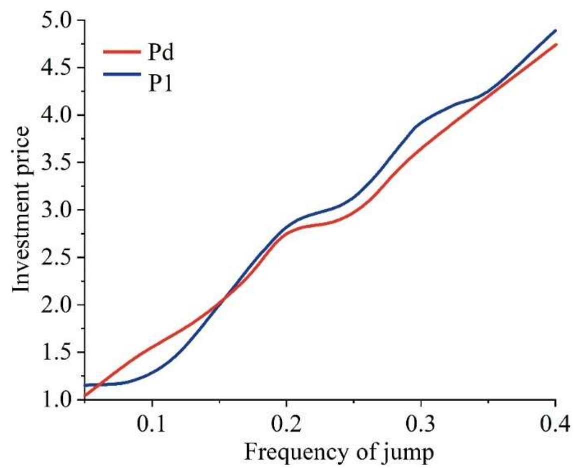

Fig. 7. Negative jump frequency affects investment price

The impact of positive jump amplitude on investment is shown in Figure 8.Under the ESG rating mechanism, the investment price level of the positive jump model gradually decreases as the jump amplitude increases, which is because the increase in the positive jump amplitude of the price, i.e., for the reduction of the risk of insolvency, will give a positive boost to the enterprise's investment. Under the assumption of complete market, the financing method of enterprise investment in plant construction has no effect on investment. However, due to information asymmetry, some enterprises through debt financing, in order to avoid the impact of information asymmetry, creditors will be constrained by the enterprise investment behavior. The enterprise investment behavior is affected by the creditor, the enterprise has to stop some in production projects or raise the investment line, which is the underinvestment we mentioned before. In addition, when the price of the product rises, enterprise managers prefer high-risk and high-yield projects for the benefit of shareholders, resulting in overinvestment.

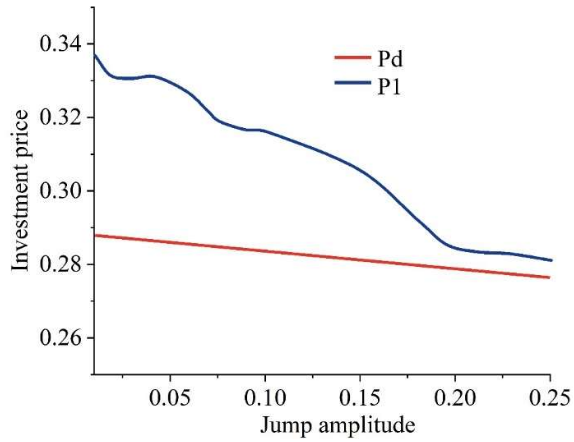

Fig. 8. The magnitude of the jump is the impact on investment

The investment price level of enterprises under different financing methods is shown in Figure 9. When the cost of investment I = 4, the investment price gap between equity financing and debt financing gradually increases, and the vertical coordinate in the figure is the ratio of the investment price gap under different financing methods to the investment price of equity financing, indicating that when the volatility, i.e., the risk, reaches a certain level, shareholders tend to overinvest. This shows that through debt financing shareholder managers in the interests of the stimulation of chasing high-risk projects will bring losses to the enterprise and creditors or even jeopardize the survival of the enterprise.

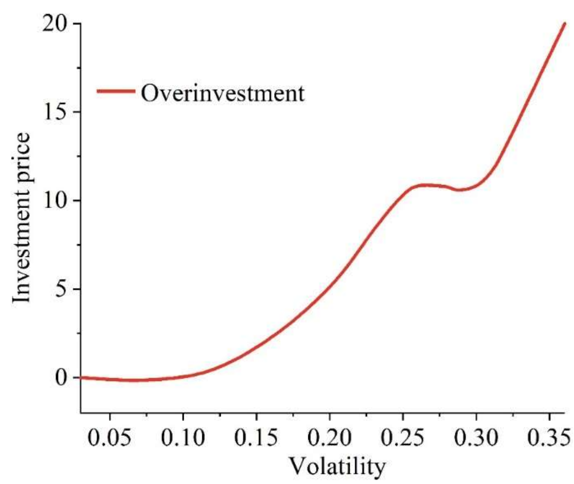

Fig. 9. Investment price level of enterprises under different financing

We define L_2 as the value of the firm under the principle of maximizing firm value, then the value of the firm that operates under the principle of maximizing shareholders' interests is L_1. Taking the positive jump diffusion model as an example, the parameters μ =0.05, ξ =0.1, σ =0.4, θ =0, λ =0.01, r =0.1, γ =1, and c =5 are selected. The difference in the firm value function of the manager under the two operating principles is calculated. The difference in the firm value function is shown in Figure 10. L_1 is the value of the firm that maximizes shareholders' interests, while L_2 is the value of the firm that operates on the principle of maximizing firm value. We find that L_1 is greater than L_2 when the price is below a certain level, while L_2 is greater than L_1 when the price rises.Based on different business principles, corporate managers consider different lines of investment, and when the price rises, the corporate decision tends to overinvest, which results in a loss of corporate value. On the contrary, forced by debt contract constraints tend to underinvest and downsize the business, resulting in a loss of investment opportunities for the firm.

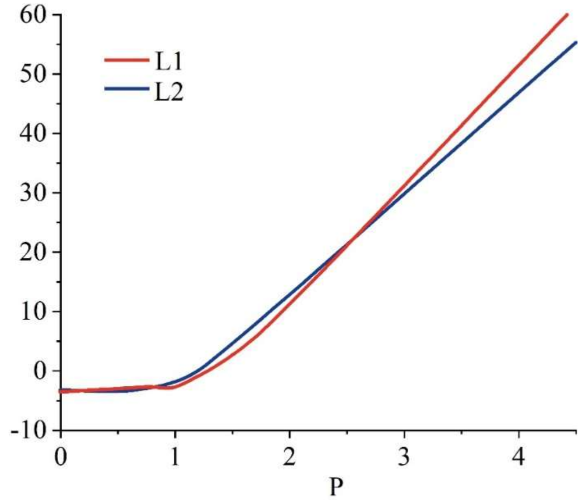

Fig. 10. The difference in the company's value function

4. Conclusion

The optimal capital structure problem is the core problem of enterprise financing decision, and the research on this problem has been paid attention by many scholars. Through experimental research, the article draws the following conclusions:

Through the Poisson jump intensity on the option value of the impact of the experiment can be obtained. Option value with the increase of debt dividend first decreases and rebound when it reaches the minimum. At the same debt dividend, the option value increases with the increase of jump intensity.

When the investment cost I = 4, the investment price gap between equity financing and debt financing gradually increases, and the investment price gap between equity financing and debt financing gradually increases, which leads to the conclusion that shareholders tend to overinvest when the volatility reaches a certain level.

Through the comparison of the investment strategies of the Jump Diffusion Model and the Geometric Brownian Motion Model, we find that the reasonable speculation and testing of the price operation law is of great significance to the enterprise's decision-making, and the adoption of improper strategies will lead to the enterprise to incur unnecessary losses or to lose good investment opportunities. In the sensitivity analysis of the relevant parameters of the jump diffusion model, we find that the different selection of parameters will produce different investment strategies. The selection of parameters will have a significant impact on the timing of new projects or expansion of business, so enterprises must accurately estimate the relevant parameters of the price operation model before making decisions, and then reasonable investment and operation.

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