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A study on roman domination lower deg-centric graphs

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ABSTRACT

The lower deg-centric graph of a simple, connected graph G, denoted by G_{ld}, is a graph constructed from G such that $V(G_{ld}) = V(G)$ and $E(G_{ld}) = \{v_iv_j : d_G(v_i, v_j) < \deg_G(v_i)\}$. This paper presents the Roman domination number of lower deg-centric graphs. Also, investigate the properties and structural characteristics of this type of graph.

Keywords: distance, deg-centric graph, lower deg-centric graph, domination, Roman domination

1. Introduction

For a basic terminology of graph theory, we refer to [15]. For further topics on graph classes, (see [2]). A graph is assumed to be a simple, connected, and undirected graph throughout this paper. The number of edges of a graph G is denoted by $\varepsilon(G)$. Recall that the distance between two distinct vertices v_i and v_j of G, denoted by $d_G(v_i, v_j)$, is the length of the shortest path joining them. The eccentricity of a vertex $v_i \in V(G)$, denoted by $e(v_i)$, is the farthest distance from v_i to some vertex of G. The *degree centric graph* or *deg-centric graph* of a graph G is the graph G_d with $V(G_d) = V(G)$ and $E(G_d) = \{v_iv_j : d_G(v_i, v_j) \leq \deg_G(v_i)\}$ [10]. Let G be a graph and G_d be the deg-centric graph of G. Then, the successive iteration *deg-centric graph* of G, denoted by G_{d^k}, is defined as the derived graph obtained by taking the deg-centric graph successively k times; that is, $G_{d^k} = ((G_d)_d \ldots)_d$, ($k$-times). This process is known as *deg-centrication process* [10]. A particular type of newly derived graphs based on the vertex degrees and distances in

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graphs called *exact deg-centric graphs* have been introduced in (see [11]) as follows, The *exact degree centric graph* or *exact deg-centric graph* of a graph G is the graph G_{ed} with $V(G_{ed}) = V(G)$ and $E(G_{ed}) = \{v_iv_j : d_G(v_i, v_j) = \deg_G(v_i)\}$ [11]. Let G be a graph and G_{ed} be the exact deg-centric graph of G. Then, the successive iteration *exact deg-centric graph* of G, denoted by G_{ed^k}, is defined as the derived graph obtained by taking the exact deg-centric graph successively k times, that is, $G_{ed^k} = ((G_{ed})_{ed} \ldots)_{ed}$, ($k$-times). This process is known as *exact deg-centrication process* [11]. The *upper degree centric graph* or *upper deg-centric graph* of a graph G and denoted by G_{ud}, is the graph with $V(G_{ud}) = V(G)$ and $E(G_{ud}) = \{v_iv_j : d_G(v_i, v_j) \geq \deg_G(v_i)\}$. This graph transformation is called upper deg-centrication (see [12]). Let G be a graph and G_{ud} be the upper deg-centric graph of G. Then the *iterated upper deg-centric graph* of G, denoted by G_{ud^k}, is defined as the graph obtained by applying *upper deg-centrication* successively k-times; That is,$G_{ud^k} = ((G_{ud})_{ud} \ldots)_{ud}$, ($k$-times) (see [12]). The coarse degree centric graph or coarse deg-centric graph of a graph G, denoted by G_{cd}, is the graph with $V(G_{cd}) = V(G)$ and $E(G_{cd}) = \{v_iv_j : d_G(v_i, v_j) > \deg_G(v_i)\}$. This graph transformation is called *coarse deg-centrication* of the graph (see [13]).

A dominating set in a graph G with vertex set $V(G)$ is a set S of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in S. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A dominating set of G of cardinality $\gamma(G)$ is called a $\gamma(G)$-set. Roman dominating functions and their variants have been in the literature for over more than two decades [4, 1, 3, 5]. Cockayne et al. [4] was the first to mathematically formulate the concept of Roman dominating functions in graphs based on the defense strategy of Roman Emperor Constantine that was mentioned in the work of Ian Stewart [6]. A *Roman Dominating Function* (RDF) on a graph $G = (V, E)$ is a function $f : V \to \{0, 1, 2\}$ such that every vertex v with $f(v) = 0$ is adjacent to at least one vertex u with $f(u) = 2$. The value $\omega(f) = \sum\limits_{v \in V} f(v)$ is called the weight of f. The least value of $\omega(f)$ among all the Roman dominating functions f on G is called the *Roman domination number* of G, denoted by $\gamma_R(G)$. A Roman dominating function f with $\omega(f) = \gamma_R(G)$ is called a γ_R-function of G.

The functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. There is always a one-one correspondence between these functions and the ordered partitions induced by them, and thus, these functions can be written as $f = (V_0, V_1, V_2)$.

Motivated by recent studies on Roman domination in upper deg-centric graphs [9], exact deg-centric graph [8] and deg-centric graphs [7], we extend the investigation to a new class: the lower deg-centric graphs. In this paper, we examine the Roman domination number and explore several related properties of lower deg-centric graphs, contributing to the broader understanding of domination parameters in specialized graph classes.

Definition 1.1. [14] The *lower degree centric graph* or *lower deg-centric graph* of a graph G, denoted by G_{ld}, is the graph with $V(G_{ld}) = V(G)$ and $E(G_{ld}) = \{v_iv_j : d_G(v_i, v_j) < \deg_G(v_i)\}$. This graph transformation is called *lower deg-centrication* of the graph.

Definition 1.2. [14] Let G be a graph and G_{ld} be the lower deg-centric graph of G. Then, the iterated *lower deg-centric graph* of G, denoted by G_{ld^k}, is defined as the graph obtained by applying *lower deg-centrication* successively k-times; That is, $G_{ld^k} = ((G_{ld})_{ld} \ldots)_{ld}$, ($k$-times).

A graph G is *D-completable* if, after a finite number of iterated lower deg-centrication, the resultant graph is complete. Let $\varphi(G)$ denote the number of iterations required to transform a D-completable graph G to complete. By convention $\varphi(K_n) = 0$, $n \geq 1$ and $\varphi(K_{1,n}) = \infty$, $n \geq 2$.

Note that in the lower deg graph G_{ld}, if a vertex has degree one in G, that vertex should not have any edge contributions in G_{ld}, this vertex would get a loop attached on the first iteration of G_{ld}, but in our study we do not want loops in our graph.

Proposition 1.3. [14] *For a connected graph G of order n, the lower deg-centric graph $G_{ld} \cong K_n$ if and only if $\deg_G(v_i) > e_G(v_i)$, for all $v_i \in V(G)$.*

Proposition 1.4. [4] *For any graph G of order n, $\gamma(G) = \gamma_R(G)$ if and only if $G = \overline{K_n}$.*

Proposition 1.5. [4] *If G is a graph of order n that contains a vertex of degree $n-1$, then $\gamma(G) = 1$ and $\gamma_R(G) = 2$.*

Proposition 1.6. [14] *For $n \geq 3$, the lower deg-centric graph of a path graph P_n is isomorphic to P_n.*

Proposition 1.7. [4] *For any graph G of order n and maximum degree Δ, then $\frac{2n}{\Delta+1} \leq \gamma_R(G)$.*

2. Roman domination number of lower deg-centric graphs

This section will address the Roman domination number of the lower deg-centric graphs. We apply this concept to the lower deg-centric graph G_{ld}, which is defined based on the vertex degrees and pairwise distances in the original graph G. The goal is to explore how the structural properties of G_{ld} influence its Roman domination number and to derive exact values or bounds for various standard graph classes.

Proposition 2.1. *For a connected graph G of order n, if $\deg_G(v_i) > e_G(v_i)$, for all $v_i \in V(G)$, then, $\gamma_R(G_{ld}) = 2$.*

Proof. Connected graph G of order n, if $deg_G(v_i) > e_G(v_i)$, then,in views of Proposition 1.3, $G_{ld} \cong K_n$. In Roman domination, the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. The lower deg-centric graph is a complete graph with n vertices, so that we can assign the value two to any vertex v_i. That is, $f(v_i) = 2$, all other vertices are adjacent to v_i, we can assign a value of zero to these vertices. Then, after summation, the least value of

$\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R(G_{ld}) = 2$. □

Proposition 2.2. *If G_{ld} is a graph of order $n \geq 3$ which contains a vertex of degree $n-1$, then $\gamma_R(G_{ld}) = 2$.*

Proof. The result is a direct consequence of Proposition 1.5. □

Proposition 2.3. *For a complete graph K_n, $n \geq 3$, $\gamma_R((K_n)_{ld}) = 2$.*

Proof. For a complete graph K_n, $\delta(K_n) > e_G(v_i)$, the lower deg-centric graph of a complete graph K_n of order $n \geq 3$ is always isomorphic to the complete graph K_n. In view of Proposition 2.1, $\gamma_R((K_n)_{ld}) = 2$. □

For convenience, a path P_n is depicted on a horizontal line, and the vertices are labelled from left to right as v_1, v_2, v_3,. . . , v_n.

Proposition 2.4. *For a path P_n, $\gamma_R((P_n)_{ld}) = \lceil \frac{2n}{3} \rceil$.*

Proof. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Consider the lower deg-centric graph P_n, if $n = 1, 2$, it is directly from Definition 1.1. For $n \geq 3$, the lower deg-centric graph of a path graph P_n is isomorphic to P_n 1.6. For any graph G of order n and maximum degree Δ, then $\frac{2n}{\Delta+1} \leq \gamma_R(G)$ 1.7. The path graph achieves the lower bound of this, that is, $\lceil \frac{2n}{3} \rceil$, $\gamma_R(P_n) = \lceil \frac{2n}{3} \rceil$ [4]. Hence, $\gamma_R((P_n)_{ld}) = \lceil \frac{2n}{3} \rceil$. □

Proposition 2.5. *For a cycle C_n, $n \geq 3$, $\gamma_R((C_n)_{ld}) = \lceil \frac{2n}{3} \rceil$.*

Proof. Let the functions $f : V \to \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. For $n \geq 3$, the lower deg-centric graph of a cycle graph C_n is isomorphic to C_n [14]. For any graph G of order n and maximum degree Δ, then $\frac{2n}{\Delta+1} \leq \gamma_R(G)$ [4]. The cycle graph achieves the lower bound of this, that is, $\lceil \frac{2n}{3} \rceil$, $\gamma_R(C_n) = \lceil \frac{2n}{3} \rceil$ [4]. Hence, $\gamma_R((C_n)_{ld}) = \lceil \frac{2n}{3} \rceil$. □

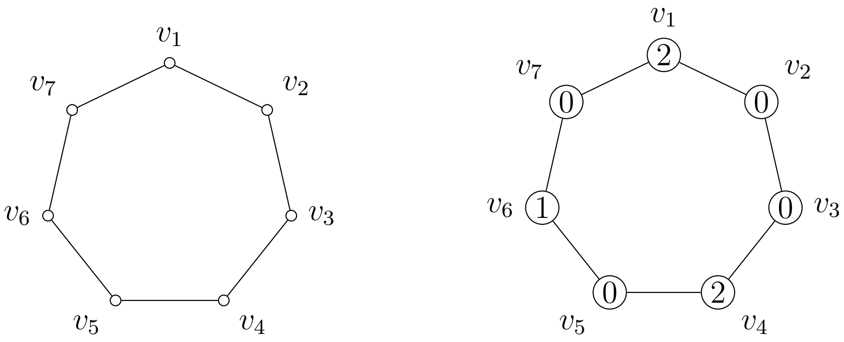
An illustration of Proposition 2.5 is given in Figure 1.

A *star graph*, denoted by $K_{1,n}$, $n \geq 0$, consists of a central vertex that is adjacent to all other vertices, which are of degree one.

Proposition 2.6. *For a star graph $K_{1,n}$, $n \geq 1$, then,*

$$\gamma_R((K_{1,n})_{ld}) = 2.$$

Proof. In view of Definition 1.1, the lower deg-centric graph of a star graph $K_{1,n}$, $n \geq 0$, is always isomorphic to the star graph. If $n = 0$, we can assign the value as one,



(a) C_7 (b) $(C_7)_{ld}$

Fig. 1. $\gamma_R((C_7)_{ld}) = \lceil \frac{2\times 7}{3} \rceil$=5

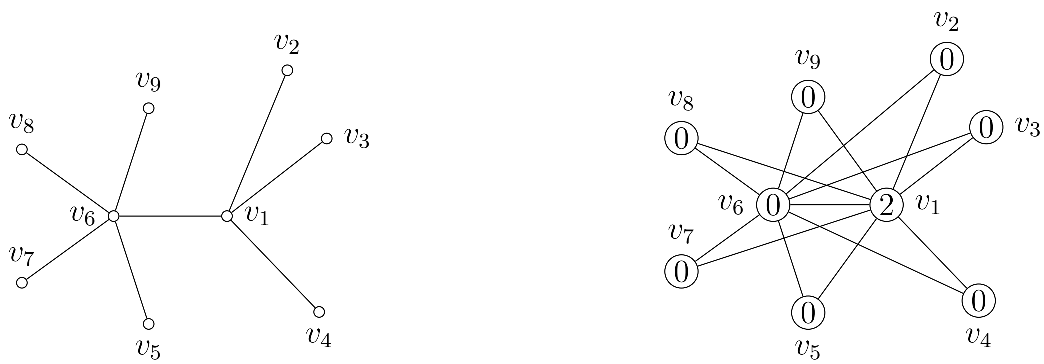
$\gamma_R((K_1)_{ld}) = 1$. If $n \geq 1$, in Roman domination, in the central vertex, assign the value as two, and all other vertices as zero. Hence, $\gamma_R((K_{1,n})_{ld}) = 2$. □

A non-trivial *bistar graph*, denoted by $S_{a,b}$, is a graph obtained by joining the centers of two non-trivial star graphs $k_{1,a}$, $a \geq 1$ and $k_{1,b}$, $b \geq 1$ with the edge v_0u_0.

Proposition 2.7. *For a bistar graph* $S_{a,b}$, $a, b \geq 2$, $\gamma_R((S_{a,b})_{ld}) = 2$.

Proof. Consider a bistar graph $S_{a,b}$, $a, b > 1$. Let the pendant vertices of $K_{1,a}$ be the set $X = \{v_1, v_2, \ldots, v_a\}$ and let the pendant vertices of $K_{1,b}$ be the set $Y = \{u_1, u_2, \ldots, u_b\}$. Finally, let $W = \{v_0, u_0\}$ be center vertices. By Definition 1.1, it follows that both v_0, u_0 are adjacent with all other $a + b + 1$ vertices. In Roman domination, we can assign the value two to any one of these vertices. That is, $f(v_0) = 2$ or $f(u_0) = 2$, assign all $a+b+1$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum\limits_{v\in V} f(v) = 2$. Hence, $\gamma_R((S_{a,b})_{ld}) = 2$. □

An illustration of Proposition 2.7 is given in Figure 2.



(a) $S_{4,3}$ (b) $(S_{4,3})_{ld}$

Fig. 2. $\gamma_R((S_{4,3})_{ld}) = 2$

Proposition 2.8. *For a complete bipartite graph* $K_{n,m}$, $n, m \geq 3$, *then*

$$\gamma_R((K_{n,m})_{ld}) = 2.$$

Proof. In view of Definition 1.1, the complete bipartite graph $K_{n,m}$, $n, m \geq 3$ of the lower deg-centric graph is complete, which implies $\varepsilon((K_{n,m})_{ld}) = K_{n+m}$. By Proposition 1.5, $\gamma_R((K_{n,m})_{ld}) = 2$. □

A *wheel graph* denoted by $W_{1,n}$, $n \geq 3$ is obtained by taking a cycle C_n, $n \geq 3$ (the rim with rim-vertices) and adding the central vertex v_0 with *spokes* namely, edges $v_0 v_i$, $1 \leq i \leq n$.

Proposition 2.9. *For a wheel graph $W_{1,n}$, $n \geq 3$, then,*

$$\gamma_R((W_{1,n})_{ld}) = 2.$$

Proof. For a wheel graph $W_{1,n}$, $n \geq 3$, note that, $deg(v_i) > e(v_i)$ in wheel graph, for all $v_i \in V(W_{1,n})$. In view of Definition 1.1, $(W_{1,n})_{ld}$ is isomorphic to K_{n+1}. In view of Proposition 1.5, $\gamma_R((W_{1,n})_{ld}) = 2$. □

A *helm graph*, denoted by $H_{1,n,}$, $n \geq 3$ is a graph obtained from a wheel graph $W_{1,n}$ by attaching a pendant vertex u_i to the corresponding*rim vertex* v_i.

Proposition 2.10. *For a helm graph $H_{1,n}$, $n \geq 3$, $\gamma_R((H_{1,n})_{ld}) = 2$.*

Proof. The helm graph $H_{1,n,}$, $n \geq 3$, the helm graph is of the order $2n+1$. Let $V(H_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. Then by Definition 1.1, there are $2n$ edge incident to v_0 and v_i in $(H_{1,n})_{ld}$. In Roman domination, we can assign value two to any of these vertices v_0 or v_i. That is, $f(v_0) = 2$ 0r $f(v_i) = 2$, assign all $2n$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((H_{1,n})_{ld}) = 2$. □

A *closed helm graph* denoted by $CH_{1,n}$, $n \geq 3$ is the graph obtained from a helm graph $H_{1,n}$ by cyclically joining the pendant vertices to form an outer rim.

Proposition 2.11. *For a closed helm graph $CH_{1,n}$, $n \geq 3$, then, $\gamma_R((CH_{1,n})_{ld}) = 2$.*

Proof. Consider a closed helm graph $CH_{1,n}$ $n \geq 3$, is clearly of the order $2n+1$. Let $V(CH_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. For all $CH_{1,n}$, $n < 6$, $\delta(CH_{1,n}) = 3$. For $n = 3, 4, 5$. In view of Definition 1.1 and Proposition 1.3, the lower deg-centric graph of a closed helm graph $CH_{1,n}$ of order $n < 6$ is the complete graph. Finally, by Proposition 2.1, $\gamma_R((CH_{1,n})_{ld}) = 2$. If $n \geq 6$, we have $\delta(CH_{1,n}) = 3$ and $diam(CH_{1,n}) = 4$, in $CH_{1,n}$ center vertex v_0, $\deg(v_0) = n$. In view of Definition 1.1, $\deg(v_0) = 2n$ in lower deg-centric graph. Now we can assign a value of two in Roman domination, $f(v_0) = 2$, all other $2n$ vertices are adjacent to v_0, and assign a value of zero to all these values. Finally, $\gamma_R((CH_{1,n})_{ld}) = 2$. □

A *sunlet graph* , denoted by Sl_n, $n \geq 3$, is a graph obtained by attaching a pendant vertex to every vertex of a cycle graph c_n, $n \geq 3$. In other words, a sunlet graph on $2n$ vertices is obtained by taking the corona product $C_n \circ K_1$.

Proposition 2.12. *For a Sunlet graph Sl_n, $n \geq 3$,*

$$\gamma_R((Sl_n)_{ld}) = \begin{cases} 2 & \text{if } n = 3, \\ 3 & \text{if } n = 4, \\ \lceil \frac{2n}{3} \rceil & \text{if } n \geq 5. \end{cases}$$

Proof. Let the functions $f : V \rightarrow \{0, 1, 2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0, 1, 2$. Consider a sunlet graph Sl_n, $n \geq 3$, of order $2n$. Let $V(Sl_n) = \{v_1, v_2, \ldots, v_n, \underbrace{u_1, u_2, \ldots, u_n}_{\text{pendant vertices}}\}$. For Sl_3, $\deg(v_i) = 3 > e(v_i) = 2$ in Sl_n, as per Definition 1.1, in Roman Domination, assign any v_i value as two, then all v_i vertices are adjacent with other $2n - 1$ vertices in the lower deg-centric graph, and assign a value of zero to all these values. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Finally, $\gamma_R((Sl_3)_{ld}) = 2$.

For Sl_4, by Definition 1.1, all v_i vertices are adjacent with other $2n - 2$ vertices. In Roman Domination, assign any v_i value as two, then all v_i vertices are adjacent with other $2n - 2$ vertices in the lower deg-centric graph, and assign a value of zero to all these values. Now, one non-assigned vertex u_i remains, and we can assign value one to this vertex. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2 + 1$. Finally, $\gamma_R((Sl_4)_{ld}) = 3$.

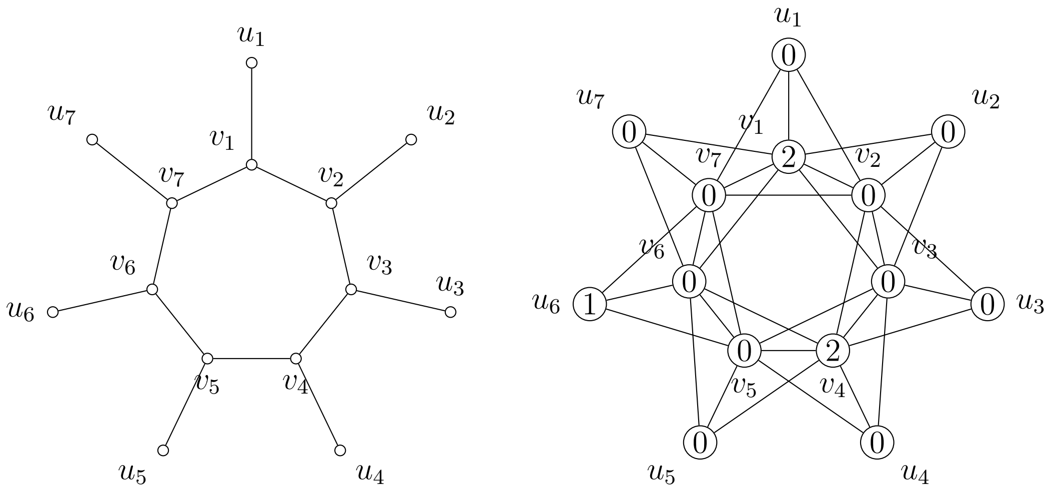
Consider $n \geq 5$, $deg(v_i) = 3$, and $deg(u_i) = 1$ in Sl_n. By Definition 1.1, all v_i vertices are adjacent with seven vertices. However, since all u_i are pendant vertices, no edge incident at u_i in $(Sl_n)_{ld}$. Then, all u_i have degree three, $\deg(v_i) = 7$, and $\deg(u_i) = 3$ in $(Sl_n)_{ld}$. In Roman Domination, consecutively labeled n rim vertices $v_1, v_2, v_3, \ldots, v_n$. Then, assign $f(v_1) = 2$, then vertex v_1 adjacent to seven vertices, four v_i vertices, and three u_i vertices that assign the values zero. Then, assign the value two to the vertices v_{1+3n}, $n = 1, 2, 3, \ldots, n$. That is, we assign $f(v_1) = 2$, $f(v_4) = 2$, $f(v_7) = 2, \ldots, f(v_{(1+3n)}) = 2$, all other vertices can assign a value of zero. Note that one special case, If Sl_{3n+1}, $n = 2, 3, 4, \ldots,$, we can assign $f(v_1) = 2$, $f(v_4) = 2$, $f(v_7) = 2, \ldots, f(v_{(1+3n)}) = 2$, and $f(u_{n-1}) = 1$, all other vertices can assign a value of zero. Then, after summation, the least value of $\omega(f) = \sum_{v \in V} f(v) = \lceil \frac{2n}{3} \rceil$. Hence, $\gamma_R((Sl_n)_{ld}) = \lceil \frac{2n}{3} \rceil$. □

An illustration to Proposition 2.12 is given in Figure 3.

A *double wheel* DW_n is obtained by taking two copies of a wheel $W_{1,n}$ $n \geq 3$ and *merging* the two central vertices.

Proposition 2.13. *For a double wheel graph DW_n, $n \geq 3$, then,*

$$\gamma_R((DW_n)_{ld}) = 2.$$



(a) Sl_7 (b) $(Sl_7)_{ld}$

Fig. 3. $\gamma_R(Sl_7)_{ld}) = 5$

Proof. For a double wheel graph DW_n, $n \geq 3$, clearly, the double wheel graph is of the order $2n+1$. Let $V(DW_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_0) = 2n > e(v_0) = 1$ and $\deg(v_i) = \deg_G(u_i) = 3 > e(v_0) = 1$ in DW_n, by Definition 1.1, $2n$ edge incident from all $2n+1$ vertices in $(DW_n)_{ld}$. That is, $(DW_n)_{ld} \cong K_{2n+1}$. In views of Proposition 1.5, $\gamma_R((DW_n)_{ld}) = 2$. □

A *djembe graph*, denoted by $D_{1,n}$, is obtained by joining the vertices $u_i's$; $1 \leq i \leq n$ of a closed helm graph $CH_{1,n}$ to its central vertex v_0.

Proposition 2.14. *For a djembe graph $D_{1,n}$, $n \geq 3$, then,*

$$\gamma_R((D_{1,n})_{ld}) = 2.$$

Proof. For a djembe graph $D_{1,n}$, $n \geq 3$, clearly, the djembe graph is of the order $2n+1$. Let $V(D_{1,n}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_0) = 2n > e(v_0) = 1$ and $\deg(v_i) = \deg_G(u_i) = 4 > e(v_0) = 1$ in $D_{1,n}$, by Definition 1.1, $2n$ edge incident at all $2n+1$ vertices in $(D_{1,n})_{ld}$. That is, $(D_{1,n})_{ld} \cong K_{2n+1}$. In view of Proposition 1.5, $\gamma_R((D_{1,n})_{ld}) = 2$. □

A *gear graph*, denoted by G_n,$n \geq 3$, is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the perimeter of a wheel graph $W_{1,n}$.

Proposition 2.15. *For a gear graph G_n, $n \geq 3$, then, $\gamma_R((G_n)_{ld}) = 2$.*

Proof. For a gear graph G_n, $n \geq 3$, clearly, the gear graph is of the order $2n+1$. Let $V(G_n) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_0) = n > e(v_0) = 2$ in G_n, by Definition 1.1, all other vertices are incident to v_0 in $(G_n)_{ld}$, $f(v_0) = 2$, and assign value zero to all these values. Finally, $\gamma_R((G_n)_{ld}) = 2$. □

A *web graph*, denoted by $Wb_{1,n}$, $n \geq 3$ is the graph obtained by attaching a pendant edge to each vertex of the outer cycle (or rim) of the closed helm graph $CH_{1,n}$.

Proposition 2.16. *For a web graph* $Wb_{1,n}$*,* $n \geq 3$*,* $\gamma_R((Wb_{1,n})_{ld}) = 2$.

Proof. The web graph $Wb_{1,n}$, $n \geq 3$, is of the order $3n + 1$.

Let $V(Wb_{1,n,}) = \{v_0, v_1, v_2, \ldots, v_{n-1}, v_n, u_1, u_2, u_3, \ldots, u_n, \underbrace{w_1, w_2, w_3, \ldots, w_n}_{\text{pendant vertices}}\}$. If $n = 3$, any vertex u_i, $\deg(u_i) = 4 > e(u_i) = 2$ in $Wb_{1,n}$, by Definition 1.1, $f(u_i) = 2$, all other vertices are incident to u_i in $(Wb_{1,3})_{ld}$, and assign value zero to all these values. Finally, $\gamma_R((Wb_{1,3})_{ld}) = 2$. If $n \geq 4$, Since $\deg(v_0) = n > e(v_0) = 2$ in $Wb_{1,n}$, by Definition 1.1, all other vertices are incident to v_0 in $(Wb_{1,n})_{ld}$, and assign value zero to all these values. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((Wb_{1,n})_{ld}) = 2$. □

A *blossom graph*, denoted by $Bl_{1,n}$, is obtained by making each u_i adjacent to the central vertex of the closed sunflower graph.

Proposition 2.17. *For a blossom graph* $Bl_{1,n}$*,* $n \geq 3$*, then,*

$$\gamma_R((Bl_{1,n})_{ld}) = 2.$$

Proof. Consider a blossom graph $Bl_{1,n,}$, $n \geq 3$. The Blossom graph is of the order $2n+1$. Let $V(Bl_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. By Proposition 1.3, $(Bl_{1,n})_{ld}$ is complete. In views of Proposition 1.5, $\gamma_R((Bl_{1,n})_{ld}) = 2$. □

A *flower graph*, $F_{1,n}$, $n \geq 3$ is a graph obtained from a helm graph $H_{1,n,}$ by joining each of its pendant vertices u_i's to its central vertex v_0.

Proposition 2.18. *For a flower graph* $F_{1,n}$*,* $n \geq 3$*,* $\gamma_R((F_{1,n})_{ld}) = 2$.

Proof. Consider a flower graph $F_{1,n,}$, $n \geq 3$. Clearly, the flower graph is of the order $2n + 1$. Let $V(F_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_0) = 2n > \deg(v_i) = n > e(v_0) = 2$ in $F_{1,n,}$, by Definition 1.1, $2n$ edge incident at v_0 and v_i in $(F_{1,n,})_{ld}$. In Roman domination, we can assign value two to any of these vertices v_0 or v_i. That is, $f(v_0) = 2$ or $f(v_i) = 2$, assign all $2n$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((F_{1,n})_{ld}) = 2$. □

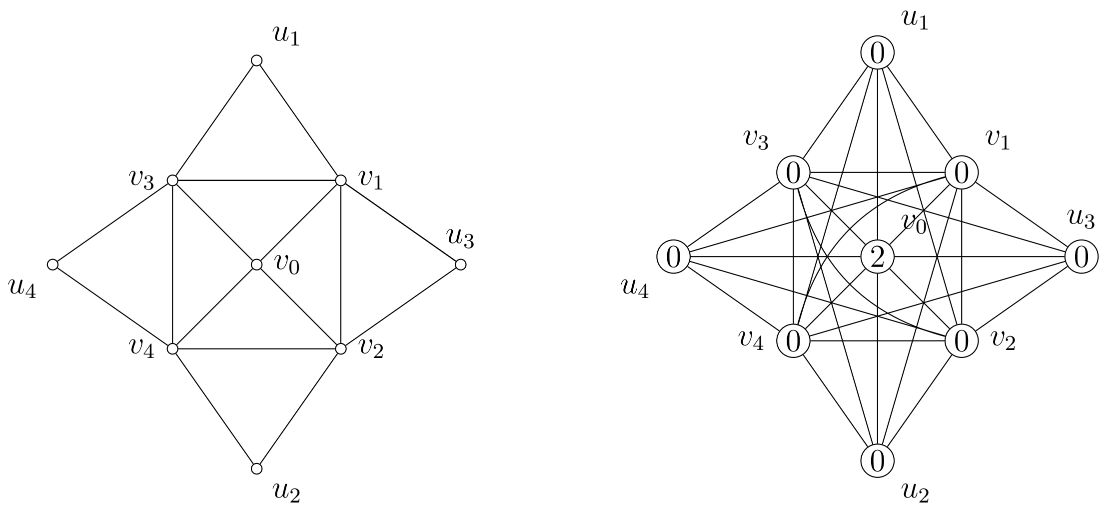
The *sunflower graph*, denoted by $SF_{1,n}$, $n \geq 3$ is obtained from the wheel $W_{1,n}$ by attaching n vertices u_i, $1 \leq i \leq n$ such that each u_i is adjacent to v_i and v_{i+1} and count the suffix is taken modulo n.

Proposition 2.19. *For a sunflower graph* $SF_{1,n}$*,* $n \geq 3$*,* $\gamma_R((SF_{1,n})_{ld}) = 2$.

Proof. For a sunflower graph $SF_{1,n,}$, $n \geq 3$, the sunflower graph is of the order $2n + 1$. Let $V(SF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_0) = n > e(v_0) = 2$ and

$\deg(v_i) = n + 1 > e(v_i) = 2$ in $SF_{1,n}$. Then by Definition 1.1, $2n$ edge incident at v_0 and v_i in $(SF_{1,n,})_{ld}$. In Roman domination, we can assign value two to any of these vertices v_0 or v_i. That is, $f(v_0) = 2$ 0r $f(v_i) = 2$, assign all $2n$ adjacent vertices value as zero. Then, the least value of $\omega(f) = \sum_{v \in V} f(v) = 2$. Hence, $\gamma_R((SF_{1,n})_{ld}) = 2$. □

An illustration of Proposition 2.19 is given in Figure 4.



(a) $SF_{1,4}$ (b) $(SF_{1,4})_{ld}$

Fig. 4. $\gamma_R((SF_{1,4})_{ld}) = 2$

A *closed sunflower graph* $CSF_{1,n}$ is obtained by adding the edge u_iu_{i+1} of the sunflower graph.

Proposition 2.20. *For a closed sunflower graph $CSF_{1,n}$, $n \geq 3$, then,*

$$\gamma_R((CSF_{1,n})_{ld}) = 2.$$

Proof. Consider a closed sun flower graph $CSF_{1,n,}$, $n \geq 3$. The closed sunflower graph is of the order $2n+1$. Let $V(CSF_{1,n,}) = v_0, v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. By Proposition 1.3, $(CSF_{1,n})_{ld}$ is complete graph. In view of Proposition 1.5, $\gamma_R((CSF_{1,n})_{ld}) = 2$. □

Consider a complete graph K_n with the vertex set $V = v_1, v_2, v_3, \ldots, v_n$. Let $U = u_1, u_2, u_3, \ldots, u_n$ be a copy of $V(G)$ such that u_i corresponds to v_i. The *sun graph*, denoted by S_n, is a graph with vertex set $V \cup U$ and two vertices x and y are adjacent in S_n if $x \sim y$ in K_n and $x = u_i$, $y \in v_i, v_{i+1}$.

Proposition 2.21. *For a sun graph S_n, $n \geq 3$, then,*

$$\gamma_R((S_n)_{ld}) = 2.$$

Proof. For a sun graph S_n, $n \geq 3$, the sun graph is of the order $2n$.

Let $V(S_n) = v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n$. Since $\deg(v_i) = n+1 > e(v_i) = 2$ in S_n, by Definition 1.1, $2n-1$ edge incident at v_i in $(S_n)_{ld}$. All other vertices are incident to v_i in $(S_n)_{ld}$, assign any one of v_i value as two, $f(v_i) = 2$, and assign a value of zero to all other $2n-1$ vertices. Finally, $\gamma_R((S_n)_{ld}) = 2$.

□

A *closed sun graph* CS_n is the graph obtained from adding the edges u_iu_{i+1} in the sun graph. In view of Definition 1.1, the lower deg-centric graph of a closed sun graph CS_n, $n \geq 3$, is complete which implies $\varepsilon((CS_n)_{ld}) = \varepsilon(K_{2n})$. That is, $\gamma_R((CS_n)_{ld}) = 2$.

A *friendship graph*, denoted by $F_n, n \geq 1$, is obtained by joining n copies of the complete graph K_3 with a common vertex. Note that, In view of Definition 1.1, the lower deg-centric graph of a friendship graph F_n, $n \geq 1$, is always isomorphic to the friendship graph F_n. All vertices are adjacent to the center vertex v_0, assign value two to the center vertex, that is, $f(v_0) = 2$. That is, $\gamma_R((F_n)_{ld}) = 2$.

An *antiprism graph*, denoted by A_n, $n \geq 3$ is a graph obtained two cycles C_n and C'_n of order n with vertex sets $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $U = \{u_1, u_2, u_3, \ldots, u_n\}$ respectively. Join the vertices u_iv_i and u_iv_{i+1} to form the additional edges.

Proposition 2.22. *For $n \geq 3$,*

$$\gamma_R((A_n)_{ld}) = \begin{cases} 2\left\lceil \frac{n}{6} \right\rceil, & \text{if } n \equiv 0,2,3,4,5 \pmod 6, \\ 2\lfloor \frac{n}{6} \rfloor + 1, & \text{if } n \equiv 1 \pmod 6. \end{cases}$$

Proof. Consider an antiprism graph A_n, $n \geq 3$, is of the order $2n$.

Let $V(A_n) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. If $3 \leq n \leq 6$, $\deg(v_i) = \deg(u_i) = 4 > e(v_i) = e(u_i)$ in A_n then by Definition 1.1, $(A_n)_{ld} \cong K_{2n}, \gamma_R((A_n)_{ld}) = 2$. Hence, $\gamma_R((A_n)_{ld}) = 2\left\lceil \frac{n}{6} \right\rceil$.

If $n > 6$, $\deg(v_i) = \deg(u_i) = 4$ in A_n then by Definition 1.1, $\deg(v_i) = \deg(u_i) = 12$ in $(A_n)_{ld}$. In Roman domination, the functions $f : V \to \{0,1,2\}$ on a graph induce an ordered partition (V_0, V_1, V_2) of the vertex set, where $V_i = \{v \in V | f(v) = i\}; i = 0,1,2$. In Roman domination, we can assign value two to any of these vertices u_i or v_i. That is, $f(u_i) = 2$ or $f(v_i) = 2$, assign all 12 adjacent vertices value as zero. In view of definition the least value of $\omega(f) = \sum_{v \in V} f(v) = 2\left\lceil \frac{n}{6} \right\rceil$, if $n \equiv 0,2,3,4,5 \pmod 6$ and $2\lfloor \frac{n}{6} \rfloor + 1$, if $n \equiv 1 \pmod 6$. □

3. Conclusion

The concept of Roman domination in lower deg-centric graphs has been introduced, along with an investigation of the Roman domination number for certain graph classes under this framework. Several exploratory results have been presented to lay the groundwork for future research in this area. As a potential direction, the study can be extended to analyze various graph-theoretical parameters in the context of lower deg-centric graphs across different graph classes. Such extensions may yield significant and insightful results. Additionally, emerging researchers may explore alternative forms of domination in graphs to further enrich this domain of study.

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