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Block-transitive 2-$(v, 5, \lambda)$ designs with sporadic socle

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ABSTRACT

This paper contributes to the classification of non-trivial 2-designs with block size 5 admitting a block-transitive automorphism group. Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a non-trivial 2-$(v, 5, \lambda)$ design and G be a block-transitive automorphism group of $\mathscr{D}$. The main aim of this paper is to determine all pairs $(\mathscr{D}, G)$ when $\mathrm{Soc}(G)$ is a sporadic simple group.

Keywords: block-transitive, 2-design, automorphism group, sporadic group

1. Introduction

Definition 1.1. A t-(v, k, λ) design $\mathscr{D}$ is a finite incidence structure $(\mathscr{P}, \mathscr{B})$ satisfying the following properties:

- $\mathscr{P}$ is a set of v elements, called *points*,
- $\mathscr{B}$ is a set of b k-subsets of $\mathscr{P}$, called *blocks*,
- Each t-subset of $\mathscr{P}$ lies in exactly λ blocks.

Since all the blocks have the same size k, it follows that each point belongs to the same number r of blocks. We call a 2-(v, k, λ) design $\mathscr{D}$ to be *non-trivial* if $2 < k < v - 1$. If $b = \binom{v}{k}$, then we speak of a *complete design*. All of the designs we will deal with in this paper will be non-trivial.

An *automorphism* of $\mathscr{D}$ is a permutation of $\mathscr{P}$ which permutes the blocks of $\mathscr{B}$. The *full automorphism group* of $\mathscr{D}$, denoted by $\mathrm{Aut}(\mathscr{D})$, is a group consisting of all automorphisms of $\mathscr{D}$. For any subgroup G of $\mathrm{Aut}(\mathscr{D})$, the design $\mathscr{D}$ is said to be *block-transitive* if G acts transitively on the blocks, and *flag-transitive* if G acts transitively on the flags, where a

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flag refers a point-block pair (α, B) of $\mathscr{D}$ with $\alpha \in B$. And the *point-primitivity*, *point-imprimitivity* and *point s-transitivity* can be defined similarly. A set of blocks of $\mathscr{D}$ is called a set of *base blocks* with respect to an *automorphism group* G of $\mathscr{D}$ if it contains exactly one block from each G-orbits on the block set. In particular, every block B can be a *base block* of $\mathscr{D}$ if G is *block-transitive* or *flag-transitive*.

In the homogeneity properties of incidence structures $\mathscr{D} = (\mathscr{P}, \mathscr{B})$, transitivity is a key and natural attribute. Hence, over recent decades, much research has been dedicated to classifying designs with specific transitivity features, particularly focusing on flag-transitivity. By contrast, research on 2-designs with block-transitive automorphism groups is relatively rare. In most cases, classification studies of block-transitive 2-designs typically focus on cases with restricted parameters. For example, in 1988, the classification for the block-transitive automorphism groups of 2-$(v, k, 1)$ designs has been completed by Camina and Siemons [5]. O'Keefe et al. are concerned about block-transitive, point-imprimitive designs with $\lambda = 1$ in their 1993 work [14]. Recently, Zhang and Zhou [19] have studied block-transitive automorphism groups of non-trivial 2-(v, k, λ) designs with $(r, k) = 1$, where r is the number of blocks incident with a given point.

Nevertheless, we would like to classify 2-designs in the other direction. We intend to study 2-design with a fixed value of k rather than λ or r. It follows from a result of Block [2] that G block-transitive implies its point-transitive. Cameron and Praeger [4] demonstrated that for block-transitive point-imprimitive t-(v, k, λ) designs, the value of t is necessarily 2 or 3. For instance, Zhan and Pang are concerned about block-transitive 3-design with block size at most 6 in [18] and have given a few results on the classification of non-trivial 3-$(v, 4, \lambda)$ designs in [15]. Additionally, building on the results of Delandtsheer and Doyen [9], it can be concluded that any block-transitive point-imprimitive 2-design must satisfy the inequality $v \leqslant (\binom{k}{2} - 1)^2$. It immediately follows that most of 2-designs are point-primitive. Hopefully, there are some new results have been shown recently. Camina and Spiezia [6] have made a contribution to the study of sporadic groups and automorphism of finite linear spaces. Zhang and Zhou [20] are concerned about block-transitive and point-primitive 2-$(v, k, 2)$ designs with sporadic socle.

In this paper, inspired by previous researchers, it is natural for us to consider the classification of non-trivial 2-(v, k, λ) designs with $k = 5$. The following proposition, which is obtained in [12], is our starting point.

Proposition 1.2. [12, Theorems 1-3] *Let G be a block-transitive automorphism group of a non-trivial 2-$(v, 5, \lambda)$ design $\mathscr{D}$. Then one of the following holds:*

(i) *If G is point-primitive then G must be of affine type, almost simple type or product type. Moreover, if G has a product action on $\mathscr{P}$, then one of the following holds:*

(1) $\mathrm{Soc}(G) = A_9 \times A_9$, *and $\mathscr{D}$ is a 2-$(81, 5, \lambda)$ design with $\lambda \in \{5880, 7056, 14112\}$.*

(2) $\mathrm{Soc}(G) = PSL(2, 8) \times PSL(2, 8)$, *and $\mathscr{D}$ is a 2-$(81, 5, \lambda)$ design with*

$$\lambda \in \{392, 784, 1176, 1568, 2352, 4704, 7056\}.$$

(3) $\mathrm{Soc}(G) = A_{19} \times A_{19}$, *and $\mathscr{D}$ is a 2-$(361, 5, 3329280)$ design.*

(ii) *If G is point-imprimitive then $\mathscr{P}$ has a invariant non-trivial partition with d classes of size c. Then (v, c, d) is one of the following:*

$$(16, 4, 4), (21, 3, 7), (21, 7, 3), (81, 9, 9).$$

Moreover,

(1) *If $(c, d) = (4, 4)$ then $\lambda \in \{4, 8, 12, 16, 24, 32, 48, 96, 144\}$.*

(2) *If $(c, d) = (3, 7)$ then*

$$\lambda \in \{1, 2, 3, 4, 6, 8, 12, 20, 24, 27, 48, 54, 60, 81, 108, 120, 162, 324, 540\}.$$

(3) *If $(c, d) = (7, 3)$ then $\lambda \in \{7, 14, 21, 42, 49, 98, 147, 196, 245\}$.*

(4) *If $(c, d) = (9, 9)$ then $\lambda \leq 40824$.*

By Proposition 1.2, we konw that all block-transitive point-imprimitive automorphism groups of 2-$(v, 5, \lambda)$ design have been determined, so we restrict our attention to G is point-primitive. Here we focus on the case where the automorphism group is of almost simple type. The well-known classification of finite simple groups that a finite non-abelian simple group is isomorphic to one of the groups in the following four types: (1) sporadic simple groups; (2) alternating groups; (3) classical simple groups; (4) exceptional simple groups of Lie type. Since we have determined all block-transitive 2-$(v, 5, \lambda)$ designs with alternating socle in [11]. As a continuation of this classification project, we will classify such block-transitive 2-designs where the automorphism group is of almost simple type with sporadic socle, and get the main theorem as follows:

Theorem 1.3. *Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a non-trivial 2-$(v, 5, \lambda)$ design. Suppose that $G \leq$ $\mathrm{Aut}(\mathscr{D})$ acts transitively on the blocks of $\mathscr{D}$ and* $\mathrm{Soc}(G)$ is a sporadic simple group. Then all pairs $(\mathscr{D}, G)$ are listed in Table 1.

Remark 1.4. (1) The notation "FT" listed in column "Note" indicated that $\mathscr{D}$ is a flag-transitive 2-$(v, 5, \lambda)$ design and all such designs can be found in [16, Table 1].

(2) The notation "CD" indicated that $\mathscr{D}$ is a complete 2-$(v, 5, \frac{(v-2)(v-3)(v-4)}{6})$ design.

The structure of this paper is organized as follows. Some notations and useful results which are important for the present paper will be collected in Section 2. In Section 3, we prove our main result.

2. Preliminaries

In this section, we review and state some preliminary results from design theory and group theory which will be used throughout the paper. Note that the notation and terminology we use are both standard (see [7, 17]). For the sake of notation, in the remainder of this paper we denote the rank of G by $rank(G)$, and let G_α denote the stabilizer of a point $\alpha \in \mathscr{P}$ in G, and G_B the setwise stabilizer of a block $B \in \mathscr{B}$ in G.

Table 1. All block-transitive 2-$(v,5,\lambda)$ designs with sporadic socle

Case	G	$\mathscr{D}$	Note	Case	G	$\mathscr{D}$	Note
1	M_{11}	2-$(11,5,12)$	FT	23		2-$(176,5,144)$	FT
2		2-$(11,5,72)$	FT	24		2-$(176,5,240)$	FT
3		2-$(12,5,20)$	FT	25		2-$(176,5,360)$	FT
4		2-$(12,5,100)$		26		2-$(176,5,900)$	
5	M_{12}	2-$(12,5,120)$	FT, CD	27		2-$(176,5,1440)$	FT
6	M_{22}	2-$(22,5,20)$	FT	28		2-$(176,5,1800)$	
7		2-$(22,5,160)$	FT	29		2-$(176,5,2400)$	
8		2-$(22,5,800)$		30		2-$(176,5,3600)$	
9		2-$(176,5,4)$		31		2-$(176,5,4800)$	
10		2-$(176,5,12)$		32		2-$(176,5,5760)$	FT
11		2-$(176,5,36)$		33		2-$(176,5,7200)$	
12		2-$(176,5,72)$		34		2-$(176,5,14400)$	
13		2-$(176,5,96)$		35		2-$(176,5,28800)$	
14		2-$(176,5,144)$		36	Co_3	2-$(276,5,1120)$	FT
15		2-$(176,5,288)$		37		2-$(276,5,22680)$	FT
16	$M_{22}:2$	2-$(22,5,20)$	FT	38		2-$(276,5,151200)$	
17		2-$(22,5,320)$	FT	39		2-$(276,5,181440)$	FT
18		2-$(22,5,800)$		40		2-$(276,5,453600)$	
19	M_{23}	2-$(23,5,210)$	FT	41		2-$(276,5,653184)$	FT
20		2-$(23,5,1120)$	FT	42		2-$(276,5,907200)$	
21	M_{24}	2-$(24,5,1540)$	FT, CD	43		2-$(276,5,1020600)$	
22	HS	2-$(176,5,40)$	FT				

Lemma 2.1. *The parameters (v,b,r,k,λ) of a 2-design satisfying the following equations:*

$$bk = vr; \tag{1}$$

$$\binom{v}{2}\lambda = b\binom{k}{2}, \tag{2}$$

and so

$$r = \frac{\lambda(v-1)}{(k-1)}. \tag{3}$$

Lemma 2.2. [12, Lemma 2.2] *Let $\mathscr{D}$ be a 2-(v,k,λ) design with a block-transitive automorphism group G. Then for all non-trivial subdegrees d of G, the parameter r satisfies*

$$r \mid k\lambda d.$$

Moreover, $v - 1 \mid k(k-1)d$.

The following lemma concerns the necessary and sufficient conditions for the existence of block-transitive 2-designs which is well known and plays a key role in our proof.

Lemma 2.3. [10, Lemma 2.4] *Let* G *be a permutation group on* $\mathscr{P}$*, having orbits* $O_1, O_2, ..., O_m$ *on the set of* 2*-element subsets of* $\mathscr{P}$*. If* B *is a* k*-subset of* $\mathscr{P}$*, then* $(\mathscr{P}, B^G)$ *is a block-transitive* 2*-design if and only if*

$$\frac{q_1}{|O_1|} = \frac{q_2}{|O_2|} = ... = \frac{q_m}{|O_m|},$$

where q_i *is the number of* 2*-element subsets of* B *which belongs to* O_i*, there* $i = 1, 2, ..., m$.

From Eq. (2) of Lemma 2.1 and Chapter 3 of [1], we have the following corollary:

Corollary 2.4. *Let* G *be a* 2*-transitive permutation group on the set* $\mathscr{P}$*, and let* B *be a* 5*-subset of* $\mathscr{P}$*. Then the structure* $\mathscr{D}$=$(\mathscr{P}, B^G)$ *is a* 2-$(v, 5, \lambda)$ *design with* $\lambda = \frac{20|B^G|}{v(v-1)}$ *admitting a block-transitive automorphism group* G.

Lemma 2.5. [11, Corollary 2.4] *Let* $\mathscr{D}$ *be a* 2-$(v, 5, \lambda)$ *design with a block-transitive automorphism group* G*, then* $rank(G) \leq 21$*. In particular,*

(i) *if* $9 \leqslant rank(G) \leqslant 11$*,* G *has a subdegree* $\frac{v-1}{10}$ *or* $\frac{v-1}{20}$*;*

(ii) *if* $12 \leqslant rank(G) \leqslant 21$*,* G *has a subdegree* $\frac{v-1}{20}$.

Corollary 2.6. *Let* $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ *be a non-trivial* 2-$(v, 5, \lambda)$ *design. Suppose that* $G \leq \mathrm{Aut}(\mathscr{D})$ *acts transitively on the blocks of* $\mathscr{D}$ *and* $\mathrm{Soc}(G)$ is a sporadic simple group. Then $rank(G) \leq 3$.

Proof. Firstly, if $rank(G) > 8$, we can identify all groups from ATLAS[8] and none of which satisfies Lemma 2.5, so we only consider those groups with rank at most 8 which are all listed in [13, Theorem 1.1]. From Lemma 2.2, we know that for each non-trivial subdegree d of G, we have $v - 1 \mid 20d$. Through this equation, we can eliminate most groups with $rank(G) \leq 8$ and list all the possible groups that meet the condition in Table 2. As can be seen from the Table 2, the $rank(G) \leq 3$. □

Remark 2.7. In the final column of Table 2, "TD" abbreviates the transitivity degree of the automorphism group G.

Next we are about to start proving our main theorem.

3. Proof

In this section, we start to prove Theorem 1.3. We always suppose that $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ is a 2-$(v, 5, \lambda)$ design and G is a block-transitive automorphism group of $\mathscr{D}$ with sporadic

Table 2. All possible automorphism groups G

Case	G	Degree	rank(G)	subdegrees	TD
1	M_{11}	11	2	1,10	4
2	M_{11}	12	2	1,11	3
3	M_{12}	12	2	1,11	5
4	M_{22}	22	2	1,21	3
5	$M_{22}:2$	22	2	1,21	3
6	M_{23}	23	2	1,22	4
7	M_{24}	24	2	1,23	5
8	HS	176	2	1,175	2
9	Co_3	276	2	1,275	2
10	M_{22}	176	3	1,70,105	1

socle.

Notice that these groups except M_{22} acting on 176 points are all 2-transitive on $\mathscr{P}$ from Table 2. Hence, from Corollary 2.4, we can determine that $\mathscr{D}$=($\mathscr{P}, B^G$) must be a 2-design, where B is an arbitrary 5-subset of $\mathscr{P}$. Now, we demonstrate our method for determining all possible parameters by analyzing each Case in the Table 2. Before this, we have discovered two complete designs, which are presented in the following proposition.

Proposition 3.1. *If $G \cong M_{12}$ or M_{24}, then $\mathscr{D}$ is a complete 2-$(12,5,120)$ design, or a complete 2-$(24,5,1540)$ design, respectively.*

Proof. Clearly, M_{12} acts 5-transitively on $\mathscr{P}$. By Corollary 2.4, $\mathscr{D}$ is a complete 2-$(12,5,120)$ design admitting M_{12} as its block-transitive automorphism group. Similarly, if $G \cong M_{24}$, then $\mathscr{D}$ is a complete 2-$(24,5,1540)$ design admitting M_{24} as its block-transitive automorphism group. □

For the Cases 1, 2, 4, 5, 6 in Table 2, as their approaches to proof are identical, we classify them into the same category and present the following proposition.

Proposition 3.2. (1) *If $G \cong M_{11}$, then $\mathscr{D}$ is a 2-$(11,5,\lambda)$ design with $\lambda \in \{12,72\}$, or a 2-$(12,5,\lambda)$ design with $\lambda \in \{20,100\}$.*

(2) *If $G \cong M_{22}$, then $\mathscr{D}$ is a 2-$(22,5,\lambda)$ design with $\lambda \in \{20,160,800\}$.*

(3) *If $G \cong M_{22}:2$, then $\mathscr{D}$ is a 2-$(22,5,\lambda)$ design with $\lambda \in \{20,320,800\}$.*

(4) *If $G \cong M_{23}$, then $\mathscr{D}$ is a 2-$(23,5,\lambda)$ design with $\lambda \in \{210,1120\}$.*

Proof. Let $G \cong M_{11}$ acts transitively on $\mathscr{P}$ with $|\mathscr{P}|= 11$. There exists 2 orbits of G acting on all 5-element subsets of $\mathscr{P}$, namely O_1, O_2, and $|O_1|= 66$, $|O_2|= 396$. From

Corollary 2.4, we can get $\mathscr{D} = (\mathscr{P}, O_1)$ is a 2-$(11, 5, 12)$ design, or $\mathscr{D} = (\mathscr{P}, O_2)$ is a 2-$(11, 5, 72)$ design. Let $G \cong M_{11}$ acts transitively on $\mathscr{P}$ with $|\mathscr{P}| = 12$. Also, G has 2 orbits on all 5-element subsets of $\mathscr{P}$ with size 132, 660. Thus, $\mathscr{D}$ is a 2-$(12, 5, 20)$ design, or a 2-$(12, 5, 100)$ design.

Let $G \cong M_{22}$ acts transitively on $\mathscr{P}$ with $|\mathscr{P}| = 22$. There exists 4 orbits of G acting on all 5-element subsets of $\mathscr{P}$, namely O_i where $i \in \{1, 2, 3, 4\}$ and $|O_1| = 462$, $|O_2| = 3696$, $|O_3| = 3696$ and $|O_4| = 18480$. By the same way, we can easily obtain $\mathscr{D} = (\mathscr{P}, O_i)$ is a 2-$(22, 5, 20)$ design, a 2-$(22, 5, 160)$ design, or a 2-$(22, 5, 800)$ design. Particularly, by using Magma[3]-command `IsIsomorphic(D2,D3)`, we found that $\mathscr{D}_2$ is isomorphic to $\mathscr{D}_3$. So $\mathscr{D}$ is a 2-$(22, 5, \lambda)$ design with $\lambda \in \{20, 160, 800\}$.

Let $G \cong M_{22} : 2$ acts transitively on $\mathscr{P}$ with $|\mathscr{P}| = 22$. Similarly, we can obtain $\mathscr{D}$ is a 2-$(22, 5, 20)$ design, a 2-$(22, 5, 320)$ design, or a 2-$(22, 5, 800)$ design. Note that the designs 2-$(22, 5, 20)$ and 2-$(22, 5, 800)$ with $G \cong M_{22}$ are isomorphic to that with $M_{22} : 2$. For the others groups, the designs with the same parameters are unique up to isomorphism.

Let $G \cong M_{23}$ acts transitively on $\mathscr{P}$ with $|\mathscr{P}| = 23$. By the same way, we can get $\mathscr{D}$ is a 2-$(23, 5, 210)$ design, or a 2-$(23, 5, 1120)$ design. □

Proposition 3.3. (1) *If $G \cong HS$, then $\mathscr{D}$ is a 2-$(176, 5, \lambda)$ design with*

$$\lambda \in \{40, 144, 240, 360, 900, 1440, 1800, 2400, 3600, 4800, 5760, 7200, 14400, 28800\}.$$

(2) *If $G \cong Co_3$, then $\mathscr{D}$ is a 2-$(276, 5, \lambda)$ design with*

$$\lambda \in \{1120, 22680, 151200, 181440, 453600, 653184, 907200, 1020600\}.$$

Proof. In these Cases, we can not calculate the orbits of G as we do in Proposition 3.2 because the cardinality of set is too large. But we can use the following method to determine all designs $\mathscr{D}$ such that G acts block-transitively on $\mathscr{D}$.

Firstly, note that $G \cong HS$ acts 2-transitively on $\mathscr{P}$ with $|\mathscr{P}| = 176$. Each 5-subsets containing points 1 and 2 is a base block. Hence, we can assume $B = \{1, 2, i, j, k\}$, where $i, j, k \in \{3, 4, \ldots, 176\}$, which are distinct. For each $\{i, j, k\}$, by using Magma, we can easily obtain

$$|G_B| \in \{1, 2, 4, 5, 6, 8, 12, 16, 20, 32, 80, 120, 200, 720\}.$$

By Eq. (2) and $b = \frac{|G|}{|G_B|}$, we can get $\mathscr{D} = (\mathscr{P}, B^G)$ is a 2-$(176, 5, \lambda)$ design with

$$\lambda \in \{40, 144, 240, 360, 900, 1440, 1800, 2400, 3600, 4800, 5760, 7200, 14400, 28800\}.$$

Especially, if $|G_B| = 1$, there are 10 2-$(176, 5, 28800)$ designs. Due to the high computational effort, it is difficult to verify whether the designs with the same parameters are isomorphic. Note that $G \cong Co_3$ acts 2-transitively on $\mathscr{P}$ with $|\mathscr{P}| = 276$. By the same way, we obtain $|G_B| \in \{128, 144, 200, 288, 720, 864, 5760, 116640\}$ and we can get $\mathscr{D} = (\mathscr{P}, B^G)$ is a 2-$(276, 5, \lambda)$ design with $\lambda \in \{1120, 22680, 151200, 181440, 453600, 653184, 907200, 1020600\}$. Particularly, let $b_i = \frac{|G|}{|G_B|}$ where $i \in \{1, 2, \ldots, 8\}$, it is easy to verify that $\sum_{i=1}^{8} b_i = \binom{v}{5}$, which exactly indicates that all the designs obtained above are unique up to isomorphism. □

Proposition 3.4. *If $G \cong M_{22}$, then $\mathscr{D}$ is a* 2-$(176, 5, \lambda)$ *design with*

$$\lambda \in \{4, 12, 36, 72, 96, 144, 288\}.$$

Proof. Let $G \cong M_{22}$ acting on 176 points. Clearly, $rank(G) = 3$ and its subdegrees are 1, 70, 105. We assume $B = \{1, 2, i, j, k\}$, where $i, j, k \in \{3, 4, \ldots, 176\}$, which are distinct. Then G has 2 orbits O_1, O_2 on all 2-subsets of $\mathscr{P}$ and $|O_1| = 6160$, $|O_2| = 9240$. From Lemma 2.3, we know that if $(\mathscr{P}, B^G)$ is a 2-design, it must satisfy $\frac{q_1}{q_2} = \frac{|O_1|}{|O_2|} = \frac{2}{3}$ (where q_i is the number of 2-element subsets of B which belongs to O_i, there $i = 1, 2$) and $q_1 + q_2 = 10$. It means that there are 4 pairs 2-element subsets of B belong to O_1 and 6 pairs 2-element subsets of B belong to O_2. By using Magma, we obtain $|G_B| \in \{1, 2, 3, 4, 8, 24, 72\}$ for all base blocks B. From Equation (2) and $b = \frac{|G|}{|G_B|}$, we have that $\mathscr{D}$ is a 2-$(176, 5, \lambda)$ design with $\lambda \in \{4, 12, 36, 72, 96, 144, 288\}$. Also, if $|G_B| = 1$, there are 672 2-$(176, 5, 288)$ designs. Due to the high computational effort, it is difficult to verify whether the designs with the same parameters are isomorphic. □

Proposition 3.1-3.4 complete the proof of Theorem 1.3.

According to the above main Corollary 2.6 and Theorem 1 in [11], we state an important corollary.

Corollary 3.5. *Let $\mathscr{D} = (\mathscr{P}, \mathscr{B})$ be a non-trivial* 2-$(v, 5, \lambda)$ *design. Suppose that $G \leq$* Aut$(\mathscr{D})$ *acts transitively on the blocks of $\mathscr{D}$ and* Soc(G) is either an alternating simple group or a sporadic simple group. Then $rank(G) \leq 3$.

Lastly, we found that the classification work for block-transitive 2-$(v, 5, \lambda)$ designs is still not complete on affine group and almost simple group of Lie type. So we have the following problem, which needs further research in the future.

Problem. Can we classify a 2-$(v, 5, \lambda)$ design admitting a block-transitive point-primitive automorphism group whose socle is an affine group or a simple group of Lie type?

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