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# Counting of lattices containing up to four comparable reducible elements and having nullity up to three

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ABSTRACT

In 1940, Birkhoff raised the open problem of computing of all posets/lattices on $n$ elements up to isomorphism for small $n$. Many authors tried to solve this problem by providing algorithms such as nauty. In 2020, Gebhardt and Tawn given an orderly algorithm for constructing unlabelled lattices of given size and explicitly obtained the number of lattices on up to 20 elements. In 2020, Bhavale and Waphare introduced the concept of nullity of a poset as the nullity of its cover graph. Recently, Bhavale and Aware counted lattices having nullity up to two. Bhavale and Aware also counted all non-isomorphic lattices on $n$ elements, containing up to three reducible elements, having arbitrary nullity $k \geq 2$. In this paper, we count up to isomorphism the class of all lattices on $n$ elements containing four comparable reducible elements, and having nullity three.

*Keywords:* poset, lattice, counting, nullity

## 1. Introduction

In 1940, Birkhoff [5] raised the open problem of computing for small $n$ all posets/lattices on a set of $n$ elements up to isomorphism. Many authors from all over the world tried to solve this problem. In 2002, Brinkmann and McKay [6] obtained the number of posets on up to 16 elements. The problem of counting of all posets on $n$ elements up to isomorphism is still open for $n \geq 17$. In 2002, Heitzig and Reinhold [9] carried out the enumeration of all non-isomorphic lattices on up to 18 elements using algorithmic approach. In 2015, Jipsen and Lawless [10] adapted and improved the algorithm of Heitzig and Reinhold [9],

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and calculated the number of lattices on up to 19 elements. In 2020, Gebhardt and Tawn [7] gave an improved orderly algorithm for constructing unlabelled lattices of given size. Further, Gebhardt and Tawn [7] obtained the number of lattices on up to 20 elements. The problem of counting of lattices on $n$ elements up to isomorphism is still open for $n \geq 21$.

In 2003, Pawar and Waphare [12] enumerated all non-isomorphic lattices with $n$ elements, containing $n$ edges, which are precisely lattices of nullity one. In 2002, Thakare et al. [11] enumerated all non-isomorphic lattices on $n$ elements, containing exactly two reducible elements, and having arbitrary nullity $k \geq 1$. Thakare et al. [11] also counted all non-isomorphic lattices on $n$ elements and up to $n + 1$ edges, which are precisely the lattices of nullity up to two. Independently, Bhavale and Aware [2] counted all non-isomorphic lattices on $n$ elements having nullity up to two. Recently, Bhavale and Aware [3] counted all non-isomorphic lattices on $n$ elements, containing up to three reducible elements, and having arbitrary nullity $k \geq 2$. According to Bhavale and Waphare [4], if a dismantlable lattice of nullity $k$ contains $r$ reducible elements then $2 \leq r \leq 2k$. So the problem of counting of all non-isomorphic lattices on $n$ elements, containing $r \geq 4$ reducible elements, and having nullity $k \geq 3$ is still open. Note that the lattices containing either up to three reducible elements, or four reducible elements and having nullity up to two are precisely the lattices in which all the reducible elements are lying on a chain. However there exists a lattice containing four reducible elements not lying on a chain and having nullity three, for example, the lattice M (see Figure 1). Now in this paper, we will restrict ourselves to the counting of all non-isomorphic lattices having nullity up to three in which all the four reducible elements are lying on a chain.

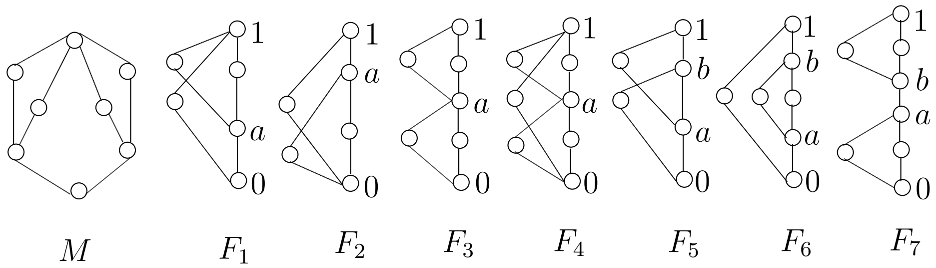


**Fig. 1.**

An element $y$ in $P$ *covers* an element $x$ in $P$ if $x < y$, and there is no element $z$ in $P$ such that $x < z < y$. Denote this fact by $x \prec y$, and say that pair $< x, y >$ is a *covering* or an *edge*. If $x \prec y$ then $x$ is called a *lower cover* of $y$, and $y$ is called an *upper cover* of $x$. If $x$ is not covered by $y$ then we denote it as $x \not\prec y$. The graph on a poset $P$ with edges as covering relations is called the *cover graph* and is denoted by $C(P)$. The number of coverings in a chain is called *length* of the chain. If a chain contains $n$ elements then we call it as $n$-chain. A *meet*(*join*) of elements $x, y$ in a poset $P$, denoted by $\wedge(\vee)$, is defined as the greatest lower bound(least upper bound) of $x$ and $y$ in $P$. An element of a poset $P$ is called *doubly irreducible* if it has at most one lower cover and at most one upper cover in $P$. Let $Irr(P)$ denote the set of all doubly irreducible elements in a poset

$P$. Let $Red(P) = P \setminus Irr(P)$ denote the set of all reducible elements in $P$. A poset $L$ is a lattice if $x \wedge y$ and $x \vee y$ exist in $L, \forall x, y \in L$. An element $x$ in a lattice $L$ is *meet-reducible*(*join-reducible*) in $L$ if there exist $y, z \in L$ both distinct from $x$, such that $y \wedge z = x(y \vee z = x)$. An element $x$ in a lattice $L$ is *reducible* if it is either meet-reducible or join-reducible. $x$ is *meet-irreducible*(*join-irreducible*) if it is not meet-reducible(join-reducible); $x$ is *doubly irreducible* if it is both meet-irreducible and join-irreducible. The set of all doubly irreducible elements in $L$ is denoted by $Irr(L)$, and its complement in $L$ is denoted by $Red(L)$. Bhavale and Waphare [4] introduced the concept of nullity of a poset $P$, denoted by $\eta(P)$, as nullity of its cover graph. Thus $\eta(P) = p - q + c$, where $p$ is the number of edges, $q$ is the number of vertices, and $c$ is the number of connected components of the cover graph $C(P)$.

**Definition 1.1.** [13] A finite lattice of order $n$ is called *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \ldots \subset L_n(= L)$ of sublattices of $L$ such that $|L_i|= i$, for all $i$.

The concept of *adjunct operation of lattices*, is introduced by Thakare et al. [11]. Suppose $L_1$ and $L_2$ are two disjoint lattices and $(p, q)$ is a pair of elements in $L_1$ such that $p < q$ and $p \not\prec q$. Define the partial order $\leq$ on $L = L_1 \cup L_2$ with respect to the pair $(p, q)$ as: $a \leq b$ in L if $a, b \in L_1$ and $a \leq b$ in $L_1$, or $a, b \in L_2$ and $a \leq b$ in $L_2$, or $a \in L_1$, $b \in L_2$ and $a \leq p$ in $L_1$, or $a \in L_2$, $b \in L_1$ and $q \leq b$ in $L_1$. It is denoted as $L = L_1]^q_p L_2$ and we say that $L$ is an *adjunct* of $L_1$ and $L_2$. The procedure of obtaining $L$ in this way is called *adjunct operation* of $L_1$ with $L_2$. The pair $(p, q)$ is called as an *adjunct pair*. A diagram of $L$ is obtained using diagram of $L_1$ and a diagram of $L_2$ by placing $L_1$, $L_2$ side by side such that the least element 0 of $L_2$ is at the higher position than $p$ and the largest element 1 of $L_2$ is at lower position than $q$ and then by adding the coverings $< p, 0 >$ and $< 1, q >$.

A lattice $L$ is called an *adjunct of lattices* $L_0, L_1, \ldots, L_t$, if it is of the form $L = L_0]^{q_1}_{p_1} L_1]^{q_2}_{p_2} L_2 \cdots]^{q_t}_{p_t} L_t$, where for each $i, 1 \leq i \leq t, (p_i, q_i)$ is an adjunct pair.

**Theorem 1.2.** [11] *A finite lattice is dismantlable if and only if it is an adjunct of chains.*

**Corollary 1.3.** [4] *A dismantlable lattice $L$ containing $n$ elements is of nullity $k$ if and only if $L$ is an adjunct of $k + 1$ chains.*

Let $L$ be a lattice with $n$ elements, $e$ edges, and having nullity $k$. By definition of nullity of a lattice we have $k = e - n + 1$, that is, $e = n + k - 1$. Thus we have the following result.

**Proposition 1.4.** *A dismantlable lattice with $n$ elements is of nullity $k$ if and only if it has $n + k - 1$ edges.*

If $P$ and $P'$ are two disjoint posets, the *direct sum* (*see* [14]), denoted by $P \oplus P'$, is defined by taking the order relation $\leq$ on $P \cup P'$ as: $x \leq y$ if and only if $x, y \in P$ and $x \leq y$ in $P$, or $x, y \in P'$ and $x \leq y$ in $P'$, or $x \in P, y \in P'$.

**Lemma 1.5.** *Let $\mathscr{L}_1(p)$ and $\mathscr{L}_2(q)$ be some classes of non-isomorphic lattices on $p \geq r \geq 1$ and $q \geq s \geq 1$ elements respectively. If $L_1 \in \mathscr{L}_1(p)$ and $L_2 \in \mathscr{L}_2(q)$ are such that $L_1 \oplus L_2 = L \in \mathscr{L}(n)$, a class of non-isomorphic lattices on $n = p + q$ elements then* $|\mathscr{L}(n)| = \sum_{p=r}^{n-s}(|\mathscr{L}_1(p)| \times |\mathscr{L}_2(n-p)|)$.

**Proof.** As $n = p + q$, for fixed $p$, there are $|\mathscr{L}_1(p)|$ non-isomorphic lattices on $p$ elements and $|\mathscr{L}_2(n-p)|$ non-isomorphic lattices on $n-p$ elements. Therefore by multiplication principle, for fixed $p$, there are up to isomorphism $|\mathscr{L}_1(p)| \times |\mathscr{L}_2(n-p)|$ lattices in $\mathscr{L}(n)$. Further, $r \leq p = n - q \leq n - s$, since $q \geq s$. Thus there are $\sum_{p=r}^{n-s}(|\mathscr{L}_1(p)| \times |\mathscr{L}_2(n-p)|)$ lattices in $\mathscr{L}(n)$ up to isomorphism. □

The following definition is due to Bhamre and Pawar [1].

**Definition 1.6.** [1] Let $P_1$ be a poset with the largest element and $P_2$ be a poset with the least element such that the greatest element of $P_1$ and the least element of $P_2$ are the same (say $\alpha$) and $P_1 \cap P_2 = \{\alpha\}$, then the vertical sum of $P_1$ with $P_2$, denoted by $P_1 \circ P_2$, is a poset $(P_1 \cup P_2, \leq)$ where $x \leq y$ if and only if $x, y \in P_1$ and $x \leq y$ in $P_1$, or $x, y \in P_2$ and $x \leq y$ in $P_2$, or $x$ in $P_1$ and $y$ in $P_2$.

**Lemma 1.7.** *Let $\mathscr{L}_1(p)$ and $\mathscr{L}_2(q)$ be some classes of non-isomorphic lattices on $p \geq r \geq 1$ and $q \geq s \geq 1$ elements respectively. If $L_1 \in \mathscr{L}_1(p)$ and $L_2 \in \mathscr{L}_2(q)$ are such that $L_1 \circ L_2 = L \in \mathscr{L}(n)$, a class of non-isomorphic lattices on $n = p + q - 1$ elements then* $|\mathscr{L}(n)| = \sum_{p=r}^{n-s+1}(|\mathscr{L}_1(p)| \times |\mathscr{L}_2(n-p+1)|)$.

**Proof.** As $n = p+q-1$, for fixed $p$, there are $|\mathscr{L}_1(p)|$ non-isomorphic lattices. For $m \in \mathbb{N}$, let $\mathscr{P}_2(m) = \{L \backslash \{0\} \mid L \in \mathscr{L}_2(m+1)\}$. Observe that there is a one to one correspondence between the class $\mathscr{P}_2(m)$ and the class $\mathscr{L}_2(m+1)$, and hence $|\mathscr{L}_2(m+1)| = |\mathscr{P}_2(m)|$. Note that for fixed $p$, there are $|\mathscr{P}(n-p)|$ non-isomorphic posets on $n-p$ elements, and hence there are $|\mathscr{L}_2(n-p+1)|$ non-isomorphic lattices on $n-p+1$ elements. Therefore by multiplication principle, for fixed $p$, there are $|\mathscr{L}_1(p)| \times |\mathscr{L}_2(n-p+1)|$ lattices up to isomorphism in $\mathscr{L}(n)$. Further, $1 \leq r \leq p = n - q + 1 \leq n - s + 1$, since $q \geq s \geq 1$. Thus there are $\sum_{p=r}^{n-s+1}(|\mathscr{L}_1(p)| \times |\mathscr{L}_2(n-p+1)|)$ lattices in $\mathscr{L}(n)$ up to isomorphism. □

Thakare et al. [11] defined a *block* as a finite lattice in which the largest element 1 is join-reducible and the smallest element 0 is meet-reducible. Moreover, if $L$ is a lattice different from a chain, then $L$ contains a unique maximal sublattice which is a block called as *maximal block*. The lattice $L$ is of the form $C \oplus \mathbf{B}$ or $\mathbf{B} \oplus C$ or $C \oplus \mathbf{B} \oplus C'$, where $C, C'$ are chains and $\mathbf{B}$ is the maximal block. Further $\eta(L) = \eta(\mathbf{B})$.

Bhavale and Waphare [4] introduced the following concepts namely, retractible element, basic retract, basic block, basic block associated to a poset, and fundamental basic block.

**Definition 1.8.** [4] Let $P$ be a poset. Let $x \in Irr(P)$. Then $x$ is called a *retractible* element of $P$ if it satisfies either of the following conditions.

(a) There are no $y, z \in Red(P)$ such that $y \prec x \prec z$.

(b) There are $y, z \in Red(P)$ such that $y \prec x \prec z$ and there is no other directed path from $y$ to $z$ in $P$.

**Definition 1.9.** [4] A poset $P$ is a *basic retract* if no doubly-irreducible element of $P$ having exactly one upper cover and exactly one lower cover is retractible in the poset $P$.

**Definition 1.10.** [4] A poset $P$ is a *basic block* if it is one element or $Irr(P) = \emptyset$ or removal of any doubly irreducible element reduces nullity by one.

**Definition 1.11.** [4] $B$ is a *basic block associated to a poset* $P$ if $B$ is obtained from the basic retract associated to $P$ by successive removal of all the pendant vertices of $C(P)$.

**Theorem 1.12.** [4] *Let $B$ be the basic block associated to a poset $P$. Then $Red(B) = Red(P)$ and $\eta(B) = \eta(P)$.*

**Definition 1.13.** [4] A dismantlable lattice $B$ is said to be a *fundamental basic block* if it is a basic block and all the adjunct pairs in the adjunct representation of $B$ into chains are distinct.

Let $\mathscr{L}(n; r, k)$ be the class of all non-isomorphic lattices on $n$ elements such that every member of it contains $r$ comparable reducible elements, and has nullity $k$. Let $\mathscr{L}(n; r, k, h)$ be the subclass of $\mathscr{L}(n; r, k)$ such that the basic block associated to a member of it is of height $h$. Let $\mathscr{B}(j; r, k)$ be the class of all non-isomorphic maximal blocks on $j$ elements such that every member of it contains $r$ comparable reducible elements, and has nullity $k$. Let $\mathscr{B}(j; r, k, h)$ be the subclass of $\mathscr{B}(j; r, k)$ such that the basic block associated to a member of it is of height $h$. Let $B^*$ denote the dual of the basic block $B$. Let $P_n^k$ denote the number of partitions of $n$ into $k$ non-decreasing positive integer parts.

In the following section, we count all non-isomorphic lattices on $n$ elements, containing four comparable reducible elements and having nullity three. For that purpose of counting, we use the following results due to Bhavale and Aware [2], [3].

**Lemma 1.14.** [3] *For the integers $m \geq 4$ and $1 \leq k \leq m-3$, $|\mathscr{B}(m; 2, k)| = P_{m-2}^{k+1}$.*

**Theorem 1.15.** ([11],[3]) *For $n \geq 4$ and for $1 \leq k \leq n-3$, $|\mathscr{L}(n; 2, k)| = \sum_{j=1}^{n-k-2} j P_{n-j-1}^{k+1}$.*

For $i = 1, 2, 3, 4$, let $\mathscr{B}_i(m; 3, k)$ denote the subclass of $\mathscr{B}(m; 3, k)$ such that $F_i$(See Figure 1) is the basic block associated to $\mathbf{B} \in \mathscr{B}_i(m; 3, k)$.

**Proposition 1.16.** [3] *For an integer* $m \geq 6$ *and for* $2 \leq k \leq m-4$,

$$|\mathscr{B}_1(m;3,k)| = |\mathscr{B}_2(m;3,k)| = \sum_{l=1}^{m-5}\sum_{i=1}^{m-l-4} P^k_{m-l-i-2} + \sum_{r=5}^{m-2}\sum_{s=1}^{k-2}\sum_{i=1}^{r-4} P^{s+1}_{r-i-2} P^{k-s}_{m-r}.$$

**Proposition 1.17.** [3] *For an integer* $m \geq 7$ *and for* $2 \leq k \leq m-5$,

$$|\mathscr{B}_3(m;3,k)| = \sum_{l=4}^{m-3}\sum_{t=1}^{k-1} P^{t+1}_{l-2} P^{k-t+1}_{m-l-1}.$$

**Proposition 1.18.** [2] *For* $j \geq 6$,

$$|\mathscr{B}(j;4,2,3)| = \binom{j-2}{4}.$$

**Proposition 1.19.** [2]*For* $j \geq 8$,

$$|\mathscr{B}(j;4,2,5)| = \sum_{m=0}^{j-8}\sum_{s=4}^{j-m-4} P^2_{s-2} P^2_{j-m-s-2}.$$

For the other preliminaries, definitions, notation, and terminology, reader may refer [3], [8] and [16].

## 2. Counting of lattices containing four comparable reducible elements and having nullity three

In this section, we count all non-isomorphic lattices on $n$ elements, containing four comparable reducible elements and having nullity three.

Bhavale and Waphare [4] obtained the formulae of counting of all non-isomorphic fundamental basic blocks and basic blocks containing $r$ comparable reducible elements and having nullity $k$. Let $\mathscr{F}_r(k)$ be the class of all non-isomorphic fundamental basic blocks such that each member in it contains $r$ comparable reducible elements, and has nullity $k$. Note that, $|\mathscr{F}_0(0)| = |\mathscr{F}_2(1)| = 1, |\mathscr{F}_3(2)| = 3$ (see Figure 1, $F_1, F_2, F_3$).

**Proposition 2.1.** [4] *For fixed* $r \geq 1$ *and for* $\lfloor \frac{r+2}{2} \rfloor \leq k \leq \binom{r+1}{2}$,

$$|\mathscr{F}_{r+1}(k)| = \sum_{l=1}^{r}\sum_{j=0}^{l}\binom{r}{j}\binom{r-j}{l-j}|\mathscr{F}_{r-j}(k-l)|.$$

Let $\mathscr{B}_r(k)$ be the class of all non-isomorphic basic blocks such that each member in it contains $r$ comparable reducible elements, and has nullity $k$.

**Proposition 2.2.** [4] *For fixed* $r \geq 2$ *and for* $\lfloor \frac{r+1}{2} \rfloor \leq m \leq k \leq \binom{r}{2}$,

$$|\mathscr{B}_r(k)| = \sum_{m=\lfloor\frac{r+1}{2}\rfloor}^{k} \binom{k-1}{m-1} |\mathscr{F}_r(m)|.$$

Using Proposition 2.1, $|\mathscr{F}_4(2)| = 3$, that is, there are exactly three fundamental basic blocks namely, $F_5, F_6$ and $F_7$ (see Figure 1) containing four comparable reducible elements and having nullity two. Using these three fundamental basic blocks, we obtain 6 basic blocks containing four comparable reducible elements and having nullity three namely, $B_1, B_2, B_4, B_5, B_{13}$, and $B_{14}$ (see Figure 2) up to isomorphism by just adding a 1-chain to $F_5, F_6$, and $F_7$. Also using Proposition 2.1, $|\mathscr{F}_4(3)| = 16$, that is, there are exactly sixteen fundamental basic blocks (which are also basic blocks) namely, $B_3$, $B_6$ to $B_{12}$, and $B_{15}$ to $B_{22}$ (see Figure 2) containing four comparable reducible elements and having nullity three. Therefore using Proposition 2.2, $|\mathscr{B}_4(3)| = 6 + 16 = 22$. That is, there are exactly twenty two basic blocks namely, $B_1$ to $B_{22}$ (see Figure 2) containing four comparable reducible elements and having nullity three up to isomorphism. In fact we have the following result.

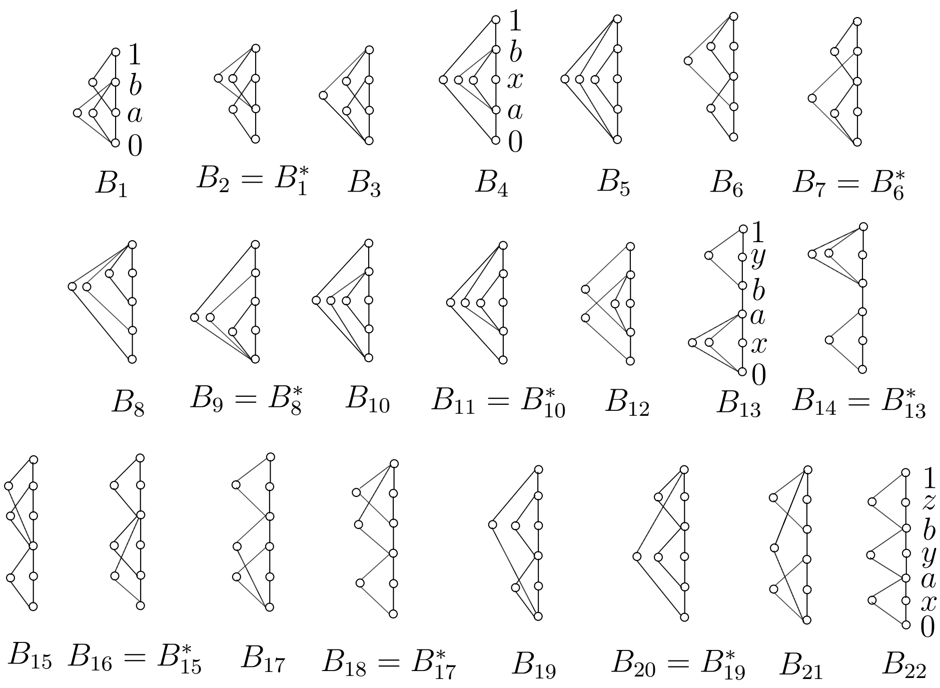


**Fig. 2.** Basic blocks on 4 comparable reducible elements and having nullity 3

**Proposition 2.3.** *If $B$ is the basic block associated to $\boldsymbol{B} \in \mathscr{B}(j;4,3)$ where $j \geq 7$ then $B \in \{B_1, B_2, B_3, \ldots, B_{22}\}$.*

Now out of 22 two basic blocks there are 3 of height three namely, $B_1, B_2, B_3$, 9 of height four namely, $B_4$ to $B_{12}$, 11 of height five namely, $B_{13}$ to $B_{21}$, and $B_{22}$ is of height six. Also, for $n \geq 7$, if $L \in \mathscr{L}(n;4,3,h)$ then $3 \leq h \leq 6$. Now in this section, firstly we count the classes $\mathscr{B}(j;4,3,h)$ for $3 \leq h \leq 6$. Secondly, we count the classes $\mathscr{L}(n;4,3,h)$

for $3 \leq h \leq 6$, and thereby we count the class $\mathscr{L}(n;4,3)$. For $1 \leq i \leq 22$, let $\mathbb{B}_i(j;4,3)$ be the subclass of $\mathscr{B}(j;4,3)$ containing all maximal blocks of type $\mathbf{B} \in \mathscr{B}(j;4,3)$ such that $B_i$ is the basic block associated to $\mathbf{B}$.

**Remark 2.4.** (a) For $j \geq 7, \mathscr{B}(j;4,3,3) = \dot{\cup}_{i=1}^{3}\mathbb{B}_i(j;4,3)$.
(b) For $j \geq 8, \mathscr{B}(j;4,3,4) = \dot{\cup}_{i=4}^{12}\mathbb{B}_i(j;4,3)$.
(c) For $j \geq 9, \mathscr{B}(j;4,3,5) = \dot{\cup}_{i=13}^{21}\mathbb{B}_i(j;4,3)$.
(d) For $j \geq 10, \mathscr{B}(j;4,3,6) = \mathbb{B}_{22}(j;4,3)$.
(e) For $j \geq 7, \mathscr{B}(j;4,3) = \dot{\cup}_{h=3}^{6}\mathscr{B}(j;4,3,h)$.
(f) For $n \geq 7, \mathscr{L}(n;4,3) = \dot{\cup}_{h=3}^{6}\mathscr{L}(n;4,3,h)$.

### 2.1. *Counting of the class $\mathscr{B}(j;4,3,3)$*

Now we count the the class $\mathscr{B}(j;4,3,3)$ by counting the classes $\mathbb{B}_i(j;4,3)$ for $i = 1$ to 3. For $x < y$ in a poset $P$, the interval $[x,y) = \{a \in P : x \leq a < y\}$, and $(x,y) = \{a \in P : x < a < y\}$; similarly, $(x,y]$ and $[x,y]$ can also be defined.

**Proposition 2.5.** *For $j \geq 7$,* $|\mathbb{B}_1(j;4,3)| = \sum_{s=1}^{j-6}\sum_{r=1}^{j-s-5}\sum_{l=2}^{j-s-r-3}(j-s-r-l-2)P_l^2.$

**Proof.** Let $\mathbf{B} \in \mathbb{B}_1(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_1$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_1) = Red(\mathbf{B})$ and $\eta(B_1) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_1$ is given by $B_1 = C]_a^1\{c_1\}]_0^b\{c_2\}]_0^b\{c_3\}$, where $C: 0 \prec a \prec b \prec 1$ is a 4-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_a^1C_1]_0^bC_2]_0^bC_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains. Let $s = |[0,a) \cap C_0| \geq 1$, $t = |[a,1] \cap C_0| \geq 3$, $r = |C_1| \geq 1$, $|C_2| = l_1, |C_3| = l_2$ with $1 \leq l_1 \leq l_2$. Let $l = l_1 + l_2$.

Now for fixed $s, r, l$, there are $t-2 = j-s-r-l-2$ choices for $b$ in $\mathbf{B}$ up to isomorphism. Also note that $l$ elements can be distributed into the chains $C_2$ and $C_3$ in $P_l^2$ ways up to isomorphism. That is, for fixed $s, r, l$, there are $(t-2) \times P_l^2 = (j-s-r-l-2) \times P_l^2$ non-isomorphic maximal blocks in $\mathbb{B}_1(j;4,3)$ up to isomorphism. Now for fixed $s, r$, $2 \leq l = j-s-t-r \leq j-s-r-3$, since $t \geq 3$. Therefore there are $\sum_{l=2}^{j-s-r-3}(j-s-r-l-2) \times P_l^2$ maximal blocks in $\mathbb{B}_1(j;4,3)$ up to isomorphism. Again for fixed $s$, $1 \leq r = j-s-t-l \leq j-s-5$, since $t \geq 3$, $l \geq 2$, and there are $\sum_{r=1}^{j-s-5}\sum_{l=2}^{j-s-r-3}(j-s-r-l-2)P_l^2$ maximal blocks in $\mathbb{B}_1(j;4,3)$ up to isomorphism. Further $1 \leq s = j-t-r-l \leq j-6$, since $t \geq 3$, $r \geq 1$, $l \geq 2$, and there are $\sum_{s=1}^{j-6}\sum_{r=1}^{j-s-5}\sum_{l=2}^{j-s-r-3}(j-s-r-l-2)P_l^2$ maximal blocks in $\mathbb{B}_1(j;4,3)$ up to isomorphism. □

Note that $\mathbf{B} \in \mathbb{B}_2(j;4,3)$ if and only if $\mathbf{B^*} \in \mathbb{B}_1(j;4,3)$. Therefore by Proposition 2.5, we have the following result.

**Corollary 2.6.** *For $j \geq 7$,* $|\mathbb{B}_2(j;4,3)| = \sum_{s=1}^{j-6}\sum_{r=1}^{j-s-5}\sum_{l=2}^{j-s-r-3}(j-s-r-l-2)P_l^2$.

**Proposition 2.7.** *For $j \geq 7$,* $|\mathbb{B}_3(j;4,3)| = \sum_{p=1}^{j-6}\binom{j-p-2}{4}$.

**Proof.** Let $\mathbf{B} \in \mathbb{B}_3(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_3$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_3) = Red(\mathbf{B})$ and $\eta(B_3) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_3$ is given by $B_3 = C]_0^b\{c_1\}]_a^1\{c_2\}]_0^1\{c_3\}$, where $C : 0 \prec a \prec b \prec 1$ is a 4-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_0^bC_1]_a^1C_2]_0^1C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = \mathbf{B}']_0^1C_3$, where $\mathbf{B}' = C_0]_0^bC_1]_a^1C_2 \in \mathscr{B}(i;4,2,3)$ for $i \geq 6$ and $|C_3| = p \geq 1$ with $j = i + p \geq 7$. Suppose $\mathbf{D} = \mathbf{D}']_0^1C_3'$, where $\mathbf{D}' = C_0']_0^bC_1']_a^1C_2' \in \mathscr{B}(i;4,2,3)$ for $i \geq 6$ and $|C_3'| = p \geq 1$ with $j = i + p \geq 7$. Then we claim that $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$ and $C_3 \cong C_3'$. To prove this, suppose $\mathbf{B} \cong \mathbf{D}$. As $|C_3| = |C_3'| = p$, $C_3 \cong C_3'$, and hence $\mathbf{B} \setminus C_3 \cong \mathbf{D} \setminus C_3'$, *that is*, $\mathbf{B}' \cong \mathbf{D}'$. The converse is obvious.

Now for fixed $p$, there are $|\mathscr{B}(j-p;4,2,3)|$ maximal blocks in $\mathbb{B}_3(j;4,3)$ up to isomorphism. Further $1 \leq p = j-i \leq j-6$, since $i \geq 6$. Therefore there are $\sum_{p=1}^{j-6}|\mathscr{B}(j-p;4,2,3)|$ maximal blocks in $\mathbb{B}_3(j;4,3)$ up to isomorphism. By Proposition 1.18, $|\mathscr{B}(i;4,2,3)| = \binom{i-2}{4}$. Therefore there are $\sum_{p=1}^{j-6}\binom{j-p-2}{4}$ maximal blocks in $\mathbb{B}_3(j;4,3)$ up to isomorphism. □

Using Proposition 2.5, Corollary 2.6, and Proposition 2.7, we have the following result.

**Theorem 2.8.** *For $j \geq 7$,*

$$|\mathscr{B}(j;4,3,3)| = \sum_{i=1}^{3}|\mathbb{B}_i(j;4,3)| = \sum_{s=1}^{j-6}\sum_{r=1}^{j-s-5}\sum_{l=2}^{j-s-r-3}2(j-s-r-l-2)P_l^2 + \sum_{p=1}^{j-6}\binom{j-p-2}{4}.$$

### *2.2. Counting of the class $\mathscr{B}(j;4,3,4)$*

Here we count the classes $\mathbb{B}_i(j;4,3)$ for $i = 4$ to $12$; consequently, we count the class $\mathscr{B}(j;4,3,4)$. For that sake, let us define $\mathscr{L}'(n;2,2)$ as the subclass of $\mathscr{L}(n;2,2)$ such that any $L \in \mathscr{L}'(n;2,2)$ is of the form $L = C \oplus \mathbf{B} \oplus C'$ where $\mathbf{B}$ is the maximal block, and $C, C'$ are chains with $|C| \geq 1, |C'| \geq 1$. Then we have the following result.

**Proposition 2.9.** *For $n \geq 7$,* $|\mathscr{L}'(n;2,2)| = \sum_{i=2}^{n-5}(i-1)P_{n-i-2}^3$.

**Proof.** Let $L \in \mathscr{L}'(n;2,2)$. Then $L = C \oplus \mathbf{B} \oplus C'$ where $\mathbf{B}$ is the maximal block, and $C, C'$ are chains with $|C| \geq 1, |C'| \geq 1$. Let $|C| + |C'| = i \geq 2$ and $\mathbf{B} \in \mathscr{B}(p;2,2)$, where

$p = n - i \geq 5$, since $i \geq 2$. For fixed $i$, using Lemma 1.14, by taking $k = 2$, we have $|\mathscr{B}(n-i;2,2)| = P^3_{n-i-2}$. Now $i-2$ (excluding 0 and 1) elements can be distributed over the chains $C$ and $C'$ in $(i-2)+1 = i-1$ ways up to isomorphism. Further, $2 \leq i = n-p \leq n-5$, since $p \geq 5$. Therefore $|\mathscr{L}'(n;2,2)| = \sum_{i=2}^{n-5}(i-1)|\mathscr{B}(n-i;2,2)| = \sum_{i=2}^{n-5}(i-1)P^3_{n-i-2}$. □

**Proposition 2.10.** *For* $j \geq 8, |\mathbb{B}_4(j;4,3)| = \sum_{t=1}^{j-7}\sum_{i=2}^{j-t-5}(i-1)P^3_{j-t-i-2}$.

**Proof.** Let $\mathbf{B} \in \mathbb{B}_4(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_4$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_4) = Red(\mathbf{B})$ and $\eta(B_4) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_4$ is given by $B_4 = C]^b_a\{c_1\}]^b_a\{c_2\}]^1_0\{c_3\}$, where $C : 0 \prec a \prec x \prec b \prec 1$ is a 5-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]^b_aC_1]^b_aC_2]^1_0C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = L]^1_0C_3$ where $L \in \mathscr{L}'(m;2,2)$ for $m \geq 7$ and $|C_3| = t \geq 1$ with $j = m+t \geq 8$. Suppose $\mathbf{D} = L']^1_0C'_3$ where $L' \in \mathscr{L}'(m;2,2)$ for $m \geq 7$ and $|C'_3| = t \geq 1$ with $j = m+t \geq 8$. Then it is clear that $\mathbf{B} \cong \mathbf{D}$ if and only if $L \cong L'$ and $C_3 \cong C'_3$.

Now for fixed $t$, there are $|\mathscr{L}'(j-t;2,2)|$ maximal blocks in $\mathbb{B}_4(j;4,3)$ of type $\mathbf{B}$ up to isomorphism. Further $1 \leq t = j-m \leq j-7$, since $m \geq 7$. Therefore $|\mathbb{B}_4(j;4,3)| = \sum_{t=1}^{j-7}|\mathscr{L}'(j-t;2,2)|$. Hence the proof follows from Proposition 2.9. □

**Proposition 2.11.** *For* $j \geq 8, |\mathbb{B}_5(j;4,3)| = \sum_{p=4}^{j-4}\sum_{t=1}^{j-p-3}tP^2_{j-p-t-1}P^2_{p-2}$.

**Proof.** Let $\mathbf{B} \in \mathbb{B}_5(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_5$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_5) = Red(\mathbf{B})$ and $\eta(B_5) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_5$ is given by $B_5 = C]^b_a\{c_1\}]^1_0\{c_2\}]^1_0\{c_3\}$, where $C : 0 \prec a \prec x \prec b \prec 1$ is a 5-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]^b_aC_1]^1_0C_2]^1_0C_3$, where $C_0$ is a maximal chain containing all the reducible element of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = \mathbf{B}']^1_0L'$ where $\mathbf{B}' = C'_2]^1_0C_3 \in \mathscr{B}(p;2,1)$ for $p \geq 4$ with $C'_2 = \{0\} \oplus C_2 \oplus \{1\}$, and $L' = C'_0]^b_aC_1 \in \mathscr{L}(q;2,1)$ for $q \geq 4$, with $C'_0 = C_0 \setminus \{0,1\}$. Note that $j = p+q \geq 8$. Suppose $\mathbf{D} = \mathbf{D}']^1_0L''$ where $\mathbf{D}' = C'''_2]^1_0C'_3 \in \mathscr{B}(p;2,1)$ for $p \geq 4$ with $C'''_2 = \{0\} \oplus C''_2 \oplus \{1\}$, and $L'' = C'''_0]^b_aC'_1 \in \mathscr{L}(q;2,1)$ for $q \geq 4$, with $C'''_0 = C''_0 \setminus \{0,1\}$. Then it is clear that $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$ and $L' \cong L''$.

Now for fixed $p$, there are $|\mathscr{B}(p;2,1)| \times |\mathscr{L}(j-p;2,1)|$ maximal blocks in $\mathbb{B}_5(j;4,3)$ up to isomorphism. Clearly for fixed $p$, by Lemma 1.14, $|\mathscr{B}(p;2,1)| = P^2_{p-2}$. Further $4 \leq p = j-q \leq j-4$, since $q \geq 4$. Therefore there are $\sum_{p=4}^{j-4}|\mathscr{B}(p;2,1)| \times |\mathscr{L}(j-$

$p;2,1)|$ maximal blocks in $\mathbb{B}_5(j;4,3)$ up to isomorphism. Also using Theorem 1.15, by taking $k=1$, we have for fixed $p$, $|\mathscr{L}(j-p;2,1)|=\sum_{t=1}^{j-p-3} tP^2_{j-p-t-1}$. Hence, there are $\sum_{p=4}^{j-4} P^2_{p-2}\times\left(\sum_{t=1}^{j-p-3} tP^2_{j-p-t-1}\right)=\sum_{p=4}^{j-4}\sum_{t=1}^{j-p-3} tP^2_{p-2}P^2_{j-p-t-1}$ maximal blocks in $\mathbb{B}_5(j;4,3)$ up to isomorphism. □

**Proposition 2.12.** *For* $j\geq 8, |\mathbb{B}_6(j;4,3)|=\sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proof.** Let $\mathbf{B}\in\mathbb{B}_6(j;4,3)$. Let $0<a<b<1$ be the reducible elements of $\mathbf{B}$. As $B_6$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_6)=Red(\mathbf{B})$ and $\eta(B_6)=\eta(\mathbf{B})=3$. Observe that an adjunct representation of $B_6$ is given by $B_6 = C]_a^1\{c_1\}]_b^1\{c_2\}]_0^b\{c_3\}$, where $C:0\prec a\prec b\prec y\prec 1$ is a 5-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B}=C_0]_a^1C_1]_b^1C_2]_0^bC_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1,C_2,C_3$ are chains.

Observe that, $\mathbf{B}=L]_0^bC_3$ where $L=C'\oplus\mathbf{B}'$ with $\mathbf{B}'\in\mathscr{B}_1(m;3,2)$ for $m\geq 6$, $C'$ is a chain with $|C'|=r\geq 1$, $C_3$ is a chain with $|C_3|=t\geq 1$, and $j=m+r+t\geq 8$. Suppose $\mathbf{D}=L']_0^bC_3'$ where $L'=C''\oplus\mathbf{D}'$ with $\mathbf{D}'\in\mathscr{B}_1(m;3,2)$ for $m\geq 6$, $C''$ is a chain with $|C''|=r\geq 1$, $C_3'$ is a chain with $|C_3'|=t\geq 1$. Then it is clear that $\mathbf{B}\cong\mathbf{D}$ if and only if $L\cong L'$ and $C_3\cong C_3'$.

Using Proposition 1.16, by taking $k=2$, we have $|\mathscr{B}_1(m;3,2)|=\sum_{l=1}^{m-5}\sum_{i=1}^{m-l-4} P^2_{m-l-i-2}$. Therefore for fixed $r$ and $t$, there are $\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$ maximal blocks of type $\mathbf{B}'$ up to isomorphism in $\mathscr{B}_1(j-t-r;3,2)$. Now $1\leq r=j-t-m\leq j-t-6$, since $m\geq 6$. Therefore for fixed $t$, there are $\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$ lattices of type $L$ up to isomorphism. Further, $1\leq t=j-m-r\leq j-7$, since $m\geq 6$, $r\geq 1$. Hence there are $\sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$ maximal blocks of type $\mathbf{B}$ up to isomorphism in $\mathbb{B}_6(j;4,3)$. □

Note that $\mathbf{B}\in\mathbb{B}_7(j;4,3)$ if and only if $\mathbf{B^*}\in\mathbb{B}_6(j;4,3)$. Therefore using Proposition 2.12, we have the following result.

**Corollary 2.13.** *For* $j\geq 8$, $|\mathbb{B}_7(j;4,3)|=\sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proposition 2.14.** *For* $j\geq 8, |\mathbb{B}_8(j;4,3)|=\sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proof.** The proof is similar to the proof of Proposition 2.12. □

Note that $\mathbf{B} \in \mathbb{B}_9(j;4,3)$ if and only if $\mathbf{B^*} \in \mathbb{B}_8(j;4,3)$. Therefore using Proposition 2.14, we have the following result.

**Corollary 2.15.** *For* $j \geq 8$, $|\mathbb{B}_9(j;4,3)| = \sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proposition 2.16.** *For* $j \geq 8, |\mathbb{B}_{10}(j;4,3)| = \sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proof.** The proof is similar to the proof of Proposition 2.12. □

Note that $\mathbf{B} \in \mathbb{B}_{11}(j;4,3)$ if and only if $\mathbf{B^*} \in \mathbb{B}_{10}(j;4,3)$. Therefore using Proposition 2.16, we have the following result.

**Corollary 2.17.** *For* $j \geq 8, |\mathbb{B}_{11}(j;4,3)| = \sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proposition 2.18.** *For* $j \geq 8, |\mathbb{B}_{12}(j;4,3)| = \sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} P^2_{j-t-r-l-i-2}$.

**Proof.** The proof is similar to the proof of Proposition 2.12. □

Using Proposition 2.10, 2.11, 2.12, Corollary 2.13, Proposition 2.14, Corollary 2.15, Proposition 2.16, Corollary 2.17, and Proposition 2.18, we have the following result.

**Theorem 2.19.** *For* $j \geq 8$, $|\mathscr{B}(j;4,3,4)| = \sum_{i=4}^{12}|\mathbb{B}_i(j;4,3)| = \sum_{t=1}^{j-7}\sum_{i=2}^{j-t-5}(i-1)P^3_{j-t-i-2}$

$$+\sum_{p=4}^{j-4}\sum_{t=1}^{j-p-3} tP^2_{j-p-t-1}P^2_{p-2} + \sum_{t=1}^{j-7}\sum_{r=1}^{j-t-6}\sum_{l=1}^{j-t-r-5}\sum_{i=1}^{j-t-r-l-4} 7P^2_{j-t-r-l-i-2}.$$

### 2.3. *Counting of the class* $\mathbb{B}(j;4,3,5)$

Here we count the classes $\mathbb{B}_i(j;4,3)$ for $i = 13$ to $21$; consequently, we count the class $\mathscr{B}(j;4,3,5)$.

**Proposition 2.20.** *For* $j \geq 9$, $|\mathbb{B}_{13}(j;4,3)| = \sum_{r=0}^{j-9}\sum_{p=5}^{j-r-4} P^3_{p-2}P^2_{j-p-r-2}$.

**Proof.** Let $\mathbf{B} \in \mathbb{B}_{13}(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_{13}$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_{13}) = Red(\mathbf{B})$ and $\eta(B_{13}) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_{13}$ is given by $B_{13} = C]^a_0\{c_1\}]^a_0\{c_2\}]^1_b\{c_3\}$, where $C : 0 \prec x \prec a \prec b \prec y \prec 1$ is a 6-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_0^a C_1]_0^a C_2]_b^1 C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = \mathbf{B}' \oplus C' \oplus \mathbf{B}''$ where $\mathbf{B}' \in \mathscr{B}(p; 2, 2)$ with $p \geq 5$, $\mathbf{B}'' \in \mathscr{B}(q; 2, 1)$ with $q \geq 4$, and $C'$ is a chain with $|C'| = r \geq 0$. Suppose $\mathbf{D} = \mathbf{D}' \oplus C'' \oplus \mathbf{D}''$ where $\mathbf{D}' \in \mathscr{B}(p; 2, 2)$ with $p \geq 5$, $\mathbf{D}'' \in \mathscr{B}(q; 2, 1)$ with $q \geq 4$, and $C''$ is a chain with $|C''| = r \geq 0$. Then it is clear that $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$, $C' \cong C''$, and $\mathbf{B}'' \cong \mathbf{D}''$. Note that $j = p + q + r \geq 9$.

As $\mathbf{B}' \in \mathscr{B}(p; 2, 2)$ and $\mathbf{B}'' \in \mathscr{B}(q; 2, 1)$ by Lemma 1.14, $|\mathscr{B}(p; 2, 2)| = P_{p-2}^3$ and $|\mathscr{B}(q; 2, 1)| = P_{q-2}^2$. Therefore for fixed $p$ and $r$, there are $|\mathscr{B}(p; 2, 2)| \times |\mathscr{B}(j-p-r; 2, 1)| = P_{p-2}^3 \times P_{j-p-r-2}^2$ maximal blocks in $\mathbb{B}_{13}(j; 4, 3)$ up to isomorphism. Now $5 \leq p = j - q - r \leq j - r - 4$, since $q \geq 4$. Therefore for fixed $r$, by Lemma 1.5, there are $\sum_{p=5}^{j-r-4} (|\mathscr{B}(p; 2, 2)| \times |\mathscr{B}(j - p - r; 2, 1)|) = \sum_{p=5}^{j-r-4} (P_{p-2}^3 \times P_{j-p-r-2}^2)$ maximal blocks in $\mathbb{B}_{13}(j; 4, 3)$ up to isomorphism. Further $0 \leq r = j - p - q \leq j - 9$, since $p \geq 5$ and $q \geq 4$. Hence there are $\sum_{r=0}^{j-9} \sum_{p=5}^{j-r-4} P_{p-2}^3 P_{j-p-r-2}^2$ maximal blocks in $\mathbb{B}_{13}(j; 4, 3)$ up to isomorphism. □

Note that $\mathbf{B} \in \mathbb{B}_{14}(j; 4, 3)$ if and only if $\mathbf{B^*} \in \mathbb{B}_{13}(j; 4, 3)$. Therefore using Proposition 2.20, we have the following result.

**Corollary 2.21.** *For* $j \geq 9, |\mathbb{B}_{14}(j; 4, 3)| = \sum_{r=0}^{j-9} \sum_{p=5}^{j-r-4} P_{p-2}^3 P_{j-p-r-2}^2$.

**Proposition 2.22.** *For* $j \geq 9$, $|\mathbb{B}_{15}(j; 4, 3)| = \sum_{p=4}^{j-5} \sum_{l=1}^{j-p-4} \sum_{i=1}^{j-p-l-3} P_{p-2}^2 P_{j-p-l-i-1}^2$.

**Proof.** Let $\mathbf{B} \in \mathbb{B}_{15}(j; 4, 3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_{15}$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_{15}) = Red(\mathbf{B})$ and $\eta(B_{15}) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_{15}$ is given by $B_{15} = C]_0^a\{c_1\}]_a^b\{c_2\}]_a^1\{c_3\}$, where $C : 0 \prec x \prec a \prec y \prec b \prec 1$ is a 6-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_0^a C_1]_a^b C_2]_a^1 C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = \mathbf{B}' \circ \mathbf{B}''$ where $\mathbf{B}' \in \mathscr{B}(p; 2, 1)$ with $p \geq 4$, $\mathbf{B}'' \in \mathscr{B}_2(q; 3, 2)$ with $q \geq 6$, and as the element $a$ is considered twice, $j = p + q - 1$. Suppose $\mathbf{D} = \mathbf{D}' \circ \mathbf{D}''$ where $\mathbf{D}' \in \mathscr{B}(p; 2, 1)$ with $p \geq 4$, $\mathbf{D}'' \in \mathscr{B}_2(q; 3, 2)$ with $q \geq 6$. Then it is clear that $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$ and $\mathbf{B}'' \cong \mathbf{D}''$.

Now for fixed $p$, there are $|\mathscr{B}(p; 2, 1)| \times |\mathscr{B}_2(q; 3, 2)|$ maximal blocks in $\mathbb{B}_{15}(j; 4, 3)$ up to isomorphism, where $q = j - p + 1$. Further $4 \leq p = j - q + 1 \leq j - 5$, since $q \geq 6$. Therefore by Lemma 1.7, $|\mathbb{B}_{15}(j; 4, 3)| = \sum_{p=4}^{j-5} (|\mathscr{B}(p; 2, 1)| \times |\mathscr{B}_2(j - p + 1; 3, 2)|)$. As $\mathbf{B}' \in \mathscr{B}(p; 2, 1)$ by Lemma 1.14, $|\mathscr{B}(p; 2, 1)| = P_{p-2}^2$. Also using Proposition 1.16, by taking

$k=2$, we have $|\mathscr{B}_2(j-p+1;3,2)| = \sum_{l=1}^{j-p-4}\sum_{i=1}^{j-p-l-3} P^2_{j-p-l-i-1}$. Therefore $|\mathbb{B}_{15}(j;4,3)| = \sum_{p=4}^{j-5}\left(P^2_{p-2}\times \sum_{l=1}^{j-p-4}\sum_{i=1}^{j-p-l-3} P^2_{j-p-l-i-1}\right) = \sum_{p=4}^{j-5}\sum_{l=1}^{j-p-4}\sum_{i=1}^{j-p-l-3}(P^2_{p-2}\times P^2_{j-p-l-i-1})$. □

Note that $\mathbf{B}\in\mathbb{B}_{16}(j;4,3)$ if and only if $\mathbf{B^*}\in\mathbb{B}_{15}(j;4,3)$. Therefore using Proposition 2.22, we have the following result.

**Corollary 2.23.** *For $j\geq 9$,* $|\mathbb{B}_{16}(j;4,3)| = \sum_{p=4}^{j-5}\sum_{l=1}^{j-p-4}\sum_{i=1}^{j-p-l-3} P^2_{p-2}P^2_{j-p-l-i-1}$.

**Proposition 2.24.** *For $j\geq 9$,* $|\mathbb{B}_{17}(j;4,3)| = \sum_{p=4}^{j-5}\sum_{l=1}^{j-p-4}\sum_{i=1}^{j-p-l-3} P^2_{p-2}P^2_{j-p-l-i-1}$.

**Proof.** The proof is similar to the proof of Proposition 2.22. □

Note that $\mathbf{B}\in\mathbb{B}_{18}(j;4,3)$ if and only if $\mathbf{B^*}\in\mathbb{B}_{17}(j;4,3)$. Therefore using Proposition 2.24, we have the following result.

**Corollary 2.25.** *For $j\geq 9$,* $|\mathbb{B}_{18}(j;4,3)| = \sum_{p=4}^{j-5}\sum_{l=1}^{j-p-4}\sum_{i=1}^{j-p-l-3} P^2_{p-2}P^2_{j-p-l-i-1}$.

**Proposition 2.26.** *For $j\geq 9$,* $|\mathbb{B}_{19}(j;4,3)| = \sum_{r=1}^{j-8}\sum_{q=1}^{j-r-7}\sum_{l=4}^{j-q-r-3} P^2_{l-2}P^2_{j-q-r-l-1}$.

**Proof.** Let $\mathbf{B}\in\mathbb{B}_{19}(j;4,3)$. Let $0<a<b<1$ be the reducible elements of $\mathbf{B}$. As $B_{19}$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_{19}) = Red(\mathbf{B})$ and $\eta(B_{19}) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_{19}$ is given by $B_{19} = C]_0^a\{c_1\}]_a^b\{c_2\}]_0^1\{c_3\}$, where $C: 0\prec x\prec a\prec y\prec b\prec 1$ is a 6-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_0^aC_1]_a^bC_2]_0^1C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = (\mathbf{B'}\oplus C')]_0^1C_3$ where $\mathbf{B'}\in\mathscr{B}_3(p;3,2)$ with $p\geq 7$, $C'$ is a chain with $|C'| = q\geq 1$, and $|C_3| = r\geq 1$. Note that $j = p+q+r\geq 9$. Suppose $\mathbf{D} = (\mathbf{D'}\oplus C'')]_0^1C_3'$ where $\mathbf{D'}\in\mathscr{B}_3(p;3,2)$ with $p\geq 7$, $C''$ is a chain with $|C''| = q\geq 1$, and $|C_3'| = r\geq 1$. Then it is clear that $\mathbf{B}\cong\mathbf{D}$ if and only if $\mathbf{B'}\cong\mathbf{D'}$, $C'\cong C''$, and $C_3\cong C_3'$.

Now for fixed $q$ and $r$, there are $|\mathscr{B}_3(j-q-r;3,2)|$ maximal blocks in $\mathbb{B}_{19}(j;4,3)$ up to isomorphism. Further $1\leq q = j-p-r\leq j-r-7$, since $p\geq 7$. Therefore for fixed $r$, there are $\sum_{q=1}^{j-r-7}|\mathscr{B}_3(j-q-r;3,2)|$ maximal blocks in $\mathbb{B}_{19}(j;4,3)$ up to isomorphism. Furthermore $1\leq r = j-p-q\leq j-8$, since $p\geq 7$, $q\geq 1$. Therefore there are $\sum_{r=1}^{j-8}\sum_{q=1}^{j-r-7}|\mathscr{B}_3(j-q-r;3,2)|$ maximal blocks in $\mathbb{B}_{19}(j;4,3)$ up to isomorphism. Now using

Proposition 1.17, by taking $k = 2$, we get $|\mathscr{B}_3(j-q-r;3,2)| = \sum_{l=4}^{j-q-r-3} P_{l-2}^2 P_{j-q-r-l-1}^2$.
Thus there are $\sum_{r=1}^{j-8}\sum_{q=1}^{j-r-7}\left(\sum_{l=4}^{j-q-r-3} P_{l-2}^2 P_{j-q-r-l-1}^2\right)$ maximal blocks in $\mathbb{B}_{19}(j;4,3)$ up to isomorphism. □

Note that $\mathbf{B} \in \mathbb{B}_{20}(j;4,3)$ if and only if $\mathbf{B^*} \in \mathbb{B}_{19}(j;4,3)$. Therefore using Proposition 2.26, we have the following result.

**Corollary 2.27.** *For* $j \geq 9$, $|\mathbb{B}_{20}(j;4,3)| = \sum_{r=1}^{j-8}\sum_{q=1}^{j-r-7}\sum_{l=4}^{j-q-r-3} P_{l-2}^2 P_{j-q-r-l-1}^2$.

**Proposition 2.28.** *For* $j \geq 9$, $|\mathbb{B}_{21}(j;4,3)| = \sum_{t=1}^{j-8}\sum_{m=0}^{j-t-8}\sum_{s=4}^{j-t-m-4} P_{s-2}^2 P_{j-t-m-s-2}^2$.

**Proof.** Let $\mathbf{B} \in \mathbb{B}_{21}(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_{21}$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_{21}) = Red(\mathbf{B})$ and $\eta(B_{21}) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_{21}$ is given by $B_{21} = C]_0^a\{c_1\}]_b^1\{c_2\}]_0^1\{c_3\}$, where $C : 0 \prec x \prec a \prec b \prec y \prec 1$ is a 6-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_0^a C_1]_b^1 C_2]_0^1 C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = \mathbf{B}']_0^1 C_3$ where $\mathbf{B}' \in \mathscr{B}(p;4,2,5)$ with $p \geq 8$ and $C_3$ is a chain with $|C_3| = t \geq 1$. Note that $j = p + t \geq 9$. Suppose $\mathbf{D} = \mathbf{D}']_0^1 C_3'$ where $\mathbf{D}' \in \mathscr{B}(p;4,2,5)$ with $p \geq 8$ and $C_3'$ is a chain with $|C_3'| = t \geq 1$. Then it is clear that $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$ and $C_3 \cong C_3'$.

Now for fixed $t$, there are $|\mathscr{B}(j-t;4,2,5)|$ maximal blocks in $\mathbb{B}_{21}(j;4,3)$ up to isomorphism. By Proposition 1.19, we have $|\mathscr{B}(p;4,2,5)| = \sum_{m=0}^{p-8}\sum_{s=4}^{p-m-4} P_{s-2}^2 P_{p-m-s-2}^2$, for $p \geq 8$. Further $1 \leq t = j - p \leq j - 8$, since $p \geq 8$. Therefore there are $\sum_{t=1}^{j-8}|\mathscr{B}(j-t;4,2,5)| = \sum_{t=1}^{j-8}\left(\sum_{m=0}^{j-t-8}\sum_{s=4}^{j-t-m-4} P_{s-2}^2 P_{j-t-m-s-2}^2\right)$ maximal blocks in $\mathbb{B}_{21}(j;4,3)$ up to isomorphism. □

Using Proposition 2.20, Corollary 2.21, Proposition 2.22, Corollary 2.23, Proposition 2.24, Corollary 2.25, Proposition 2.26, Corollary 2.27, and Proposition 2.28, we have the following result.

**Theorem 2.29.** *For $j \geq 9$,*

$$\begin{aligned}|\mathscr{B}(j;4,3,5)| &= \sum_{i=13}^{21} |\mathbb{B}_i(j;4,3)| \\ &= \sum_{p=4}^{j-5} \sum_{l=1}^{j-p-4} \sum_{i=1}^{j-p-l-3} 4P^2_{p-2}P^2_{j-p-l-i-1} + \sum_{r=0}^{j-9} \sum_{p=5}^{j-r-4} 2P^3_{p-2}P^2_{j-p-r-2} \\ &\quad + \sum_{r=1}^{j-8} \sum_{q=1}^{j-r-7} \sum_{l=4}^{j-q-r-3} 2P^2_{l-2}P^2_{j-q-r-l-1} + \sum_{t=1}^{j-8} \sum_{m=0}^{j-t-8} \sum_{s=4}^{j-t-m-4} P^2_{s-2}P^2_{j-t-m-s-2}.\end{aligned}$$

### 2.4. *Counting of the class $\mathscr{L}(n;4,3)$*

Now in this subsection, firstly we count the class $\mathscr{B}(j;4,3,6)$, which is same as the class $\mathbb{B}_{22}(j;4,3)$, secondly we count the class $\mathscr{B}(j;4,3)$, and finally we count the class $\mathscr{L}(n;4,3)$.

**Theorem 2.30.** *For $j \geq 10$, $|\mathscr{B}(j;4,3,6)| = |\mathbb{B}_{22}(j;4,3)| = \sum_{p=7}^{j-3} \sum_{l=4}^{p-3} P^2_{j-p-1}P^2_{l-2}P^2_{p-l-1}$.*

**Proof.** Let $\mathbf{B} \in \mathbb{B}_{22}(j;4,3)$. Let $0 < a < b < 1$ be the reducible elements of $\mathbf{B}$. As $B_{22}$(see Figure 2) is the basic block associated to $\mathbf{B}$, by Theorem 1.12, $Red(B_{22}) = Red(\mathbf{B})$ and $\eta(B_{22}) = \eta(\mathbf{B}) = 3$. Observe that an adjunct representation of $B_{22}$ is given by $B_{22} = C]_0^a\{c_1\}]_a^b\{c_2\}]_b^1\{c_3\}$, where $C: 0 \prec x \prec a \prec y \prec b \prec z \prec 1$ is a 7-chain. Also by Corollary 1.3, $\mathbf{B}$ has an adjunct representation $\mathbf{B} = C_0]_0^aC_1]_a^bC_2]_b^1C_3$, where $C_0$ is a maximal chain containing all the reducible elements of $\mathbf{B}$, and $C_1, C_2, C_3$ are chains.

Observe that, $\mathbf{B} = \mathbf{B}' \circ \mathbf{B}''$ where $\mathbf{B}' \in \mathscr{B}_3(p;3,2)$ with $p \geq 7$, $\mathbf{B}'' \in \mathscr{B}(q;2,1)$ with $q \geq 4$, and as the element $b$ is considered twice, $j = p+q-1 \geq 10$. If $\mathbf{D} = \mathbf{D}' \circ \mathbf{D}''$ where $\mathbf{D}' \in \mathscr{B}_3(p;3,2)$ with $p \geq 7$, $\mathbf{D}'' \in \mathscr{B}(q;2,1)$ with $q \geq 4$. Then it is clear that $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$ and $\mathbf{B}'' \cong \mathbf{D}''$.

Now for fixed $p$, there are $|\mathscr{B}_3(p;3,2)| \times |\mathscr{B}(j-p+1;2,1)|$ maximal blocks in $\mathbb{B}_{22}(j;4,3)$ up to isomorphism. Further $7 \leq p = j-q+1 \leq j-3$, since $q \geq 4$. Hence by Lemma 1.7, $|\mathbb{B}_{22}(j;4,3)| = \sum_{p=7}^{j-3} (|\mathscr{B}_3(p;3,2)| \times |\mathscr{B}(j-p+1;2,1)|)$. Now as $\mathbf{B}'' \in \mathscr{B}(j-p+1;2,1)$, by Lemma 1.14, $|\mathscr{B}(j-p+1;2,1)| = P^2_{j-p-1}$. Also using Proposition 1.17 by taking $k = 2$, we have $|\mathscr{B}_3(p;3,2)| = \sum_{l=4}^{p-3} P^2_{l-2}P^2_{p-l-1}$. Therefore

$$|\mathbb{B}_{22}(j;4,3)| = \sum_{p=7}^{j-3} \left( P^2_{j-p-1} \times \sum_{l=4}^{p-3} P^2_{l-2} \times P^2_{p-l-1} \right) = \sum_{p=7}^{j-3} \sum_{l=4}^{p-3} (P^2_{j-p-1} \times P^2_{l-2} \times P^2_{p-l-1}). \qquad \square$$

By Remark 2.4, using Theorem 2.8, Theorem 2.19, Theorem 2.29, and Theorem 2.30, we have the following result.

**Theorem 2.31.** *For $j \geq 7$,*

$$
\begin{aligned}
|\mathscr{B}(j;4,3)| &= \sum_{h=3}^{6} |\mathscr{B}(j;4,3,h)| \\
&= \sum_{s=1}^{j-6} \sum_{r=1}^{j-s-5} \sum_{l=2}^{j-s-r-3} 2(j-s-r-l-2)P_l^2 + \sum_{p=4}^{j-5} \sum_{l=1}^{j-p-4} \sum_{i=1}^{j-p-l-3} 4P_{p-2}^2 P_{j-p-l-i-1}^2 \\
&\quad + \sum_{t=1}^{j-7} \sum_{r=1}^{j-t-6} \sum_{l=1}^{j-t-r-5} \sum_{i=1}^{j-t-r-l-4} 7P_{j-t-r-l-i-2}^2 \\
&\quad + \sum_{r=1}^{j-8} \sum_{q=1}^{j-r-7} \sum_{l=4}^{j-q-r-3} 2P_{l-2}^2 P_{j-q-r-l-1}^2 + \sum_{p=7}^{j-3} \sum_{l=4}^{p-3} P_{j-p-1}^2 P_{l-2}^2 P_{p-l-1}^2 \\
&\quad + \sum_{t=1}^{j-7} \sum_{i=2}^{j-t-5} (i-1)P_{j-t-i-2}^3 + \sum_{p=4}^{j-4} \sum_{t=1}^{j-p-3} tP_{j-p-t-1}^2 P_{p-2}^2 \\
&\quad + \sum_{r=0}^{j-9} \sum_{p=5}^{j-r-4} 2P_{p-2}^3 P_{j-p-r-2}^2 + \sum_{t=1}^{j-8} \sum_{m=0}^{j-t-8} \sum_{s=4}^{j-t-m-4} P_{s-2}^2 P_{j-t-m-s-2}^2 \\
&\quad + \sum_{p=1}^{j-6} \binom{j-p-2}{4}.
\end{aligned}
$$

Using Theorem 2.31, we have the following main result (Note that for the sake of brevity, we avoid the explicit formula over there).

**Theorem 2.32.** *For $n \geq 7$, $|\mathscr{L}(n;4,3)| = \sum_{i=0}^{n-7} (i+1)|\mathscr{B}(n-i;4,3)|$.*

**Proof.** Let $L \in \mathscr{L}(n;4,3)$ with $n \geq 7$. Then $L = C \oplus \mathbf{B} \oplus C'$, where $C$ and $C'$ are the chains with $|C|+|C'| = i \geq 0$, and $\mathbf{B} \in \mathscr{B}(j;4,3)$ with $j = n-i \geq 7$. For fixed $i$, there are $|\mathscr{B}(n-i;4,3)|$ maximal blocks up to isomorphism. Also there are $i+1$ ways to arrange $i$ elements on the chains $C$ and $C'$ up to isomorphism. Further $0 \leq i = n-j \leq n-7$, since $j \geq 7$. Hence $|\mathscr{L}(n;4,3)| = \sum_{i=0}^{n-7} (i+1)|\mathscr{B}(n-i;4,3)|$. □

Although there is no explicit formula for $P_n^k$ in general, it is known that $P_n^2 = \lfloor \frac{n}{2} \rfloor$ (see [14]) and $P_n^3$ is the nearest integer to $\frac{n^2}{12}$ (see [15]). In order to obtain simplest formulae for the cardinalities $|\mathscr{B}(j;4,3,h)|, 3 \leq h \leq 6$, $|\mathscr{B}(j;4,3)|$, and $|\mathscr{L}(n;4,3)|$, we take the approximate expressions for $P_n^2$ and $P_n^3$ as $\frac{n}{2}$ and $\frac{n^2}{12}$ respectively. Further, for this purpose of counting, we take the help of online platform of Wolfram Mathematica (see [17]), and obtain the approximate integer values of the respective cardinalities for $7 \leq n \leq 35$ in the above table (see Table 1).

**Table 1.** Number of non-isomorphic lattices on $n$ elements containing 4 comparable reducible elements and having nullity 3, where $7 \leq n \leq 35$.

| $n$ | $\lvert\mathscr{B}(n;4,3,3)\rvert$ | $\lvert\mathscr{B}(n;4,3,4)\rvert$ | $\lvert\mathscr{B}(n;4,3,5)\rvert$ | $\lvert\mathscr{B}(n;4,3,6)\rvert$ | $\lvert\mathscr{B}(n;4,3)\rvert$ | $\lvert\mathscr{L}(n;4,3)\rvert$ |
|---|---|---|---|---|---|---|
| 7 | 3 | 0 | 0 | 0 | 3 | 3 |
| 8 | 17 | 8 | 0 | 0 | 25 | 31 |
| 9 | 57 | 47 | 9 | 0 | 113 | 173 |
| 10 | 147 | 152 | 41 | 1 | 341 | 656 |
| 11 | 322 | 380 | 124 | 5 | 831 | 1969 |
| 12 | 630 | 811 | 294 | 13 | 1749 | 5031 |
| 13 | 1134 | 1555 | 602 | 29 | 3320 | 11414 |
| 14 | 1914 | 2751 | 1114 | 57 | 5836 | 23632 |
| 15 | 3069 | 4575 | 1913 | 102 | 9659 | 45510 |
| 16 | 4719 | 7241 | 3103 | 172 | 15234 | 82621 |
| 17 | 7007 | 11008 | 4806 | 272 | 23093 | 142825 |
| 18 | 10101 | 16183 | 7169 | 414 | 33867 | 236897 |
| 19 | 14196 | 23122 | 10366 | 608 | 48291 | 379259 |
| 20 | 19516 | 32237 | 14594 | 866 | 67213 | 588834 |
| 21 | 26316 | 44000 | 20081 | 1204 | 91600 | 890010 |
| 22 | 34884 | 58945 | 27085 | 1638 | 122552 | 1313737 |
| 23 | 45543 | 77673 | 35899 | 2187 | 161301 | 1898766 |
| 24 | 58653 | 100857 | 46846 | 2871 | 209227 | 2693022 |
| 25 | 74613 | 129243 | 60291 | 3715 | 267861 | 3755139 |
| 26 | 93863 | 163657 | 76632 | 4743 | 338894 | 5156150 |
| 27 | 116886 | 205006 | 96311 | 5985 | 424188 | 6981348 |
| 28 | 144210 | 254286 | 119811 | 7472 | 525779 | 9332327 |
| 29 | 176410 | 312582 | 147660 | 9237 | 645889 | 12329194 |
| 30 | 214110 | 381072 | 180431 | 11319 | 786932 | 16112993 |
| 31 | 257985 | 461034 | 218745 | 13757 | 951521 | 20848314 |
| 32 | 308763 | 553849 | 263274 | 16595 | 1142480 | 26726115 |
| 33 | 367227 | 661001 | 314741 | 19879 | 1362848 | 33966764 |
| 34 | 434217 | 784088 | 373922 | 23660 | 1615887 | 42823300 |
| 35 | 510632 | 924817 | 441651 | 27992 | 1905093 | 53584928 |

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