J. COMBIN. MATH. COMBIN. COMPUT. 127 (2025) 111–123

Journal of Combinatorial Mathematics
and Combinatorial Computing
www.combinatorialpress.com/jcmcc

Combinatorial Press

Group distance magic set of group vertex magic graphs

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ABSTRACT

Let G be a graph of order n and let A be an additive Abelian group with identity 0. A mapping $l : V(G) \to A \setminus \{0\}$ is said to be a A-vertex magic labeling of G if there exists a $\mu \in A$ such that $w(v) = \sum_{u \in N_G(v)} l(u) = \mu$ for all $v \in V$ and μ is called a magic constant of ℓ. The group distance magic set of an A-vertex magic graph $gdms(G, A)$ is defined as $gdms(G, A) := \{\lambda : \lambda \text{ is a magic constant of some } A\text{-vertex magic labeling}\}$. In this paper, we investigate under what conditions $gdms(G, A)$ is a subgroup of A. We also introduce the concept of the reduced group distance magic set, $rgdms(G, A)$, which can be used as a tool to determine $gdms(G, A)$.

Keywords: Group vertex magic; group distance magic set; reduced graph; tensor product

1. Introduction

Throughout this paper, we consider finite, simple and connected graphs with vertex set $V(G)$ and edge set $E(G)$. For a vertex $v \in V(G)$, let $N_G(v)$ be the set of all vertices adjacent to v in G and $|N_G(v)| = deg_G(v)$. Let A be an additive Abelian group with identity 0. A mapping $l : V(G) \to A \setminus \{0\}$ is said to be a A-vertex magic labeling of G if there exists a μ in A such that $w(v) = \sum_{u \in N_G(v)} l(u) = \mu$ for any vertex v of G. If G admits such a labeling, then it is called an A-vertex magic graph and μ is called magic constant. Several works have been done on A-vertex magic graphs in [1, 3, 2, 7].

For basic graph-theoretic notion, we refer to Bondy and Murty [5]. Let R be a commu-

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Received 24 Jun 2025; Revised 15 Aug 2025; Accepted 29 Aug 2025; Published Online 28 Sep 2025.

DOI: 10.61091/jcmcc127-08

tative ring with unity, we denote the multiplicative group of all units in R by $U(R)$. For concepts in group theory, we refer to Herstein [6].

Definition 1.1. Let G be an A-vertex magic graph. Then $gdms(G,A) = \{\lambda \in A : \lambda$ is a magic constant of some A-vertex magic labeling of $G\}$ is called the group distance magic set of G with respect to the abelian group A.

Since $gdms(G,A) \subseteq A$, the following fundamental problem arises naturally, which is the main focus of this paper.

Problem 1.2. *For a given graph G and an Abelian group A, when $gdms(G,A)$ is a subgroup of A?*

We need the following definitions and results.

Definition 1.3. The tensor product $G \otimes H$ of graphs G and H is a graph with vertex set $V(G) \times V(H)$ and vertices (g,h) and (g',h') are adjacent in $G \otimes H$ if and only if g is adjacent to g' in G and h is adjacent to h' in H.

Observation 1.4. *Let G_1 and G_2 be two simple graphs. The graphs G_1 and G_2 have odd degree vertices if and only if $G_1 \otimes G_2$ has odd degree vertices.*

Observation 1.5. *Let G_i $(1 \leq i \leq k\,;\, k \geq 2)$ be simple graphs and $G = \otimes_{i=1}^{k} G_i$. Then G is Eulerian if and only if at least one of the G_i's is Eulerian.*

Definition 1.6 ([4, 8])**.** Let H be a graph with $V(H) = \{v_1, v_2, \ldots, v_k\}$. Let $\mathcal{F} = \{G_1, G_2, \ldots, G_k\}$ be a family of graphs. Then the graph G with $V(G) = \bigcup_{i=1}^{k} V(G_i)$ and $E(G) = (\bigcup_{i=1}^{k} E(G_i)) \cup \{uv : u \in V(G_i), v \in V(G_j)$ and $v_i v_j \in E(H)\}$ is called the graph obtained by H-join operation of the graph $G_1, G_2, \ldots, G_k$ is denoted as $G = H[G_1, G_2, \ldots, G_k]$. If $G' \cong G_i$, for $1 \leq i \leq k$, then $H[G_1, G_2, \ldots, G_k]$ is denoted by $H[G']$. We observe that $H[G']$, is the lexicographic product of H and G'.

Lemma 1.7. [2] *Let A be an Abelian group with at least three elements. If $n \geq 2$ and $a \in A$, then there exists $a_1, a_2, \ldots, a_n$ in $A \setminus \{0\}$ such that $a = \sum_{i=1}^{n} a_i$.*

Observation 1.8. [3] *A graph G is $\mathbb{Z}_2$ magic if and only if degree of every vertex in G is of same parity.*

The following Proposition is an interesting result in group theory whose proof is trivial.

Proposition 1.9. *Let p_i's be distinct prime numbers and $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_m^{\alpha_m}, \alpha_i > 0$ for* *all $i = 1, 2, \ldots, m$. The group A has an element a such that $o(a)|n$ if and only if A has a subgroup isomorphic to $\mathbb{Z}_{p_i}$ for some $i = 1, 2, \ldots, m$.*

2. Basic results on $gdms(G, A)$

In this section, we present a few necessary conditions for $gdms(G, A)$ to be a subgroup of a given group A.

Lemma 2.1. *Let G be an A-vertex magic graph with minimum degree $\delta = 1$. Then $gdms(G, A)$ is not a subgroup of A.*

Proof. Let v be a pendent vertex of G and let $N_G(v) = \{u\}$. Let l be any A-vertex magic labeling of G with magic constant μ. Then $\mu = w(v) = l(u) \neq 0$. Thus $0 \notin gdms(G, A)$ and hence the result follows. □

Lemma 2.2. *Let G be an A-vertex magic graph. If $a \in gdms(G, A)$, then $-a \in gdms(G, A)$.*

Proof. Let l be an A-vertex magic labeling of G with magic constant a. It can be easily verified that $l' : V(G) \to A \setminus \{0\}$ defined by $l'(v) = -l(v)$, is an A-vertex magic labeling with magic constant $-a$. Hence $-a \in gdms(G, A)$. □

Corollary 2.3. *$gdms(G, A)$ is a subgroup of A if and only if $0 \in gdms(G, A)$ and $gdms(G, A)$ is closed under addition.*

Lemma 2.4. *Let G be a $\mathbb{Z}_2$-vertex magic graph. Then $gdms(G, \mathbb{Z}_2)$ is a subgroup of $\mathbb{Z}_2$ if and only if G is Eulerian. In this case, $gdms(G, \mathbb{Z}_2) = \{0\}$.*

Proof. Let l be a $\mathbb{Z}_2$-vertex magic labeling of G. Clearly,

$$w(v) = \begin{cases} 0 & \text{if } deg(v) \text{ is even} \\ 1 & \text{if } deg(v) \text{ is odd.} \end{cases}$$

Hence

$$gdms(G, \mathbb{Z}_2) = \begin{cases} \{0\} & \text{if } G \text{ is Eulerian} \\ \{1\} & \text{if } deg(v) \text{ is odd for all } v \in V(G). \end{cases}$$

Since G is a $\mathbb{Z}_2$-vertex magic, by observation 1.8 it follows that the degree of all vertices have same parity, $gdms(G, \mathbb{Z}_2) = \{0\}$ if and only if G is Eulerian. □

Lemma 2.5. *Let R be a commutative ring with identity and let a be a unit in R. Let $A = (R, +)$. Let G be an A-vertex magic graph. If $\mu \in gdms(G, A)$, then $a\mu \in gdms(G, A)$.*

Proof. Let l be an A-vertex magic labeling of G with magic constant μ. Then $l' : V(G) \to A \setminus \{0\}$ defined by $l'(v) = al(v)$ is an A-vertex magic labeling of G with magic constant $a\mu$. Hence $a\mu \in gdms(G, A)$. □

Corollary 2.6. *Let $A = (\mathbb{Z}_p, \oplus)$, where p is a prime number. Let G be an A-vertex magic graph. If $a \in gdms(G, A)$ and $a \neq 0$, then $\mathbb{Z}_p \setminus \{0\} \subseteq gdms(G, A)$.*

Proof. Since every non-zero element of $\mathbb{Z}_p$ is a unit, the result follows. □

Corollary 2.7. *Let G be an A-vertex magic graph where $A = (\mathbb{Z}_p, \oplus)$. Then $gdms(G, A)$ is a subgroup of G if and only if $0 \in gdms(G, A)$.*

We now proceed to prove that a similar result holds for the Klein 4-group V_4.

Lemma 2.8. *Let G be a V_4-vertex magic graph. Then $gdms(G, V_4)$ is a subgroup of V_4 if and only if $0 \in gdms(G, V_4)$.*

Proof. Let $V_4 = \{0, a, b, c\}$. If $gdms(G, V_4) = \{0\}$, there is nothing to prove. Suppose $a \in gdms(G, V_4)$ and let l be a V_4-vertex magic labeling of G with magic constant a. Let $v \in V(G)$. Since $2a = 2b = 2c = 0$, it follow that $a = w(v) = r_1a + r_2b + r_3c$, where $r_i \in \{0, 1\}$. If $r_1a \neq 0$, then $r_1a = a$ and $r_2b + r_3c = a = b + c$. Now, the labeling l_1 obtained from l by replacing the labels a, b, c by b, c, a respectively is a V_4-vertex magic labeling of G and $w(v) = c + a = b$. Hence $b \in gdms(G, V_4)$. Similarly, $c \in gdms(G, V_4)$ and the result follows. □

Corollary 2.9. *If G is V_4-vertex magic and $gdms(G, V_4)$ is a subgroup of V_4, then $gdms(G, V_4) = V_4$ or $\{0\}$.*

Theorem 2.10. *Let p be a prime number and $a \in \mathbb{Z}_p \setminus \{0\}$. If $a \in gdms(G, \mathbb{Z}_p)$, then $\mathbb{Z}_p \setminus \{0\} \subset gdms(G, \mathbb{Z}_p)$.*

Proof. Let $a \in \mathbb{Z}_p \setminus \{0\}$. Then there exists a labeling l such that the label of every vertex is a multiple of a and $w(v) = a$, for all $v \in V(G)$. Let $v \in V(G)$. Since a generates $\mathbb{Z}_p$, we have

$$w(v) = r_1a + r_2 2a + \cdots + r_i ia + \cdots + r_{p-1}(p-1)a,\ 0 \leq r_j \leq deg_G(v),$$

for all $j \in \{1, 2, \ldots, p-1\}$.

Hence,

$$w(v) = (r_1 + 2r_2 + \cdots + ir_i + \cdots + (p-1)r_{p-1})a = a.$$

Therefore, $(r_1 + 2r_2 + \cdots + ir_i + \cdots + (p-1)r_{p-1}) \equiv 1 (\bmod p)$.
Hence $(r_1 + 2r_2 + \cdots + ir_i + \cdots + (p-1)r_{p-1})ka = ka$, where $0 < k \leq p-1$.

Now $l' : V(G) \to \mathbb{Z}_p \setminus \{0\}$ defined by $l'(v_i) = kl(v_i)$ is a $\mathbb{Z}_p$-vertex magic labeling of G with magic constant ka. Hence $ka \in gdms(G, \mathbb{Z}_p)$ and therefore $\mathbb{Z}_p \setminus \{0\} \subset gdms(G, \mathbb{Z}_p)$. □

Corollary 2.11. *Let G be a $\mathbb{Z}_p$-vertex magic graph. Then $gdms(G, \mathbb{Z}_p)$ is a subgroup of $\mathbb{Z}_p$ if and only if $0 \in gdms(G, \mathbb{Z}_p)$.*

Since the existence of A-vertex magicness of any cycle C_n and any complete k-partite graph are studied in [3], it is interesting to study their group distance magic set.

Theorem 2.12. *$gdms(C_n, A)$ is a subgroup of A if and only if one of the following holds.*
(i) $n \equiv 0(mod\ 4)$.
(ii) $n \not\equiv 0(mod\ 4)$ *and A has a subgroup isomorphic to $\mathbb{Z}_2$.*

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$. Let l be an A-vertex magic labeling of C_n. A simple computation shows that,

$$l(v_i) = l(v_{(i+4)(\text{mod } n)}). \tag{1}$$

Assume that $gdms(C_n, A)$ is a subgroup of A. If $n \equiv 0(\text{mod } 4)$, then there is nothing to prove. Let $n \not\equiv 0(\text{mod } 4)$. we consider two cases.

Case 1. n is odd.

It follows from Eq. (1) that $l(v_i) = l(v_j) = a$(say) for all $i, j \in \{1, 2, \ldots, n\}$. Hence $w(v_i) = 2a$ for all i. Therefore $gdms(C_n, A) = \{2a : a \in A \setminus \{0\}\}$. Since $gdms(C_n, A)$ is subgroup of A, it follows that $2a = 0$ for some $a \in A \setminus \{0\}$. Hence $\{0, a\}$ is subgroup of A which is isomorphic to $\mathbb{Z}_2$.

Case 2. $n \not\equiv 0(\text{mod } 4)$ and n is even.

It follows from Eq. (1) that $l(v_1) = l(v_3) = l(v_5) = \cdots = l(v_{(n-1)})$ and $l(v_2) = l(v_4) = l(v_6) = \cdots = l(v_n)$. Assume that $l(v_1) = a$ and $l(v_2) = b$. Then

$$w(v_i) = \begin{cases} 2a & \text{if } i \text{ is even,} \\ 2b & \text{if } i \text{ is odd.} \end{cases}$$

Therefore, $gdms(C_n, A) = \{2a : a \in A \setminus \{0\}\}$. Since $gdms(C_n, A)$ is a subgroup of A, it follows that $2a = 0$ for some $a \in A \setminus \{0\}$. Hence $\{0, a\}$ is subgroup of A which is isomorphic to $\mathbb{Z}_2$.

Conversely, first let us assume that $n \not\equiv 0(\text{mod } 4)$ and A has an element x of order 2. Then $l_1 : V(C_n) \to A \setminus \{0\}$ defined by $l_1(v_i) = x$, for all i, $1 \le i \le n$ is an A-vertex magic labeling of C_n with magic constant 0. Hence, $0 \in gdms(C_n, A)$. Also, $gdms(C_n, A) = \{2a : a \in A \setminus \{0\}\}$. Clearly $gdms(C_n, A)$ is closed under addition and $0 \in gdms(C_n, A)$. Hence, by Corollary 2.3, $gdms(C_n, A)$ is a subgroup of A. Now, let $n \equiv 0(\text{mod } 4)$. Let l be any A-vertex magic labeling of C_n with magic constant a. Then $a = l(v_1) + l(v_3) = l(v_2) + l(v_4)$. Thus $gdms(C_n, A) = \{a + b : a, b \in A \setminus \{0\}\}$. Clearly $x - y \in gdms(C_n, A)$ for all $x, y \in gdms(C_n, A)$ and hence $gdms(C_n, A)$ is a subgroup of A. □

We now proceed to study the group distance magic set of a complete k-partite graph. Let $G = K_{n_1,n_2,\ldots,n_k}$. Let $V_1, V_2, \ldots, V_k$ be the partite sets of G and let $V_i = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$, $1 \le i \le k$. For any function $l : V(G) \to A \setminus \{0\}$, let $l(V_i) = \sum\limits_{v \in V_i} l(v)$.

Proposition 2.13. *A complete k-partite graph G is A-vertex magic if and only if there exists a function $l : V(G) \to A \setminus \{0\}$ such that $l(V_i) = a$, for all i, $1 \le i \le k$.*

Proof. Suppose G is A-vertex magic and let $l : V(G) \to A \setminus \{0\}$ be an A-vertex magic labeling of G. Then $w(v_{ir}) = l(V_1) + l(V_2) + \ldots + l(V_{i-1}) + l(V_{i+1}) + \ldots + l(V_k)$, for all $v_{ir} \in V_i$. Since $w(v_{ir}) = w(v_{is})$, it follows that $l(V_i) = l(V_j)$. Thus, $l(V_i) = a$ for all i, $1 \le i \le k$. Conversely, if there exists a labeling ℓ with $l(V_i) = a$, $1 \le i \le k$, then $w(v) = (k-1)a$, for all $v \in V(G)$. Hence l is an A-vertex magic labeling of G with magic constant $(k-1)a$. □

Theorem 2.14. *Let $G = K_{n_1,n_2,\ldots,n_k}$ where $1 = n_1 \le n_2 \le \ldots \le n_k$ and $k \ge 3$. Let A be an Abelian group with $|A| \ge 3$. Then $gdms(G, A)$ is a subgroup of A if and only if there exists an element x in A such that $o(x)|(k-1)$.*

Proof. Let $V_1 = \{v_{11}\}$. It follows from Proposition 2.13 that for any A-vertex magic labeling l of G, the magic constant is $(k-1)l(v_{11})$. Hence $gdms(G, A) = \{(k-1)a : a \in A \setminus \{0\}\}$. Clearly, $gdms(G, A)$ is closed under addition. Hence $gdms(G, A)$ is a subgroup of A if and only if there exists $x \in A$ such that $(k-1)x = 0$. Clearly, $o(x)|(k-1)$. □

Corollary 2.15. *Let $G = K_n$, where $n \ge 3$. Then $gdms(G, A)$ is a subgroup of A if and only if there exists an element $x \in A$ such that $o(x)|n-1$.*

Corollary 2.16. *Let $G = K_{n_1,n_2,\ldots,n_k}$ where $2 \le n_1 \le n_2 \ldots \le n_k$. Let A be an Abelian group with $|A| \ge 3$. Then $gdms(G, A)$ is a subgroup of A. If G is complete bipartite, then $gdms(G, A) = A$.*

Proof. It follows from Proposition 2.13 and Theorem 2.14 that $gdms(G, A) = \{(k-1)a : a \in A\}$. Clearly, $gdms(G, A)$ is a subgroup of A and $gdms(G, A) = A$ if $k = 2$. □

Thus, if A is an Abelian group with $|A| \ge 3$, then $gdms(G, A) = A$ if $G = C_{4n}$ or G is complete bipartite. Hence the following problem arises.

Problem 2.17. *Characterize A-vertex magic graphs G for which $gdms(G, A) = A$, where $|A| \ge 3$.*

3. The reduced graph : A tool for finding $gdms(G, A)$

In this section, we recall the concept of reduced graph H of a given graph G and prove that $gdms(H, A) = gdms(G, A)$, for any Abelian group A with $|A| \ge 3$. Since $|V(H)| < |V(G)|$, this gives a nice tool for determining $gdms(G, A)$.

Let $G = (V(G), E(G))$ be a graph. The relation $\sim$ defined on $V(G)$ by $u \sim v$ if and only if $N_G(u) = N_G(v)$ is an equivalence relation (see [4]). The equivalence class containing u is denoted by $[u]$. Let $S = \{[u_1], [u_2], \ldots, [u_k]\}$ be the set of all equivalence classes. The reduced graph H of G is the graph with $V(H) = S$ and two vertices $[u_i]$ and $[u_j]$ are adjacent in H if and only if u_i and u_j are adjacent in G. Clearly, H is isomorphic to the induced subgraph $G[\{u_1, u_2, \ldots, u_k\}]$ of G and each $[u_i]$ is an independent set in G. Hence, the induced subgraph $G_i = G[[u_i]]$ is $\overline{K_{n_i}}$ where $n_i = |[u_i]|$. Also, G is isomorphic to the H-join $H[G_1, G_2, \ldots, G_k]$.

Example 3.1. For the graph G given in Figure 1a, the reduced graph H and representation of G as H-join are given in Figure 1b.

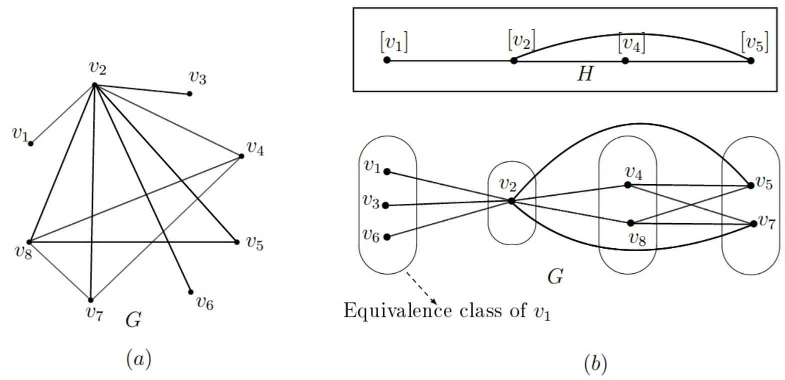

Fig. 1.

Observation 3.2. *Let A be an Abelian group with $|A| \geq 3$. If G is a graph and if $|[u]| \geq 2$ for every equivalence class $[u]$ in G, then $0 \in gdms(G, A)$.*

Proof. Using Lemma 1.7, we can label the vertices of each equivalence class with elements of A whose sum is zero. Hence $0 \in gdms(G, A)$. □

Observation 3.3. *Let A be an Abelian group with $|A| \geq 2$. Let $G = H[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_k}}]$ where H be any graph of order k and $|n_i| \geq 2$ for each i. Then $0 \in gdms(G, A)$.*

Proof. Since the reduced graph of G is H with $V(\overline{K_{n_i}})$ as equivalence classes, the result follows from the Observation 3.2. □

Sabeel et al. [7] proved that any graph G is an induced subgraph of an A-vertex magic graph, where A is a finite Abelian group. The following theorem is a generalization of the above result.

Theorem 3.4. *Any graph can be embedded as an induced subgraph of a group vertex* *magic graph.*

Proof. Let G be any graph of order n and let A be an Abelian group. Let H be the reduced graph of G. Let H' be the graph obtained from H by adding a new vertex x_i to each equivalence class $[u_i]$ where $|[u_i]|$ is odd and joining x_i to all the vertices in $N_G(u_i)$. Clearly, H' is an Eulerian graph. It follows from Lemma 2.4 and Observation 3.2 that H' is A-vertex magic and G is an induced subgraph of H'. □

It has been proved in [2] that if the reduced graph H of G is A-vertex magic where $|A|\geq 3$, then G is A-vertex magic.

Lemma 3.5. *Let H be the reduced graph of G. Then $gdms(H, A) \subseteq gdms(G, A)$.*

Proof. Let $b \in gdms(H, A)$ and let $V(H) = \{[u_1], [u_2], \ldots, [u_k]\}$. Let us assume that l is an A-vertex magic labeling of H with magic constant b and $l([u_i]) = a_i$, where $a_i \in A\setminus\{0\}$, for $i = 1, \ldots, k$. Let $u_{ij} \in [u_i]$ for $1 \leq j \leq n_i$. Define $l' : V(G) \to A \setminus \{0\}$ as follows, If $|[u_i]|$ is odd, then

$$l'(u_{ij}) = \begin{cases} a_i & \text{if } j = 1, \\ a & \text{if } j > 1 \text{ and } j \text{ is odd}, \\ -a & \text{if } j \text{ is even}, \end{cases}$$

for some $a \in A \setminus \{0\}$. If $|[u_i]|$ is even, then define

$$l'(u_{ij}) = \begin{cases} a_i + a & \text{if } j = 1, \\ a & \text{if } j > 1 \text{ and } j \text{ is odd}, \\ -a & \text{if } j \text{ is even}, \end{cases}$$

where $a_i \neq -a$. Let $v \in V(G)$. Then $v \in [u_i]$, for some i. By our definition of l', we see that $w(v) = w([u_i])$, corresponding to the labeling l in H. Hence $w(v) = b$. Thus G is an A-vertex magic graph with magic constant b and we have $gdms(H, A) \subseteq gdms(G, A)$. □

We now introduce the reduced group distance magic set(rgdms) of a graph G and prove that the gdms and rgdms of a graph G with respect to an abelian group A are the same when $|A|\geq 3$.

Definition 3.6. Let G be a graph and let A be an Abelian group. Let H be the reduced graph of G. We say that H is A'-vertex magic if there exists a labeling $l : V(H) \to A$ such that 0 can also be used to label a vertex $[u] \in V(H)$ whenever $|[u]|\geq 2$ and $\sum_{[u]\in N_H([v])} l([u]) = \mu$, for all $[v] \in V(H)$. The collection of all such μ is called the reduced group distance magic set of the graph G with respect to the Abelian group A and is denoted by $rgdms(G, A)$.

Proposition 3.7. *Let G be any graph and let A be an Abelian group with $|A|\geq 3$. Then $gdms(G, A) = rgdms(G, A)$.*

Proof. Let $a \in gdms(G, A)$. Let l be an A-vertex magic labeling of G with magic constant a. Then $l' : V(H) \to A$ defined by $l'([u]) = \sum_{v \in N_G(u)} l(v)$ is an A'-vertex magic labeling of H with magic constant a. Hence $a \in rgdms(G, A)$ and so $gdms(G, A) \subseteq rgdms(G, A)$.

Now, let $a \in rgdms(G, A)$. Let l' be an A'-vertex magic labeling of the reduced graph H with magic constant a. We now define $l : V(G) \to A \setminus \{0\}$ as follows. If $|[v]| = 1$, define $l(v) = l'(v)$. If $|[v]| = n_i \geq 2$, by Lemma 1.7, we can choose $a_1, a_2, \ldots, a_{n_i} \in A \setminus \{0\}$ such that $a_1 + a_2 + \ldots + a_{n_i} = a$. Now label the elements of $[v]$ with $a_1, a_2, \ldots, a_{n_i}$. It can be easily verified that $w(v) = a$ for all $v \in V(G)$ and hence $a \in gdms(G, A)$. Thus $rgdms(G, A) \subseteq gdms(G, A)$ and so $gdms(G, A) = rgdms(G, A)$. □

The above theorem provides a nice tool for finding $gdms(G, A)$ by transforming G to its reduced graph. We illustrate this with an example.

Example 3.8. For the graph G given in Figure 2, the reduced graph H is C_4. Let A be any Abelian group with $|A| \geq 3$. Figure 3 gives an A'-vertex magic labeling of C_4 with magic constant 0. Figure 4 gives an A'-vertex magic labeling of C_4 with magic constant a for any $a \in A \setminus \{0\}$. Hence by Proposition 3.7, $gdms(G) = rgdms(G) = A$.

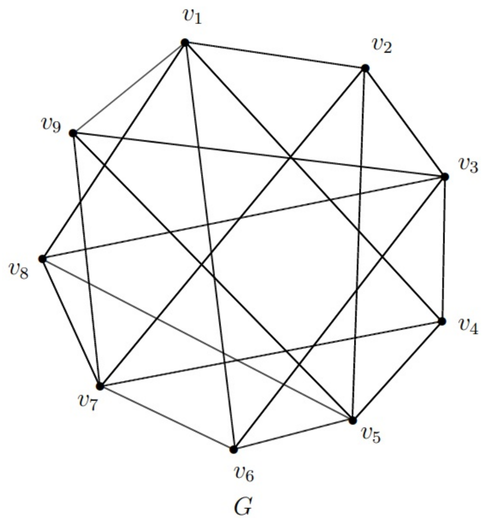

Fig. 2. A graph G with reduced graph C_4

Theorem 3.9. *Let A be an Abelian group with $|A| \geq 3$. Let $P_k = (u_1, u_2, \ldots, u_k)$ be the path on k vertices where k is even and let $G = P_k[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_k}}]$. Then $0 \in gdms(G, A)$ if and only if $n_i \geq 2$ for each i, $1 \leq i \leq k$.*

Proof. Let $0 \in gdms(G, A)$. Let l be an A-vertex magic labeling of G with magic constant 0. Clearly, $[u_i] = V(\overline{K_{n_i}})$, $1 \leq i \leq k$ are the equivalence classes of G. Let $v \in [u_k]$. Since $N_G(v) = [u_{k-1}]$, we have $w(v) = \sum_{u \in [u_{k-1}]} l(u) = 0$. Hence $|[u_{k-1}]| = n_{k-1} \geq 2$.

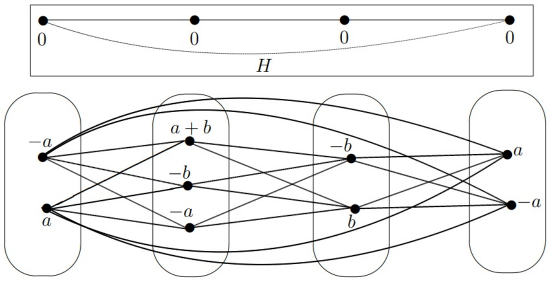

Fig. 3. Magic constant zero

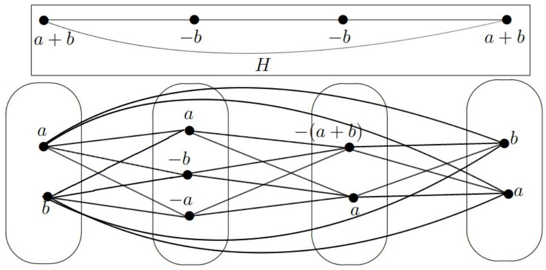

Fig. 4. Magic constant $a \neq 0$ and $a, b \in A \setminus \{0\}$ & $a \neq -b$

Now let $v \in [u_{k-2}]$. Since $w(v) = 0$, $N_G(v) = [u_{k-1}] \cup [u_{k-3}]$, $\sum_{u \in [u_{k-1}]} l(u) = 0$, and $w(v) = 0$, it follows that $\sum_{u \in [u_{k-3}]} l(u) = 0$. Hence $n_{k-1} = |[u_{k-2}]| \geq 2$. Continuing this argument, we get $n_i \geq 2$ if i is odd. Now let $v \in [u_1]$. Since $N_G(v) = [u_2]$, it follows that $n_2 \geq 2$ and proceeding as before we get $n_i \geq 2$ if i is even. The converse follows from Observation 3.3. □

Theorem 3.10. *Let A be an Abelian group with $|A| \geq 3$. Let $P_k = (u_1, u_2, \ldots, u_k)$ be the path on k vertices, where k is odd, and let $G = P_k[\overline{K_{n_1}}, \overline{K_{n_2}}, \ldots, \overline{K_{n_k}}]$. Then $0 \in gdms(G, A)$ if and only if $n_i > 1$ if i is even and $1 \leq i \leq k$.*

Proof. If $0 \in gdms(G, A)$, then proceeding as in Theorem 3.9, we get $n_i \geq 2$ if i is even. Conversely, suppose $n_i \geq 2$ if i is even. The reduced graph of G is P_k. Let $l : V(P_k) \to A$ defined by

$$l(u_i) = \begin{cases} a & \text{if } i \equiv 1 (\text{mod } 4), \\ -a & \text{if } i \equiv 3 (\text{mod } 4), \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, l is an A'-vertex magic labeling of P_k with magic constant 0. Hence $0 \in rgdms(G, A) = gdms(G, A)$. □

Theorem 3.11. *Let A be an Abelian group with $|A| \geq 3$. Let G be any graph with equiv-*

alence classes $[v_1], [v_2], \ldots, [v_k]$, *where* $|[v_i]| \geq 2$ *for each* i. *Let* G' *be the graph obtained from* G *by adding* r_i *vertices to* $[v_i]$, *where* $r_i > 0$ *for at least one* i *and joining these* r_i *vertices to all vertices in* $N_G(v_i)$. *Then* $gdms(G, A) = gdms(G', A)$.

Proof. Since G and G' have the same reduced graph, $rgdms(G) = rgdms(G')$ and the result follows from Proposition 3.7. □

Theorem 3.12. *Let* $\{G_1, G_2, \ldots, G_n\}$ *be a collection of graphs. If* $0 \in gdms(G_k, A)$, *for some* k, *then* $0 \in gdms(\otimes_{k=1}^{n} G_k, A)$.

Proof. Without loss of generality, we assume that $0 \in gdms(G_1, A)$. Let l be an A-vertex magic labeling of G_1 with magic constant 0. Let $v_i \in V(G_i)$, where $i = 1, 2, \ldots, n$. Let $N_{G_i}(v_i) = \{u_{i_1}, u_{i_2} \ldots, u_{i_{n_i}}\}$. Then $N_{\otimes_{k=1}^{n} G_k}((v_1, v_2, \ldots, v_n)) = \{(u_{1_{j_1}}, u_{2_{j_2}}, \ldots, u_{k_{j_k}}, \ldots, u_{n_{j_n}}) : j_k = 1, 2, \ldots, n_k$ and $k = 1, 2, \ldots, n\}$. Now, define $l' : V(\otimes_{k=1}^{n} G_k) \to A \setminus \{0\}$ by $l'((v_1, v_2, \ldots, v_n)) = l(v_1)$. Then

$$
\begin{aligned}
w((v_1, v_2, \ldots, v_n)) &= \sum_{j_1=1}^{n_1} \cdots \sum_{j_n=1}^{n_n} l'((u_{1_{j_1}}, u_{2_{j_2}}, \ldots, u_{n_{j_n}})) \\
&= n_2 \cdots n_i \cdots n_n \sum_{j_1=1}^{n_1} l(v_{j_i}) \\
&= 0.
\end{aligned}
$$

Since $(v_1, v_2, \ldots, v_n)$ is arbitrary, $w(v) = 0$, for all $v \in V(\otimes_{i=1}^{n} G_i)$. □

Definition 3.13. [8] Let H be a graph of order k and let $V(H) = \{v_1, v_2, \ldots, v_k\}$. Let $\{G_1, G_2, \ldots, G_k\}$ be a collection of k graphs. Then the graph obtained by joining v_i to all the vertices in G_i, where $1 \leq i \leq k$, is called the generalized corona and is denoted by $H \tilde{\circ} \wedge_{i=1}^{k} G_i$. If $G_i \cong G$, for each i, then the resulting graph is called the corona of H and G and is denoted by $H \circ G$.

Theorem 3.14. *Let* A *be an Abelian group with* $|A| \geq 3$. *Let* $G = H \tilde{\circ} \wedge_{i=1}^{k} K_{n_i}^c$ *where* $n_i \geq 2$, *for all* i. *Then* $gdms(G, A) = A \setminus \{0\}$.

Proof. Let $V(H) = \{v_1, v_2, \ldots, v_k\}$ and $V(\overline{K_{n_i}}) = \{v_{i1}, v_{i2}, \ldots, v_{in_i}\}$. Since $\delta(G) = 1$, it follows Lemma 2.1, $0 \notin gdms(G, A)$. Now, let $a \in A \setminus \{0\}$. Define $l : V(G) \to A \setminus \{0\}$ as follows. $l(v_i) = a$ for all i, $1 \leq i \leq k$. By Lemma 1.7, we label the vertices v_{ij} satisfying the condition $\sum_{j=1}^{n_i} l(v_{ij}) = (1 - deg_H(v_i))a$. Clearly, l is an A-vertex magic labeling of G with magic constant a. Hence $gdms(G, A) = A \setminus \{0\}$. □

4. Conclusion and scope

In this paper, we have introduced the concept of group distance magic set of a graph. Furthermore, we have constructed an infinite family of graphs with gdms containing zero. Additionally, we have established sufficient conditions for the presence of 0 in $gdms(G, A)$, specifically in certain product graphs. We propose the following problems for further investigation.

1) Identify family of graphs whose gdms forms a subgroup.

2) Establish necessary conditions for a graph's group distance magic set to be a subgroup.

3) Characterize graph G for which $gdms(G, A) = A$, where $|A| \geq 3$.

Acknowledgments

The authors would like to thank the Department of Science and Technology, Government of India, for providing support to carry out this work under the scheme FIST (SR/FST/MS- I/2023/131).The work of S. Balamoorthy is supported by a Junior Research Fellowship from CSIR-UGC, India (UGC-Ref.No.:1085/(CSIR-UGC NET JUNE 2019)).

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