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Integer partitions of zero and compositional structures in finite groups

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ABSTRACT

This paper addresses the enumeration of two types of zero-sum sequences. The first type is defined by constraints on both the number of terms and the values they may assume; the second type is constrained by the number of terms and the total absolute value of the sequence. For both cases, exact enumeration formulas are derived in terms of restricted partition functions, and asymptotic estimates are obtained. In the final section, an algebraic analogue of restricted compositions is introduced, and its enumerative properties are analyzed.

Keywords: zero-sum sequences, restricted partitions

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1. Introduction and definitions

The notion of zero sum is prevalent in many branches of Mathematics, including Game theory, Group theory and Graph theory. One can find a variety of zero sum problems of graph theory in a survey article by Caro [2]. In [6], Gil Kaplan et al. found an application of partitions of zero sum subsets in abelian groups in the study of anti-magic trees. A survey article by Gao and Geroldinger [4] depict zero sum problems in finite abelian groups. In this connection, we introduce a new idea namely *zero partition in integers* and we stick on to two of its enumerative functions.

Definition 1.1. A finite non increasing sequence of non zero integers is said to be a

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partition of zero if its sum equals zero. Let Z_p denote the set of partitions of zero. If $\pi \in Z_p$, then $\eta(\pi)$ denote the number of parts in π and $S(\pi)$ denote the set of parts in π.

Example 1.2. $\pi = (5, 4, 4, 2, 1, 1, -1, -3, -5, -8)$ is a partition of zero, wherein $\eta(\pi) = 10$ and $S(\pi) = \{5, 4, 2, 1, -1, -3, -5, -8\}$.

In the absence of any restrictions, the number of partitions of zero is infinite. So, it is natural to impute restrictions over the zero partitions. In this note, we treat two zero partition functions; one with conditions over the number and value of the parts and the other with the conditions over the number of parts and the absolute sum of the parts.

Definition 1.3. Let n and k be two positive integers. We define the function $Z(n, k)$ by

$$Z(n, k) = |\{\pi \in Z_p : \eta(\pi) = k \text{ and } a \in S(\pi) \text{ implies } 1 \leq |a| \leq n\}|.$$

Definition 1.4. Let $\pi = (a_1, a_2, \cdots, a_k)$ be a partition of zero. The absolute sum of π is defined as $|a_1| + |a_2| + \cdots + |a_k|$.

Definition 1.5. Define $A_z(n)$ to be the number of partitions of zero whose absolute sum of its parts is $2n$. Similarly, define $A_z(n, k)$ to be the number of partitions of zero consisting of exactly k parts and having absolute sum $2n$.

This paper is concerned with deriving exact expressions (at specific instances) and asymptotic estimates for the functions $Z(n, k)$, $A_z(n, k)$, and $A_z(n)$.

2. Partitions of zero with k bounded parts

It is immediate to observe that

$$Z(n, 2) = n.$$

In order to derive exact formulas for $Z(n, 3)$ and $Z(n, 4)$, we require a preliminary definition and a key lemma.

Definition 2.1. Let $p(r, a)$ denotes the number of partitions of r into exactly a parts.

The following expressions are classical and well known (see, for example, Andrews and Eriksson [1]):

$$p(r, 1) = 1, \qquad p(r, 2) = \left\lfloor \frac{r}{2} \right\rfloor, \qquad p(r, 3) = \left\lfloor \frac{r^2 + 3}{12} \right\rfloor.$$

It is well established that $p(r, a)$ is a quasi-polynomial in r of degree $a-1$ with quasi-period $a!$.

Lemma 2.2. (Christopher et al. [3]). *The function $p(m!\, j + r, m)$ is a polynomial in j of*

degree $m-1$ with leading coefficient

$$\frac{m!^{m-2}}{(m-1)!},$$

for each $r \in \{0, 1, \ldots, m-1\}$.

A principal result of this paper is the expression of $Z(n,k)$ in terms of $p(r,a)$.

Lemma 2.3. *Let n, k be two positive integers. If k is odd, then*

$$Z(n,k) \;=\; 2 \sum_{\substack{a+b=k \\ 1\le a\le \lfloor k/2\rfloor}} \sum_{r=b}^{nb} p(r,a)\, p(r,b).$$

If k is even, then

$$Z(n,k) \;=\; 2 \sum_{\substack{a+b=k \\ 1\le a< k/2}} \sum_{r=b}^{nb} p(r,a)\, p(r,b) \;+\; \sum_{r=k/2}^{n(k/2)} p\left(r, \frac{k}{2}\right)^2.$$

Proof. Let

$$\pi = (\gamma_1, \gamma_2, \ldots, \gamma_a, \beta_1, \beta_2, \ldots, \beta_b),$$

be a zero partition with exactly k parts, where

$$1 \le |\gamma_i|, |\beta_j| \le n, \quad \gamma_i > 0,\ \beta_j < 0.$$

Case 1. k odd.

If $a > \lfloor k/2 \rfloor$, then to each such zero partition

$$(\gamma_1, \ldots, \gamma_a, \beta_1, \ldots, \beta_b),$$

there corresponds a unique zero partition

$$(-\beta_1, \ldots, -\beta_b, -\gamma_1, \ldots, -\gamma_a),$$

and conversely. Thus it suffices to restrict to $1 \le a \le \lfloor k/2 \rfloor$, and then double the count.

Fix such an a. Let $r = |\beta_1| + \cdots + |\beta_b|$, so $b \le r \le nb$. Each partition $(|\beta_1|, \ldots, |\beta_b|)$ of r contributes $p(r,a)$ possible partitions $(\gamma_1, \ldots, \gamma_a)$. Hence we obtain

$$Z(n,k) = 2 \sum_{\substack{a+b=k \\ 1\le a\le \lfloor k/2\rfloor}} \sum_{r=b}^{nb} p(r,a)\, p(r,b).$$

Case 2. k even.

The same argument applies when $a \ne k/2$, yielding

$$2 \sum_{\substack{a+b=k \\ 1\le a< k/2}} \sum_{r=b}^{nb} p(r,a) p(r,b).$$

When $a = b = k/2$, the above doubling would overcount. In this case we simply count directly, obtaining the extra term

$$\sum_{r=k/2}^{n(k/2)} p\left(r, \frac{k}{2}\right)^2.$$

This proves the lemma. □

Applying this lemma, we obtain exact formulas for $Z(n,3)$ and $Z(n,4)$. In particular, it is noteworthy that $Z(n,3)$ is a polynomial in n of degree 2, while $Z(n,4)$ is a quasi-polynomial in n of degree 3 with quasi-period 2.

Theorem 2.4. *For every positive integer n,*
(1) $Z(n,3) = 2n^2$.
(2) $Z(n,4) = \frac{13}{6}n^3 + \frac{3}{4}n^2 - \frac{1}{6}n + \frac{1}{8}(1-(-1)^n)$.

Proof. By Lemma 2.3 for odd k,

$$Z(n,3) = 2\sum_{r=2}^{2n} p(r,1)p(r,2) = 2\sum_{r=2}^{2n} 1 \cdot \left\lfloor \frac{r}{2} \right\rfloor.$$

The sequence $\lfloor r/2 \rfloor$ for $r = 2, \ldots, 2n$ is

$$1, 1, 2, 2, \ldots, n, n,$$

so

$$\sum_{r=2}^{2n} \left\lfloor \frac{r}{2} \right\rfloor = 1 + 1 + 2 + 2 + \cdots + n + n = 2(1 + 2 + \cdots + n) - n.$$

Evaluating gives

$$2 \cdot \frac{n(n+1)}{2} - n = n^2.$$

Hence $Z(n,3) = 2n^2$, proving part (1).

From Lemma 2.3 for even k,

$$Z(n,4) = 2\sum_{r=3}^{3n} p(r,1)p(r,3) \; + \; \sum_{r=2}^{2n} p(r,2)^2.$$

Since $p(r,1) = 1$ and $p(r,3) = \lfloor (r^2+3)/12 \rfloor$, we may write

$$Z(n,4) = 2\sum_{r=1}^{3n} \left\lfloor \frac{r^2+3}{12} \right\rfloor \; + \; \sum_{r=2}^{2n} \left\lfloor \frac{r}{2} \right\rfloor^2,$$

(the extra $r = 1$ term vanishes).

First, compute the quadratic sum:

$$\sum_{r=2}^{2n} \left\lfloor \frac{r}{2} \right\rfloor^2 = \sum_{t=1}^{n} (t^2 + t^2) - n^2 = 2\sum_{t=1}^{n} t^2 - n^2 = \frac{2n^3 + n}{3}.$$

Next, set

$$S(n) = \sum_{r=1}^{3n} \left\lfloor \frac{r^2+3}{12} \right\rfloor.$$

Writing

$$\left\lfloor \frac{r^2+3}{12} \right\rfloor = \frac{r^2+3-r_r}{12}, \quad 0 \le r_r < 12,\ r_r \equiv r^2+3 \pmod{12},$$

we obtain

$$S(n) = \frac{1}{12}\left(\sum_{r=1}^{3n}(r^2+3) - \sum_{r=1}^{3n} r_r\right).$$

Since

$$\sum_{r=1}^{3n}(r^2+3) = \frac{(3n)(3n+1)(6n+1)}{6} + 3(3n) = 9n^3 + \frac{9}{2}n^2 + \frac{19}{2}n,$$

it follows that

$$S(n) = \frac{1}{12}\left(9n^3 + \frac{9}{2}n^2 + \frac{19}{2}n - \sum_{r=1}^{3n} r_r\right).$$

The sequence of residues $r_r \pmod{12}$ is periodic with period 12:

$$(4,7,0,7,4,3,4,7,0,7,4,3), \quad \text{sum} = 50.$$

Let $3n = 12q + t$ with $t \in \{0,3,6,9\}$. Then

$$\sum_{r=1}^{3n} r_r = 50q + s,$$

where s is the partial sum of the first t terms, namely

$$s = \begin{cases} 0 & (t=0), \\ 11 & (t=3), \\ 25 & (t=6), \\ 36 & (t=9). \end{cases}$$

Now write $n = 4q + m$ with $m = n \bmod 4$. Then $t = 3m$ and

$$\frac{50q+s}{12} = \frac{25(n-m)}{24} + \frac{s}{12}.$$

Thus

$$S(n) = \frac{3}{4}n^3 + \frac{3}{8}n^2 - \frac{1}{4}n + b(n),$$

where

$$b(n) = \frac{25m}{24} - \frac{s}{12}.$$

Evaluating for each $m \in \{0,1,2,3\}$ gives

$$b(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ \frac{1}{8} & \text{if } n \text{ is odd.} \end{cases}$$

Finally, combining results,

$$Z(n,4) = 2S(n) + \frac{2n^3+n}{3} = \frac{13}{6}n^3 + \frac{3}{4}n^2 - \frac{1}{6}n + \frac{1}{8}(1-(-1)^n),$$

which proves part (2). □

Combining Lemma 2.2 with Lemma 2.3, we obtain the following asymptotic estimate for $Z(n,k)$.

Theorem 2.5. *For large n,*

$$Z(n,k) \sim \frac{n^{k-1}}{k-1}\left(2\sum_{a=1}^{\lfloor (k-1)/2 \rfloor} \frac{(k-a)^{k-1}}{a!\,(a-1)!\,(k-a)!\,(k-a-1)!} + \mathbf{1}_{k\ even} \cdot \frac{(k/2)^{k-1}}{((k/2)!\,(k/2-1)!\,)^2}\right),$$

where $\mathbf{1}_{k\ even}$ denotes the indicator of the parity of k.

Proof. In view of lemma 2.2, for large r:

$$p(r,a)\,p(r,b) \sim \frac{r^{a-1}}{a!\,(a-1)!} \cdot \frac{r^{b-1}}{b!\,(b-1)!} = \frac{r^{k-2}}{a!\,(a-1)!\;b!\,(b-1)!},$$

since $a+b=k$.

Define

$$S(a,b) := \sum_{r=b}^{nb} p(r,a)\,p(r,b).$$

Using the asymptotic approximation above, we have

$$S(a,b) \sim \frac{1}{a!\,(a-1)!\;b!\,(b-1)!}\sum_{r=b}^{nb} r^{k-2}.$$

For large n, the sum can be approximated by an integral:

$$\sum_{r=b}^{nb} r^{k-2} \sim \int_0^{nb} x^{k-2}\,dx = \frac{(nb)^{k-1}}{k-1}.$$

Hence,

$$S(a,b) \sim \frac{(nb)^{k-1}}{(k-1)\,a!\,(a-1)!\;b!\,(b-1)!}.$$

Case 1. k odd.

Here $b = k - a$, with $1 \le a \le \lfloor k/2 \rfloor$. Therefore, by Lemma 2.3,

$$Z(n,k) \sim 2 \sum_{a=1}^{\lfloor k/2 \rfloor} \frac{(n(k-a))^{k-1}}{(k-1)\, a!\, (a-1)!\, (k-a)!\, (k-a-1)!}.$$

Case 2. k even.
We obtain, by Lemma 2.3

$$Z(n,k) \sim 2 \sum_{a=1}^{k/2-1} \frac{(n(k-a))^{k-1}}{(k-1)\, a!\, (a-1)!\, (k-a)!\, (k-a-1)!} + \frac{(n \cdot k/2)^{k-1}}{(k-1)\, ((k/2)!\, (k/2-1)!\,)^2}.$$

Simplification gives the required estimate. □

3. Partitions of Zero with fixed absolute sum of its parts

Our next focus is on partitions of zero in which the absolute sum of the parts equals $2n$. In this section, we investigate the functions $A_z(n)$ and $A_z(n,k)$. We begin by deriving an asymptotic estimate for $A_z(n)$.

Theorem 3.1. *For $n \to \infty$,*

$$A_z(n) \sim \frac{1}{48n^2}\, e^{\pi\sqrt{\frac{8n}{3}}}. \tag{1}$$

Proof. Let

$$\pi = (\gamma_1, \gamma_2, \dots, \gamma_a,\ \beta_1, \beta_2, \dots, \beta_b)\,,$$

be a zero partition with absolute sum $2n$, where $\gamma_i > 0$ for each i and $\beta_j < 0$ for each j. Observe that, for every partition $(\gamma_1, \gamma_2, \dots)$ of n, there are exactly $p(n)$ corresponding partitions $(\beta_1, \beta_2, \dots)$, and conversely, where $p(n)$ denotes the partition function. Hence,

$$A_z(n) = p(n)^2.$$

The stated asymptotic estimate then follows immediately from the Hardy–Ramanujan–Rademacher formula [9] for $p(n)$. □

The next result expresses $A_z(n,k)$ in terms of $p(r,a)$.

Lemma 3.2. *For integers $k \ge 3$ and $n \ge k$, we have*

$$A_z(n,k) = \sum_{a+b=k} p(n,a)\, p(n,b). \tag{2}$$

Proof. Let

$$\pi = (\gamma_1, \gamma_2, \dots, \gamma_a,\ \beta_1, \beta_2, \dots, \beta_b)\,,$$

be a zero partition with absolute sum $2n$, where each $\gamma_i > 0$ and each $\beta_j < 0$. Here the total number of parts is $a + b = k$.

Since

$$|\gamma_1|+|\gamma_2|+\cdots+|\gamma_a|+|\beta_1|+|\beta_2|+\cdots+|\beta_b|=2n,$$

and

$$\gamma_1+\gamma_2+\cdots+\gamma_a-(|\beta_1|+|\beta_2|+\cdots+|\beta_b|)=0,$$

it follows that

$$|\gamma_1|+|\gamma_2|+\cdots+|\gamma_a|=n \quad \text{and} \quad |\beta_1|+|\beta_2|+\cdots+|\beta_b|=n.$$

Thus, to each a-tuple $(\gamma_1,\ldots,\gamma_a)$ forming a partition of n, there correspond exactly $p(n,b)$ possible b-tuples $(\beta_1,\ldots,\beta_b)$, and conversely. Hence,

$$A_z(n,k)=\sum_{a+b=k} p(n,a)\,p(n,b),$$

as claimed. □

From Lemma 3.2 it follows readily that:

$$A_z(n,2)=1,$$

and

$$A_z(n,3)=\begin{cases} n, \text{ if } n \text{ is even;} \\ n-1, \text{ if } n \text{ is odd.}\end{cases}$$

Observing the formula for $A_z(n,m)$ established in Lemma 3.2, and the recurrence relation

$$p(n,m)=p(n-m,m)+p(n-1,m-1),$$

we obtain the following recurrence for $A_z(n,m)$.

Theorem 3.3. *For integers $n>0$ and $0<m\le n$,*

$$A_z(n,m) \;=\; A_z(n-m,m)+A_z(n-1,m-1) \;+\; \sum_{a+b=m} p(n-m,a)\,p(n-1,b).$$

Proof. Start from the generating function identity obtained in Lemma 3.2:

$$\sum_{m\ge 0} A_z(n,m)x^m \;=\; \Big(P_n(x)\Big)^2, \qquad P_n(x):=\sum_{r=0}^{n} p(n,r)x^r.$$

Use the standard recurrence for the partition numbers into r parts:

$$p(n,r)=p(n-r,r)+p(n-1,r-1),$$

to split $P_n(x)$. Writing

$$Q_n(x):=\sum_{r=0}^{n} p(n-r,r)x^r,$$

we obtain the decomposition

$$P_n(x) = Q_n(x) + x\,P_{n-1}(x).$$

Squaring both sides gives

$$P_n(x)^2 = Q_n(x)^2 + 2x\,Q_n(x)P_{n-1}(x) + x^2\,P_{n-1}(x)^2.$$

Collecting the coefficient of x^m on both sides yields the recurrence

$$A_z(n,m) = A_z(n-m,m) + A_z(n-1,m-1) + \sum_{a+b=m} p(n-m,a)\,p(n-1,b),$$

valid for $0 < m \le n$. □

Lemmas 2.2 and 3.2 together yield the following asymptotic estimate for $A_z(n,k)$.

Theorem 3.4. *For $k \ge 3$, we have*

$$A_z(n,k) \sim \frac{1}{k!\,(k-2)!}\binom{2k-2}{k-1} n^{k-2}. \tag{3}$$

Proof. Write $n = (k-1)!\,l + r$ with $0 \le r < (k-1)!$. By Lemma 3.2,

$$\begin{aligned} &A_z((k-1)!\,l + r, k) \\ &\quad = \sum_{a+b=k} p((k-1)!\,l + r, a)\,p((k-1)!\,l + r, b) \\ &\quad = \sum_{a+b=k} p\left(a!\left(\frac{(k-1)!}{a!}l + q_a\right) + r_a,\ a\right) p\left(b!\left(\frac{(k-1)!}{b!}l + q_b\right) + r_b,\ b\right), \end{aligned}$$

where q_a and r_a (resp. q_b and r_b) are uniquely determined by

$$a!\,q_a + r_a = r, \quad 0 \le r_a < a! \qquad (\text{resp. } b!\,q_b + r_b = r,\ 0 \le r_b < b!\,),$$

in accordance with the division algorithm.

Set

$$l'_a = \frac{(k-1)!}{a!}l + q_a, \quad l'_b = \frac{(k-1)!}{b!}l + q_b, \quad C_m = \frac{m!^{\,m-2}}{(m-1)!}, \quad m \ge 1.$$

By Lemma 2.2, $p(a!\ l'_a + r_a, a)$ (resp. $p(b!\ l'_b + r_b, b)$) is a polynomial in l'_a (resp. l'_b) of degree $a-1$ (resp. $b-1$), with leading coefficient C_a (resp. C_b). Therefore,

$$p((k-1)!\,l + r, a)\,p((k-1)!\,l + r, b),$$

is a polynomial in l of degree $(a-1)+(b-1) = k-2$ with leading coefficient

$$C_a C_b \Big(\frac{(k-1)!}{a!}\Big)^{a-1} \Big(\frac{(k-1)!}{b!}\Big)^{b-1}.$$

It follows that $A_z((k-1)!\,l+r,k)$ is itself a polynomial in l of degree $k-2$, whose leading coefficient is

$$\sum_{a+b=k} C_a C_b \Big(\frac{(k-1)!}{a!}\Big)^{a-1}\Big(\frac{(k-1)!}{b!}\Big)^{b-1}.$$

Consequently,

$$\begin{aligned}\lim_{l\to\infty}\frac{A_z((k-1)!\,l+r,k)}{((k-1)!\,l+r)^{k-2}} &= \frac{1}{((k-1)!\,)^{k-2}}\sum_{a+b=k} C_a C_b\Big(\frac{(k-1)!}{a!}\Big)^{a-1}\Big(\frac{(k-1)!}{b!}\Big)^{b-1}\\ &= \sum_{a+b=k}\frac{1}{a!\,b!\,(a-1)!\,(b-1)!}.\end{aligned}$$

Re-indexing with $a=1,\dots,k-1$ gives

$$\sum_{a=1}^{k-1}\frac{1}{a!\,(k-a)!\,(a-1)!\,(k-a-1)!} = \frac{1}{k!\,(k-2)!}\sum_{a=1}^{k-1}\binom{k}{a}\binom{k-2}{a-1}.$$

Let

$$S=\sum_{j=1}^{k-1}\binom{k-2}{j-1}\binom{k}{j}.$$

Changing index $i=j-1$, we have

$$S=\sum_{i=0}^{k-2}\binom{k-2}{i}\binom{k}{i+1}.$$

By comparing coefficients of x^{k-1} in

$$(1+x)^{k-2}(1+x)^k=(1+x)^{2k-2},$$

it follows that

$$S=\binom{2k-2}{k-1}.$$

Hence,

$$\frac{1}{k!\,(k-2)!}\sum_{j=1}^{k-1}\binom{k-2}{j-1}\binom{k}{j}=\frac{1}{k!\,(k-2)!}\binom{2k-2}{k-1}.$$

This establishes the claimed estimate. □

4. Compositions in finite groups

A *composition* of a positive integer n is an ordered sequence of positive integers $(x_1,x_2,\dots,x_m)$ such that $x_1+\cdots+x_m=n$, where each x_i is called a *part*. In this section, we investigate a natural multiplicative analogue of integer compositions within the framework of finite groups.

The motivation for introducing such an algebraic analogue of compositions in finite groups arises from combinatorial methods in group theory, such as McKay's proof of

Cauchy's theorem [7], as well as from recent studies on the Davenport constant, which is the minimum integer l such that every sequence in a finite group G of length at least l contains a zero-sum subsequence [5, 8].

We now proceed to introduce the fundamental enumerative objects considered in this section.

4.1. Compositions with k elements

Definition 4.1. Let G be a finite group and let $k \geq 1$. A *composition of $g \in G$* is an ordered tuple $(g_1, g_2, \ldots, g_k) \in G^k$ such that

$$g_1 g_2 \cdots g_k = g.$$

Definition 4.2. For $g \in G$ and $k \geq 1$ define

$$p(g,k) := \#\{(g_1, \ldots, g_k) \in G^k : \ g_1 \cdots g_k = g\}.$$

In the sequel we first establish basic enumeration formula for $p(g,k)$.

Theorem 4.3. *Let G be a finite group and let k be a positive integer. Then we have*

$$p(g,k) = o(G)^{k-1}$$

for all $g \in G$.

Proof. Proof by induction on k. When $k = 1$, it is easy to see that the assertion is true.

Assume that the result is true up to some $r \geq 1$. By the induction assumption $p(g,r) = o(G)^{r-1}$ for each $g \in G$. Now we show that $p(g, r+1) = o(G)^r$ for each $g \in G$.

let $g \in G$. By the induction assumption, any member of G say g can be written in the form $g = g_1 g_2 \cdots g_r$ with $g_i \in G \ \forall i$ in $o(G)^{r-1}$ ways. We see that, each $h \in G$ can be written as $h = gg^{-1}h = g_1 g_2 \cdots g_r (g^{-1}h)$. By induction assumption the representation $g = g_1 g_2 \cdots g_r$ can be done in $o(G)^{r-1}$ ways. As g runs over G, $g^{-1}h$ maps to each element of G. Hence the total number of representation $h = gg^{-1}h = g_1 g_2 \cdots g_r (g^{-1}h)$ is $o(G)^{r-1} o(G)$. The proof is now completed. □

4.2. Compositions with distinct elements

Definition 4.4. Let G be a finite group and let k be a positive integer such that $k \leq o(G)$. For $g \in G$, the enumerative function $d_k(g)$ is defined as

$$d_k(g) = |\{(g_1, g_2, \ldots, g_k) \in G^k : g_1 g_2 \cdots g_k = g \text{ and } g_i \neq g_j \forall i \neq j\}|.$$

Lemma 4.5. *Let G be a finite group. Then we have*

$$\sum_{g \in G} d_k(g) = \frac{o(G)!}{(o(G)-k)!}. \tag{4}$$

Proof. Let $(g_1, g_2, \cdots, g_k) \in G^k$ with $g_i \neq g_j \forall i \neq j$. Then, we have $g_1 g_2 \cdots g_k = g$ for some $g \in G$. Since the number of ways of choosing k distinct elements from G is $\frac{o(G)!}{(o(G)-k)!}$, we get the desired formula. □

Theorem 4.6. *Let G be a finite abelian group of odd order. Then we have*

$$d_2(g) = o(G) - 1,$$

for every $g \in G$.

Proof. *Case 1.* Assume $g = e$. Since $o(G)$ is odd, for each $h \neq e$, we have $h \neq h^{-1}$ and $hh^{-1} = e$. The result follows in this case.

Case 2. Assume $g \neq e$. Consider the equation $xh = g$, which has unique solution gh^{-1} for every $h \in G$. Notice that $gh^{-1} = h$ if and only if h is a solution of the equation $x^2 = g$. We observe that such a solution is unique. For if $x_1^2 = x_2^2 = g$, then $o(G)$ implies that both x_1, x_2 have same odd order, say k. This gives $x_1 = (x_1^2)^{\frac{k+1}{2}} = (x_2^2)^{\frac{k+1}{2}} = x_2$. Now the result follows. □

4.3. Compositions under a homomorphism

Definition 4.7. Let G and G' be two groups and $f : G \to G'$ be a homomorphism. Let $g' \in f(G)$. Then there exist some $g \in G$ such that $f(g) = g'$. For a fixed positive integer k, if $(g_1, g_2, \ldots, g_k)$ is a composition of g, then we have $f(g_1 g_2 \cdots g_k) = f(g_1) f(g_2) \cdots f(g_k) = f(g) = g'$. The composition $(f(g_1), f(g_2), \ldots, f(g_k))$ of g' is called the k- composition of g' under f. The number of k-compositions of g' under f is denoted by $C_f(g', k)$.

Theorem 4.8. *Let G and G' be two finite groups and let $f : G \to G'$ be a homomorphism. Then for $g' \in f(G)$ with $f(g) = g'$, we have*

$$C_f(g', k) = |g' \ker f| o(G)^{k-1}, \tag{5}$$

for every positive integer $k \leq o(G)$.

Proof. Let $h \in g' \ker f$. We obtain

$$\begin{aligned} f(h) &= f(g's) \text{ for some } s \in \ker f \\ &= f(g') f(s) \\ &= f(g'). \end{aligned}$$

By Theorem 4.3 there are $o(G)^{k-1}$ number of k tuples $(h_1, h_2, \cdots, h_k) \in G^k$ which satisfies the equality $h_1 h_2 \cdots h_k = h$.

We have

$$f(h) = f(h_1 h_2 \cdots h_k) = f(h_1) f(h_2) \cdots f(h_k);$$

which is a k- composition of g' under f. Thus, we have $C_f(g') \geq |g' \ker f| o(G)^{k-1}$.

Let $(f(g_1), f(g_2), \cdots, f(g_k))$ be a k composition of g' under f. Then we have $f(g_1)f(g_2) \cdots f(g_k) = g'$. Letting $g = g_1 g_2 \cdots g_k$ gives $f(g) = g'$. This implies $g \in g' \ker f$, and by Theorem 4.3 it follows that there are $o(G)^{k-1}$ number of k tuples $(g_1, g_2, \cdots, g_k)$ that satisfies the relation $g_1 g_2 \cdots g_k = g$. Thus, we have $P_f(g') \leq |g' \ker f| o(G)^{k-1}$. Accordingly, expected equality follows. □

5. Further work

The development of additional zero partition functions beyond those considered in this article may lead to a deeper understanding of the structure and behavior of zero partitions. Moreover, extending the concept to other analogues of compositions, particularly within broader algebraic frameworks such as rings, modules, or infinite groups, could yield significant insights and applications in algebraic theory.

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