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$(s, s+1, s+1, \dots)$-packing colorings and $(1, s, s, \dots)$-packing colorings and d-distance colorings of distance graphs $D(1,t)$

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ABSTRACT

An S-packing k-coloring of a graph G (with $S = (s_1, s_2, \dots)$ is a non-decreasing sequence of positive integers) is a mapping f from $V(G)$ to $\{1, \dots, k\}$ (the set of colors) such that for every two distincts vertices x and y in $V(G)$ with $f(x) = f(y) = i$ the distance between x and y in G is bigger than s_i. The S-packing chromatic number $\chi_S(G)$ of G is the smallest integer k such that G has an S-packing k-coloring. Given a set $D \subset \mathbb{N}^*$, a distance graph $G(\mathbb{Z}, D)$ with distance set D is a graph with vertex set $\mathbb{Z}$ and two distincts vertices u and v are adjacents if $|u-v| \in D$. In this paper, for $S = (s, s+1, s+1, \dots)$ with $s \geq \lceil \frac{t}{2} \rceil$ we give a lower bound of $\chi_S(G(\mathbb{Z}, \{1, t\}))$, and a lower bound of $\chi_d(G(\mathbb{Z}, \{1, t\}))$ with $d \geq \lceil \frac{t}{2} \rceil$, for $S = (s_1, s_2, \dots, s_i, a, a, \dots)$ with $a \geq \max(1, t-2)$ we give an upper bound of $\chi_S(G(\mathbb{Z}, \{1, t\}))$, and we determine the exact values of $\chi_S(G(\mathbb{Z}, \{1, t\}))$ and also of $\chi_d(G(\mathbb{Z}, \{1, t\}))$ for $s \geq \max(\lceil \frac{t}{2} \rceil, t-3)$ and $d \geq \max(\lceil \frac{t}{2} \rceil, t-2)$. And we give a lower and an upper bound of $\chi_S(G(\mathbb{Z}, \{1, t\}))$ for $S = (1, s, s, \dots)$ with conditions on s and t, which in the cases $s \geq \max(t-2, \lceil \frac{t}{2} \rceil)$ we determine the exact values of $\chi_S(G(\mathbb{Z}, \{1, t\}))$.

Keywords: s-packing coloring, d-distance chromatic number, s-packing chromatic number

1. Introduction

Goddard, Hedetniemi, Hedetniemi, Harris, and Rall are the first researchers who introduced the concept of S-packing [7], under the name broadcast coloring. This concept was

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introduced to generalize the notion of a proper coloring and came about as a result of potential applications in problem of distributing broadcast frequencies to radio stations. There are several domains for the applications of S-packing coloring [3]. The S-packing chromatic number (with $S = (s_1, s_2, \dots)$ is a non-decreasing sequence of positive integers) of a graph G is the smallest number of the colors such that the distance between two vertices have the same color i is bigger than s_i.

In this paper, we are studying the undirected simple graphs only. Let G be a graph, we denote by $V(G)$ the set of vertices of G, and we call the distance between two vertices u and v of $V(G)$ the length of a shortest path between u and v in G, denoted by $d_G(u, v)$. Let $S = (s_1, s_2, \dots)$ be a non-decreasing sequence of positive integers, a mapping $f : V(G) \to \{1, \dots, k\}$ with $k \in \mathbb{N}^*$ called S-packing k-coloring of G if for all differents vertices u and v belongs to $V(G)$ with $f(u) = f(v) = i$ the distance between u and v is bigger than s_i. The smallest integer k such that G admits an S-packing k-coloring is called the S-packing chromatic number of G, and we note it $\chi_S(G)$. for $S = (1, 1, 1, \dots)$ it's about the proper coloring and the chromatic number of G denoted by $\chi(G)$, for $S = (d, d, d, \dots)$ with $d \in \mathbb{N}$ it's about the d-distance coloring and the d-distance chromatic number of G denoted by $\chi_d(G)$, if $S = (1, 2, 3, \dots)$ yields the packing coloring and the packing chromatic number of G denoted by $\chi_\rho(G)$.

Let $D \subset \mathbb{N}^*$. A distance graph $G(\mathbb{Z}, D)$ with distance set D is the graph with $\mathbb{Z}$ as the vertex set, and two vertices x and y are adjacent if and only if $|x - y| \in D$, for simplification, $G(\mathbb{Z}, \{d_1, d_2, \dots, d_k\})$ be noted $D(d_1, d_2, \dots, d_k)$.

Holub and Hofman studied the S-packing colorings of the distance graphs $D(1, t)$ and $D(1, 2, t)$ for S having all elements from $\{1, 2\}$ [9]. Benmedjdoub et al. [1] investigated the 2-distance chromatic number of integer distance graphs for several types of sets D. Distance graphs were first studied systematically by Eggleton et al. in [8]. There exist many results about S-packing chromatic number of distance graphs, see [1, 2, 4, 5, 6, 10, 9, 11], but most results concern small-valued sequences. This motivates our focus here on the higher-valued sequences $(1, s, s, \dots)$ and $(s, s + 1, s + 1, \dots)$ with $s \geq \lceil \frac{t}{2} \rceil$.

In this paper, we are studying the S-packing chromatic number and the d-distance chromatic number of graph $D(1, t)$ with $t > 1$ for $S = (s, s + 1, s + 1, \dots)$ and for $S = (1, s, s, \dots)$ with $s \geq \lceil \frac{t}{2} \rceil$ and $d \geq \lceil \frac{t}{2} \rceil$.

For integers $x \leq y$, let $[\![x, y]\!] = \{x, x + 1, ..., y\}$ (in particular, $[\![x, x]\!] = \{x\}$).

2. Preliminaries

We begin by recalling a fundamental distance formula for $D(1, t)$, established by Togni [11].

Lemma 2.1. [11] *The distance between two vertices u and v of $D(1, t)$ is $d_{D(1,t)}(u, v) = \min(q + r, q + 1 + t - r)$, where $|u - v| = qt + r$ with $0 \leq r < t$.*

We also recall the following observation, due to Goddard and Xu [8].

Observation 2.2. [8] *If G_2 is a subgraph of G_1, then $\chi_S(G_2) \leq \chi_S(G_1)$ for every sequence*

S.

Lemma 2.3. *Let $t>1$ be an integer and $s \geq \lceil \frac{t}{2} \rceil$ be an integer. Let u and v two vertices of $D(1,t)$. If $d_{D(1,t)}(u,v) > s$, then $|v-u| \geq t(s-\lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil$.*

Proof. Suppose that $d_{D(1,t)}(u,v) > s$. Let $|v-u| = qt+r$ where $q,r \in \mathbb{N}$ and $0 \leq r < t$. By Lemma 2.1, $\min(q+r, q+1+t-r) = d_{D(1,t)}(u,v) > s$, then $q+r > s$ and $q+1+t-r > s$, it follows that $q \geq s + \frac{1}{2} - \frac{t}{2}$. Hence $q \geq s+1-\lceil \frac{t}{2} \rceil$. If $q > s+1-\lceil \frac{t}{2} \rceil$, then $|v-u| \geq t(s-\lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil$. If $q = s+1-\lceil \frac{t}{2} \rceil$, we have $r > s-q \geq \lceil \frac{t}{2} \rceil - 1$, implies that $|v-u| \geq t(s-\lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil$. Therefore we obtain the result. □

The previous lemma implies that if $s_i \geq \lceil \frac{t}{2} \rceil$, then any two vertices of $D(1,t)$ colored by color i in an S-packing coloring must be at least

$$L_i := t\left(s_i - \left\lceil \frac{t}{2} \right\rceil + 1\right) + \left\lceil \frac{t}{2} \right\rceil,$$

apart on the integer line. In particular, any integer interval of size at most L_i contains *at most one* vertex of color i.

Lemma 2.4. *Let $S=(s_1,s_2,\dots)$ be a non-decreasing sequence of positive integers. Let H be a subgraph of $D(1,t)$ with vertex set $X = [\![a,b]\!] \subset \mathbb{Z}$, let f be an S-packing k-coloring of subgraph H from X to $[\![1,k]\!]$ with $k \in \mathbb{N}^*$. Suppose that there exists $i \in [\![1,k]\!]$ such that $s_i \geq \lceil \frac{t}{2} \rceil$, then*

$$|\{x \in X : f(x) = i\}| \leq q + \min\{1,r\},$$

where $|X| = \left(t(s_i - \lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil\right) q + r$ with $q,r \in \mathbb{N}$ and $0 \leq r < t(s_i - \lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil$.

Proof. We prove that $|\{x \in X : f(x) = i\}| \leq q + \min\{1,r\}$. Suppose to the contrary that $|\{x \in X : f(x) = i\}| \geq q + \min\{1,r\} + 1$.

Set $X' = \{x \in X : f(x) = i\}$ and $m = |X'|$. Let $x_1 < x_2 < \cdots < x_m$ be the elements of X'. Since $f(x_1) = f(x_2) = \cdots = f(x_m) = i$, it follows that $d_{D(1,t)}(x_n, x_{n+1}) > s_i$ for all $n \in [\![1,m]\!]$. Hence, by Lemma 2.3, we have $x_{n+1} - x_n \geq t(s_i - \lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil$ for all $n \in [\![1,m]\!]$, which implies $x_m - x_1 \geq (m-1)\left(t(s_i - \lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil\right)$.

Thus $b - a \geq x_m - x_1 \geq (q + \min\{1,r\})\left(t(s_i - \lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil\right)$, a contradiction since $b-a+1 = |X| < (q+\min\{1,r\})\left(t(s_i - \lceil \frac{t}{2} \rceil + 1) + \lceil \frac{t}{2} \rceil\right) + 1$. □

Lemma 2.5. *Let $t>1$ be an integer and $a \geq t-1$ be an integer, and let $l \in \mathbb{N}^*$. Then $d_{D(1,t)}(0, lt(a - \lceil \frac{t}{2} \rceil) + l\lceil \frac{t}{2} \rceil) \geq a$.*

Proof. Let $lt(a - \lceil \frac{t}{2} \rceil) + l\lceil \frac{t}{2} \rceil = qt + r$ with $r,q \in \mathbb{N}$ and $0 \leq r < t$.

If $l=1$, then $q = a - \lceil \frac{t}{2} \rceil$ and $r = \lceil \frac{t}{2} \rceil$. Thus

$$d_{D(1,t)}\left(0, t\left(a - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil\right) = \min(q+r, q+1+t-r) \geq a.$$

Otherwise, since $l\left\lceil\frac{t}{2}\right\rceil \geq t$, $q > l(a - \left\lceil\frac{t}{2}\right\rceil)$. It follows that

$$q+r \geq l\left(a - \left\lceil\frac{t}{2}\right\rceil\right) + 1 \geq a + t - 2\left\lceil\frac{t}{2}\right\rceil \geq a \text{ and } q+1+t-r > q+1 \geq a.$$

Hence $d_{D(1,t)}\left(0, lt\left(a - \left\lceil\frac{t}{2}\right\rceil\right) + l\left\lceil\frac{t}{2}\right\rceil\right) = \min(q+r, q+1+t-r) \geq a$. □

This lemma shows that if the difference between two vertices of $D(1,t)$ is exactly

$$lt\left(a - \left\lceil\frac{t}{2}\right\rceil\right) + l\left\lceil\frac{t}{2}\right\rceil,$$

then, provided that $a \geq t-1$, their distance in $D(1,t)$ is at least a.

3. $(s, s+1, s+1, \dots)$-packing colorings of distance graphs $D(1,t)$

In this section we study $(s, s+1, s+1, \dots)$–packing colorings of the distance graphs $D(1,t)$ for $s \geq \left\lceil\frac{t}{2}\right\rceil$. To the best of our knowledge, this family has not been studied systematically; previous studies have focused on other sequences with smaller distance requirements, most notably $(1,2,2,\dots)$ on $D(1,t)$ (see Holub and Hofman [9]). As a preliminary step, we establish a general upper bound for eventually constant sequences.

Theorem 3.1. *Let $t > 1$ be an integer, $a \geq \max(1, t-2)$ be an integer and $s_1, s_2, \dots, s_i$ be integers with $1 \leq s_1 \leq s_2 \leq \dots \leq s_i \leq a$, and let $S = (s_1, s_2, \dots, s_i, a, a, \dots)$. Then $\chi_S(D(1,t)) \leq t\left(a+1-\left\lceil\frac{t}{2}\right\rceil\right) + \left\lceil\frac{t}{2}\right\rceil$.*

Proof. We set $k = t\left(a+1-\left\lceil\frac{t}{2}\right\rceil\right) + \left\lceil\frac{t}{2}\right\rceil$ and $J = [\![1,k]\!]$. Define $f : \mathbb{Z} \to J$ by: $f(u) = j$ where $u \equiv j-1 \pmod{k}$ and $j \in J$.

It is clear that f is well defined. Let u and v be two vertices of $\mathbb{Z}$ with $u < v$ such that $f(u) = f(v) = j$ with $j \in J$, then there exists $l, l' \in \mathbb{Z}$ such that $u = lk + j - 1$ and $v = l'k + j - 1$, which implies $v - u = (l'-l)k$. It follows that $d_{D(1,t)}(u,v) = d_{D(1,t)}(0, (l'-l)k)$. Since $l'-l \in \mathbb{N}^*$ and $a+1 \geq t-1$, hence, by Lemma 2.5, we have $d_{D(1,t)}(0, (l'-l)k) \geq a+1$, then $d_{D(1,t)}(u,v) \geq a+1$.

Thus any two vertices of the same color are at distance at least $a+1$, and since every $s_i \leq a$, the S-packing constraints are satisfied. Therefore f is an S-packing k-coloring of $D(1,t)$, which yields the claimed bound. □

We now derive a lower bound for $\chi_S(D(1,t))$ when $S = (s, s+1, s+1, \dots)$ and $s \geq \left\lceil\frac{t}{2}\right\rceil$.

Theorem 3.2. *Let $t > 1$ be an integer and $s \geq \left\lceil\frac{t}{2}\right\rceil$ be an integer, and let $S = (s, s+1, s+1, \dots)$. Then $\chi_S(D(1,t)) \geq t\left(s+2-\left\lceil\frac{t}{2}\right\rceil\right) + \left\lceil\frac{t}{2}\right\rceil$.*

Proof. We prove that $\chi_S(D(1,t)) \geq t\left(s+2-\left\lceil\frac{t}{2}\right\rceil\right) + \left\lceil\frac{t}{2}\right\rceil$.

Case 1. $s = \left\lceil\frac{t}{2}\right\rceil$. Let $V = \left[\!\left[0, 2t+3\left\lceil\frac{t}{2}\right\rceil\right]\!\right]$. Let H be a subgraph of $D(1,t)$ with vertex set V, let f be an S-packing k-coloring of subgraph H from V to $\{1,2,\dots,k\}$ with $k \in \mathbb{N}^* \setminus \{1\}$, let $J' = [\![1,k]\!]$, $J = J' \setminus \{1\}$, $V_j = \{i \in V : f(i) = j\}$ where $j \in J'$, and let $\alpha = |V_1|$.

By Lemma 2.4, for all $j \in J$, $|V_j| \le 2$. Let $\beta = |\{j \in J : |V_j| = 2\}|$.

First we claim that $\beta \le 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2$. Suppose to the contrary that $\beta > 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2$. Since for all $j_0 \in \{j \in J : |V_j| = 2\}$, there exist u and v in V such that $u < v$ and $f(u) = f(v) = j_0 > 1$, then by Lemma 2.3, $v - u \ge 2t + \left\lceil \frac{t}{2} \right\rceil$, imply that $u \in \llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ and $v \in \llbracket 2t + \left\lceil \frac{t}{2} \right\rceil, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$. Hence there exist β vertices in $\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ and β vertices in $\llbracket 2t + \left\lceil \frac{t}{2} \right\rceil, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$ whose colors are each used exactly twice in H, this implies that $\beta \le |\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket| = 2\left\lceil \frac{t}{2} \right\rceil + 1$.

By Lemma 2.4, three vertices in V can be colored with color 1. Then $1 \le \alpha \le 3$.

Case 1.1. $\alpha = 1$.

We have $\beta \le 2\left\lceil \frac{t}{2} \right\rceil + 1 = 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2$, a contradiction since $\beta > 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2$.

Case 1.2. $\alpha = 2$.

Then there exist two vertices u_0 and u_0' in V such that $u_0 < u_0'$ and $f(u_0) = f(u_0') = 1$. Then $d_{D(1,t)}(u_0, u_0') \ge s + 1$. Hence by Lemma 2.3, we have $u_0' - u_0 \ge t + \left\lceil \frac{t}{2} \right\rceil$, which implies $u_0 \in \llbracket 0, t + 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ and $u_0' \in \llbracket t + \left\lceil \frac{t}{2} \right\rceil + u_0, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$.

If $u_0 \in \llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$, then $\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ contains a vertex colored with color 1, and $\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ also contains β vertices colored with colors from $\{j \in J : |V_j| = 2\}$. This implies $\beta + 1 \le |\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket| = 2\left\lceil \frac{t}{2} \right\rceil + 1$, a contradiction since $\beta > 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2 = 2\left\lceil \frac{t}{2} \right\rceil$.

If $u_0 \in \llbracket 2\left\lceil \frac{t}{2} \right\rceil, t + 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$, then $u_0' \in \llbracket t + 3\left\lceil \frac{t}{2} \right\rceil, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$. Hence $\llbracket t + 3\left\lceil \frac{t}{2} \right\rceil, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$ contains a vertex of color 1 and β other vertices colored with colors from $\{j \in J : |V_j| = 2\}$. This implies $\beta + 1 \le |\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket| = 2\left\lceil \frac{t}{2} \right\rceil + 1$, a contradiction since $\beta > 2\left\lceil \frac{t}{2} \right\rceil$.

Case 1.3. $\alpha = 3$.

Then there exist three vertices u_0, u_0' and u_0'' in V such that $u_0 < u_0' < u_0''$ and $f(u_0) = f(u_0') = f(u_0'') = 1$. Then $d_{D(1,t)}(u_0, u_0') \ge s+1$ and $d_{D(1,t)}(u_0', u_0'') \ge s+1$. Hence by Lemma 2.3, we have $u_0' - u_0 \ge t + \left\lceil \frac{t}{2} \right\rceil$ and $u_0'' - u_0' \ge t + \left\lceil \frac{t}{2} \right\rceil$, which implies that $u_0'' - u_0 \ge 2t + 2\left\lceil \frac{t}{2} \right\rceil$. Therefore $u_0 \in \llbracket 0, \left\lceil \frac{t}{2} \right\rceil \rrbracket$ and $u_0'' \in \llbracket 2t + 2\left\lceil \frac{t}{2} \right\rceil, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$.

It follows that $\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ contains a vertex colored with color 1, and $\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ also contains β vertices colored with colors from $\{j \in J : |V_j| = 2\}$. This implies $\beta + 1 \le |\llbracket 0, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket| = 2\left\lceil \frac{t}{2} \right\rceil + 1$. Since $\beta > 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2$, we deduce that $\beta = 2\left\lceil \frac{t}{2} \right\rceil$.

Let $j \in \{j \in J : |V_j| = 2\}$. Since $|V_j| = 2$, there exists u_j and u_j' in V such that $u_j < u_j'$ and $f(u_j) = f(u_j') = j > 1$, then $u_j' \ge 2t + \left\lceil \frac{t}{2} \right\rceil + u_j$. Since $u_0'' > 2t + \left\lceil \frac{t}{2} \right\rceil + u_0$, and for all $u_{j'} \in \llbracket u_0 + 1, 2\left\lceil \frac{t}{2} \right\rceil \rrbracket$ with $j' \in \{j \in J : |V_j| = 2\}$, we have $u_{j'}' \ge 2t + \left\lceil \frac{t}{2} \right\rceil + u_{j'} \ge 2t + \left\lceil \frac{t}{2} \right\rceil + u_0 + 1$. Thus $u_0'', u_{j'}' \in \llbracket 2t + \left\lceil \frac{t}{2} \right\rceil + u_0 + 1, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$. Hence there exists at least $2\left\lceil \frac{t}{2} \right\rceil - u_0 + 1$ elements in $\llbracket 2t + \left\lceil \frac{t}{2} \right\rceil + u_0 + 1, 2t + 3\left\lceil \frac{t}{2} \right\rceil \rrbracket$, which implies $2\left\lceil \frac{t}{2} \right\rceil - u_0 + 1 \le 2\left\lceil \frac{t}{2} \right\rceil - u_0$, a contradiction.

In all cases we obtain a contradiction. Therefore, the claim follows.

Since the number of vertices of H whose colors are different from 1 and occur exactly twice in H is 2β, and the number of vertices of H with color 1 in H is α, hence the number of vertices of H whose colors are different from 1 and occur exactly once in H is $k - \beta - 1$. It follows that $2t + 3\left\lceil \frac{t}{2} \right\rceil + 1 = |V| = 2\beta + \alpha + k - \beta - 1 = \beta + \alpha + k - 1 \le 2\left\lceil \frac{t}{2} \right\rceil - \alpha + 2 + \alpha + k - 1 \le k + 2\left\lceil \frac{t}{2} \right\rceil + 1$, which implies $k \ge 2t + \left\lceil \frac{t}{2} \right\rceil$. Thus $\chi_S(D(1,t)) \ge 2t + \left\lceil \frac{t}{2} \right\rceil$.

Case 2. $s > \left\lceil \frac{t}{2} \right\rceil$.

Let $V = \llbracket 0, 2t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + 2\left\lceil \frac{t}{2} \right\rceil - 1 \rrbracket$. Let H be a subgraph of $D(1,t)$ with vertex set V, let f be an S-packing k-coloring of subgraph H from V to $\{1, 2, \dots, k\}$ with

$k \in \mathbb{N}^* \setminus \{1\}$, let $J' = [\![1, k]\!]$, $J = J' \setminus \{1\}$, $V_j = \{i \in V : \ f(i) = j\}$ where $j \in J'$, and let $\alpha = |V_1|$.

By Lemma 2.4, for all $j \in J$, $|V_j| \leq 2$. Let $\beta = |\{j \in J : \ |V_j| = 2\}|$. Lemma 2.4 also implies that three vertices of V can be colored with color 1. Then $1 \leq \alpha \leq 3$.

Now we claim that $\beta \leq t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil - \alpha + 1$. Suppose to the contrary that $\beta > t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil - \alpha + 1$.

Since the number of vertices of H whose colors are different from 1 and occur exactly twice in H is 2β, and the number of vertices of H with color 1 in H is α, hence the number of vertices of H whose colors are different from 1 and occur exactly once in H is $k - \beta - 1 = |V| - 2\beta - \alpha \geq 0$. By the assumption, we have $|V| - 2\beta - \alpha < 2t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + 2\left\lceil \frac{t}{2} \right\rceil - 2t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) - 2\left\lceil \frac{t}{2} \right\rceil + 2\alpha - 2 - \alpha = \alpha - 2$, then $|V| - 2\beta - \alpha \leq \alpha - 3$. If $\alpha \leq 2$, we obtain a contradiction since $|V| - 2\beta - \alpha \geq 0$. If $\alpha = 3$, then $|V| - 2\beta - \alpha = 0$, which implies $|V| = 2\beta + 3$, a contradiction since $|V|$ is even. Consequently, the claim is true.

Since $2t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + 2\left\lceil \frac{t}{2} \right\rceil = |V| = k - \beta - 1 + 2\beta + \alpha$, then $k \geq t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$. Therefore $\chi_S(D(1,t)) \geq t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$. □

Theorem 3.1 and Theorem 3.2 yield the following result.

Corollary 3.3. *Let $t > 1$ be an integer and $s \geq \max\left(\left\lceil \frac{t}{2} \right\rceil, t - 3\right)$ be an integer, and let $S = (s, s+1, s+1, \ldots)$. Then $\chi_S(D(1,t)) = t\left(s + 2 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$.*

These formulas determine $\chi_S(D(1,t))$ over a broad, previously untreated range and complement earlier exact results for small–valued sequences (e.g., $(1, 2, 2, \ldots)$) [9]. In particular, they show that, once the sequence has stabilized at $s + 1$ with s not too small relative to t, $\chi_S(D(1,t))$ is exactly given by the explicit formula above.

4. d-distance colorings of distance graphs $D(1,t)$

In this section, we give a lower bound for $\chi_d(D(1,t))$ where $d \geq \left\lceil \frac{t}{2} \right\rceil$, and we determine the exact value of $\chi_d(D(1,t))$ for $d \geq \max\left(\left\lceil \frac{t}{2} \right\rceil, t - 2\right)$.

Theorem 4.1. *Let $t > 1$ be an integer and $d \geq \left\lceil \frac{t}{2} \right\rceil$ be an integer. Then $\chi_d(D(1,t)) \geq t\left(d + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$.*

Proof. Let $V = \left[\!\left[0, t\left(d + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil - 1\right]\!\right]$, let H be a subgraph of $D(1,t)$ with vertex set V, let f be an S-packing k-coloring of subgraph H from V to $[\![1, k]\!]$ with $k \in \mathbb{N}^*$, and let $J = [\![1, k]\!]$ and $V_j = \{i \in V : \ f(i) = j\}$ with $j \in J$.
Let $j \in J$. From Lemma 2.4, $|V_j| \leq 1$. Then $|V| = t\left(d + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil = \sum_{j \in J} |V_j| \leq k$, thus $k \geq t\left(d + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$.

Therefore $\chi_d(D(1,t) \geq t\left(d + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$. □

Combining Theorem 3.1 and Theorem 4.1 yields the following corollary.

Corollary 4.2. *Let $t > 1$ be an integer and $d \geq \max\left(\left\lceil \frac{t}{2} \right\rceil, t-2\right)$ be an integer. Then $\chi_d(D(1,t)) = t\left(d + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil$.*

5. $(1, s, s, \dots)$-packing colorings of distance graphs $D(1,t)$

In this section, we investigate $(1, s, s, \dots)$–packing colorings of the distance graphs $D(1,t)$ for $s \geq \left\lceil \frac{t}{2} \right\rceil$. Prior work has focused mainly on small–valued or specially structured sequences. Přemysl Holub and Jakub Hofman showed that $\chi_{(1,2,2,\dots)}(D(1,t)) = 5$ for $t > 1$, and that $\chi_{(1,3,3,\dots)}(D(1,t)) = 5$ when t is odd [9]. As far as we are aware, no systematic study of the broader family $(1, s, s, \dots)$ with $s \geq \left\lceil \frac{t}{2} \right\rceil$ has been undertaken.

Here we derive general lower bounds for $\chi_S(D(1,t))$ with $S = (1, s, s, \dots)$, and, under suitable conditions on s and t, provide constructive upper bounds that match them, thereby yielding exact values in several cases.

We first bound how many vertices can receive color i with $s_i = 1$ inside an interval; this will be used in both lower and upper bounds.

Lemma 5.1. *Let $t > 1$ be an integer and let $S = (s_i)_{i \geq 1}$ be a non-decreasing sequence in $\mathbb{N}^*$. Let $x, y \in \mathbb{Z}$ with $x < y$, and let H be a subgraph of the distance graph $D(1,t)$ with vertex set $V = [\![x, y]\!]$. Let f be an S-packing k-coloring of H, where $k \in \mathbb{N}^*$, and let $y - x = (t+1)q + r$ with $q, r \in \mathbb{N}$ and $0 \leq r < t+1$. If there exists an integer i with $1 \leq i \leq k$ such that $s_i = 1$, then:*

(a) *If t is even and $r \neq t$, then*

$$|\{\, v \in V : f(v) = i \,\}| \;\leq\; q \cdot \frac{t}{2} + \left\lfloor \frac{r}{2} \right\rfloor + 1.$$

(b) *If t is even and $r = t$, then*

$$|\{\, v \in V : f(v) = i \,\}| \;\leq\; (q+1)\frac{t}{2}.$$

(c) *If t is odd, then*

$$|\{\, v \in V : f(v) = i \,\}| \;\leq\; 1 + \left\lfloor \frac{y - x}{2} \right\rfloor.$$

Proof. Let $V' = \{v \in V : \ f(v) = i\}$ be the set of vertices of V colored i. Let $u_0, v_0 \in V$ with $v_0 > u_0$, and set $U = \{v \in [\![u_0, v_0]\!] : \ f(v) = i\}$ and $m = |U|$.

For any distinct $u, v \in U$ with $v > u$, we have $f(u) = f(v) = i$. Since $s_i = 1$, this implies $v - u \geq 2$. Let $x_1 < x_2 < \dots < x_m$ be the elements of U. It follows that $x_m \geq x_1 + 2(m-1)$. Since $x_m \leq v_0$ and $x_1 \geq u_0$, we conclude that $m \leq \left\lfloor \frac{v_0 - u_0}{2} \right\rfloor + 1$. Therefore, $m \leq \left\lfloor \frac{y-x}{2} \right\rfloor + 1$. This completes the proof for the case when t is odd.

We claim that if $v_0 - u_0 = t$, then $m \leq \left\lfloor \frac{v_0 - u_0}{2} \right\rfloor$.

Suppose that $v_0 - u_0 = t$. Since x_1 and x_m are both colored i, then $x_m - x_1 \neq t$, which implies $v_0 \neq x_m$ or $u_0 \neq x_1$.

Suppose to the contrary that $m \geq \left\lfloor \frac{v_0 - u_0}{2} \right\rfloor + 1 = \frac{t}{2} + 1$.

Since $x_m \geq x_1 + 2(m-1) = x_1 + t$ and ($x_m < v_0$ or $x_1 > u_0$), then $v_0 > u_0 + t$, a contradiction, as claimed.

Case 1. t is even and $r \neq t$.

If $q = 0$, then $y - x = r$. Since $x, y \in V$, $|V'| \leq \lfloor \frac{y-x}{2} \rfloor + 1 = \lfloor \frac{r}{2} \rfloor + 1$.

If $q \neq 0$, then let $V_q = [\![x + q(t+1), x + q(t+1) + r]\!]$ and $V_n = [\![x + n(t+1), x + n(t+1) + t]\!]$ where $n \in [\![0, q-1]\!]$. Let $V'_n = \{v \in V_n : f(v) = i\}$ where $n \in [\![0, q]\!]$. Observe that $V = \bigcup_{n=0}^{q} V_n$ and $V' = \bigcup_{n=0}^{q} V'_n$ where the unions are disjoint. Then $|V'| = \sum_{n=0}^{q} |V'_n|$. Since for every $n \in [\![0, q-1]\!]$, $|V'_n| \leq \frac{t}{2}$ and $\left|V'_q\right| \leq \lfloor \frac{r}{2} \rfloor + 1$, hence $|V'| \leq q\frac{t}{2} + \lfloor \frac{r}{2} \rfloor + 1$.

Case 2. t is even and $r = t$.

Let $V_n = [\![x + n(t+1), x + n(t+1) + t]\!]$ where $n \in [\![0, q]\!]$. Let $V'_n = \{v \in V_n : f(v) = i\}$ where $n \in [\![0, q]\!]$. As before, $V = \bigcup_{n=0}^{q} V_n$ and $V' = \bigcup_{n=0}^{q} V'_n$ where the unions are disjoint. Since for every $n \in [\![0, q]\!]$, $|V'_n| \leq \frac{t}{2}$, hence $|V'| \leq (q+1)\frac{t}{2}$. □

We now derive a general lower bound for $\chi_S(D(1,t))$ when t is odd and $S = (1, s, s, \dots)$ with $s \geq \frac{t+1}{2}$. The following theorem provides an explicit expression that depends on the parity of s.

Theorem 5.2. *Let $t > 1$ be an odd integer and $s \geq \dfrac{t+1}{2}$ be an integer, and let $S = (1, s, s, \dots)$. Then:*

(a) *If s is odd, then*

$$\chi_S(D(1,t)) \;\geq\; \frac{t\left(s + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil}{2} \;+\; 1.$$

(b) *If s is even, then*

$$\chi_S(D(1,t)) \;\geq\; \frac{t\left(s + 1 - \left\lceil \frac{t}{2} \right\rceil\right) + \left\lceil \frac{t}{2} \right\rceil + 1}{2}.$$

Proof. We set $N = t(s + 1 - \lceil \frac{t}{2} \rceil) + \lceil \frac{t}{2} \rceil$ and $V = [\![0, N-1]\!]$. Let H be a subgraph of $D(1,t)$ with vertex set V. Let f be an S-packing k-coloring of subgraph H with $k \in \mathbb{N}^*$. Clearly $k \geq 2$. Set $J' = [\![1, k]\!]$, $J = J' \setminus \{1\}$ and $V_j = \{i \in V : f(i) = j\}$ where $j \in J'$.

By Lemma 5.1 we obtain $|V_1| \leq \left\lfloor \dfrac{N-1}{2} \right\rfloor + 1$, and by Lemma 2.4 it follows that $|V_j| \leq 1$. Since $N = |V| = \sum_{j \in J'} |V_j| = |V_1| + \sum_{j=2}^{k} |V_j|$, hence $N \leq \left\lfloor \dfrac{N-1}{2} \right\rfloor + 1 + k - 1$, which implies $k \geq N - \left\lfloor \dfrac{N-1}{2} \right\rfloor$. It follows that $\chi_S(H) \geq N - \left\lfloor \dfrac{N-1}{2} \right\rfloor$. Thus, if s is odd, then N is even, then $\chi_S(H) \geq \dfrac{N}{2} + 1$, otherwise, if s is even, then N is odd $\chi_S(H) \geq \dfrac{N+1}{2}$. Therefore, by Observation 2.2, we deduce the result. □

Theorem 5.3. *Let $t > 1$ be an odd integer and $s \geq t-2$ be an integer, and let $S = (1, s, s, \ldots)$. If s is odd, then* $\chi_S(D(1,t)) \leq \dfrac{t(s+1-\lceil \frac{t}{2} \rceil) + \lceil \frac{t}{2} \rceil}{2} + 1.$

Proof. Suppose that s is odd. Set $k = t(s+1-\lceil \frac{t}{2} \rceil) + \lceil \frac{t}{2} \rceil + 1$ and $V = [\![0, k-2]\!]$. Let g be a mapping from V to $[\![1, k]\!]$, defined by

$$g(i) = \begin{cases} 1, & \text{if } i \textit{ is even}; \\ j+2, & \text{if } i = 2j+1 \textit{ with } j \in \mathbb{N}, \end{cases}$$

for all $i \in V$.

And let f be a mapping from $\mathbb{Z}$ to $[\![1, k]\!]$, defined by $f(i) = g(i_0)$ for all $i \equiv i_0 \pmod{k-1}$ with $0 \leq i_0 < k-1$.

It is clear that g and f are well defined. Now we show that f is S-packing k-coloring.

Let $i, i' \in \mathbb{Z}$ with $i \neq i'$, such that $f(i) = f(i') = j_0$. There exists $l, l' \in \mathbb{Z}$ and $i_0, i'_0 \in [\![0, k-2]\!]$ such that $i = l(k-1) + i_0$ and $i' = l'(k-1) + i'_0$, then $g(i_0) = f(i) = j_0 = f(i') = g(i'_0)$.

For color 1. Suppose that $j_0 = 1$. Since $g(i_0) = g(i'_0) = 1$, both i_0 and i'_0 are even. Because s is odd, it follows that $k-1$ is even, which implies that both i and i' are even. Hence $|i'-i|$ is even, and therefore $|i'-i| \notin \{1, t\}$. Consequently, i and i' are not adjacent, and thus $d_{D(1,t)}(i, i') > 1$.

For color $j_0 \neq 1$. Suppose that $j_0 \neq 1$. Since $g(i_0) = g(i'_0) = j_0 > 1$, both i_0 and i'_0 are odd. Hence, by definition of mapping g, we have $i_0 = 2(j_0-2)+1 = i'_0$. It follows that $|i'-i| = |(l'-l)(k-1) - 0|$, then $d_{D(1,t)}(i, i') = d_{D(1,t)}(0, |l'-l|(k-1))$. We have $|l'-l| \in \mathbb{N}^*$ and $s+1 \geq t-1$, hence it follows from Lemma 2.5 that $d_{D(1,t)}(0, |l'-l|(k-1)) = d_{D(1,t)}(0, |l'-l|t(s+1-\lceil \frac{t}{2} \rceil) + |l'-l|\lceil \frac{t}{2} \rceil) \geq s+1$, thus $d_{D(1,t)}(i, i') \geq s+1$.

Therefore f is S-packing k-coloring. □

Theorem 5.2 and Theorem 5.3 imply the following result.

Corollary 5.4. *Let $t > 1$ be an odd integer and let $s \geq \max(t-2, \frac{t+1}{2})$ be an integer. If s is odd, then* $\chi_S(D(1,t)) = \dfrac{t(s+1-\lceil \frac{t}{2} \rceil) + \lceil \frac{t}{2} \rceil}{2} + 1.$

We now establish an upper bound for $\chi_S(D(1,t))$ when t is even and $s \geq \max(1, t-2)$.

Theorem 5.5. *Let $t > 1$ be an even integer and let $s \geq \max(1, t-2)$ be an integer, and let $S = (1, s, s, \ldots)$. If r is even and $r \neq t$, then $\chi_S(D(1,t)) \leq (\frac{t}{2}+1)q + r + 1$, where $t(s+1-\frac{t}{2}) + \frac{t}{2} = (t+1)q + r$ with $q, r \in \mathbb{N}$ and $0 \leq r < t+1$.*

Proof. Suppose that r is even and $r \neq t$. Then $0 \leq r < t-1$. And we have $s \geq \max(1, t-2)$, then $q \geq 1$.

Let $k = (1+\frac{t}{2})q + r + 1$, $V = [\![0, (t+1)q + r - 1]\!]$,

$$A_1 = \left\{ i \in V : \ i = (t+1)q' + 2r' + r + 2 \textit{ with } 0 \leq q' < q \textit{ and } 0 \leq r' < \frac{t}{2} \right\},$$

and $A_2 = V \setminus A_1$. Clearly $A_1 \neq \emptyset$ and $A_2 \neq \emptyset$. Since A_2 is a nonempty finite set, implies that there exists a strictly increasing sequence $(y_p)_{1\le p\le p_0}$ in V (with $p_0 = |A_2| = |V| - |A_1| = \left(\frac{t}{2}+1\right)q + r$) such that $A_2 = \{y_1, y_2, \dots, y_{p_0}\}$.

Let g be a mapping from V to $[\![1,k]\!]$, defined by

$$g(i) = \begin{cases} 1, \ if \ i \in A_1; \\ j+1, \ if \ i = y_j \in A_2 \ with \ j \in [\![1, p_0]\!], \end{cases}$$

for all $i \in V$.

And let f be a mapping from $\mathbb{Z}$ to $[\![1,k]\!]$, defined by $f(i) = g(i_0)$ for all $i \equiv i_0 \pmod{(t+1)q+r}$ with $0 \le i_0 < (t+1)q + r$.

It is clear that g and f are well defined. Now we show that f is S-packing k-coloring.

Let $i, i' \in \mathbb{Z}$ with $i < i'$, such that $f(i) = f(i') = j_0$. There exists $l, l' \in \mathbb{Z}$ and $i_1, i_2 \in V$ such that $i = l((t+1)q+r)+i_1$ and $i' = l'((t+1)q+r)+i_2$, then $g(i_1) = f(i) = j_0 = f(i') = g(i_2)$. Since $i < i'$, then $l' \ge l$, hence $l' - l \ge 0$.

Case 1. $j_0 = 1$ and $l' - l \ge 2$.

We have $i' - i = (l'-l)((t+1)q+r)+i_2-i_1 \ge (t+1)q+r+1 > t$ (since $i_2 - i_1 \ge -(t+1)q-r+1$ and $q \ge 1$), which implies i' and i are not adjacent. Therefore $d_{D(1,t)}(i,i') > 1$.

Case 2. $j_0 = 1$ and $l' - l = 1$ and $i_1 \le i_2$.

Since $i' - i = (t+1)q + r + i_2 - i_1 \ge (t+1)q + r > t$, thus i' and i are not adjacent. Therefore $d_{D(1,t)}(i,i') > 1$.

Case 3. $j_0 = 1$ and $l' - l = 1$ and $i_1 > i_2$.

Since $i_1, i_2 \in A_1$, then there exists $q_1, q_2 \in [\![0, q-1]\!]$ and $r_1, r_2 \in [\![0, \frac{t}{2}-1]\!]$ such that $i_1 = (t+1)q_1+2r_1+r+2$ and $i_2 = (t+1)q_2+2r_2+r+2$. Hence $i'-i = (t+1)q+r+i_2-i_1 \ge r+3 > 1$.

We prove by contradiction that $i' - i \neq t$. Suppose that $i' - i = t$. Then $i_1 = (t+1)q + r + i_2 - t$. Since $i_2 = (t+1)q_2+2r_2+r+2$, hence $i_1 = (t+1)(q+q_2)+2(\frac{r}{2}+r_2-\frac{t}{2})+r+2$.

If $\frac{r}{2} + r_2 - \frac{t}{2} \ge 0$, then $i_1 \ge (t+1)q + r + 2 > (t+1)q + r - 1$, hence $i_1 \notin V$, a contradiction since $i_1 \in V$.

If $\frac{r}{2}+r_2-\frac{t}{2} < 0$, then $0 \le \frac{r}{2}+r_2 < \frac{t}{2}$, then $i_1 = (t+1)(q+q_2-1)+2\left(\frac{r}{2}+r_2\right)+r+2+1$, hence $i_1 \notin A_1$, a contradiction since $i_1 \in A_1$.

Thus i' and i are not adjacent. Therefore $d_{D(1,t)}(i,i') > 1$.

Case 4. $j_0 = 1$ and $l' - l = 0$.

Since $g(i_1) = g(i_2) = 1$, $i_1, i_2 \in A_1$. Then there exists $q_1, q_2 \in [\![0, q-1]\!]$ and $r_1, r_2 \in [\![0, \frac{t}{2}-1]\!]$ such that $i_1 = (t+1)q_1 + 2r_1 + r + 2$ and $i_2 = (t+1)q_2 + 2r_2 + r + 2$. If $q_2 - q_1 = 0$, then $i' - i = i_2 - i_1 = 2r_2 - 2r_1 < t$, which implies $0 < i' - i < t$ and $i' - i$ is even, hence $i' - i \notin \{-t, -1, 1, t\}$. If $q_2 - q_1 = 1$, then $i' - i = t + 1 + 2r_2 - 2r_1 < t$, which implies $i' - i > 1$ and $i' - i$ is odd, hence $i' - i \notin \{-t, -1, 1, t\}$. If $q_2 - q_1 > 1$, then $i' - i > t$, which implies $i' - i \notin \{-t, -1, 1, t\}$.

Thus i' and i are not adjacent. Therefore $d_{D(1,t)}(i,i') > 1$.

Case 5. $j_0 > 1$.

Since $j_0 \neq 1$, implies that i_1 and i_2 are in A_2. And we have $g(i_1) = g(i_2) = j_0$, then $i_1 = y_{j_0-1} = i_2$. Hence $i' - i = (l' - l)((t+1)q + r)$, then $d_{D(1,t)}(i,i') = d_{D(1,t)}(0,(l' -$

$l)((t+1)q+r))$. We have $l' - l \in \mathbb{N}^*$ and $s+1 \geq t-1$, hence it follows from Lemma 2.5 that $d_{D(1,t)}(0, (l'-l)((t+1)q+r)) = d_{D(1,t)}(0, (l'-l)t(s+1-\lceil \frac{t}{2} \rceil) + (l'-l)\lceil \frac{t}{2} \rceil) \geq s+1$, then $d_{D(1,t)}(i, i') \geq s+1$.

Thus f is S-packing k-coloring. Therefore $\chi_S(D(1,t)) \leq (\frac{t}{2}+1)q+r+1$. $\square$

Theorem 5.6. *Let $t > 1$ be an even integer and $s \geq \frac{t}{2}$ be an integer, and let $S = (1, s, s, \ldots)$. Then,*

$$\chi_S(D(1,t)) \geq \begin{cases} \left(\frac{t}{2}+1\right)q + \left\lceil \frac{r+1}{2} \right\rceil, & \text{if } r \neq 0; \\ \left(\frac{t}{2}+1\right)q + 1, & \text{otherwise}, \end{cases}$$

where $t\left(s+1-\frac{t}{2}\right) + \frac{t}{2} = (t+1)q + r$ with $q, r \in \mathbb{N}$ and $0 \leq r < t+1$.

Proof. Let $V = [\![0, (t+1)q+r-1]\!]$, and $(t+1)q+r-1-0 = (t+1)q'+r'$. Let H be a subgraph of $D(1,t)$ with vertex set V, and let f be an S-packing $k-coloring$ of subgraph H with $k \in \mathbb{N}^*$. Clearly $k \geq 2$. Let $J' = [\![1, k]\!]$ and $J = J' \setminus \{1\}$ and $V_j = \{v \in V : f(v) = j\}$ and $a_j = |V_j|$ with $j \in J'$.

By Lemma 2.4, $a_j \leq 1$ for every $j \in J$.

Case 1. $r \neq 0$.

Since $(t+1)q+r-1 = (t+1)q'+r'$ and $r \neq 0$, $q' = q$ and $r' = r-1 < t$. Then by Lemma 5.1, $a_1 \leq q'\frac{t}{2} + \lfloor \frac{r'}{2} \rfloor + 1 = q\frac{t}{2} + \lfloor \frac{r-1}{2} \rfloor + 1$. Since

$$(t+1)q + r = |V| = \sum_{j \in J'} |V_j| = a_1 + \sum_{j=2}^{k} a_j \leq q\frac{t}{2} + \left\lfloor \frac{r-1}{2} \right\rfloor + 1 + k - 1,$$

then $k \geq \left(\frac{t}{2}+1\right)q + \lceil \frac{r+1}{2} \rceil$. Hence $\chi_S(H) \geq \left(\frac{t}{2}+1\right)q + \lceil \frac{r+1}{2} \rceil$. Therefore, by Observation 2.2, $\chi_S(D(1,t)) \geq \left(\frac{t}{2}+1\right)q + \lceil \frac{r+1}{2} \rceil$.

Case 2. $r = 0$. Then $q' = q-1$ and $r' = t$. Hence, from Lemma 5.1 we obtain $a_1 \leq q\frac{t}{2}$. Again

$$(t+1)q + r = |V| = a_1 + \sum_{j=2}^{k} a_j \leq q\frac{t}{2} + k - 1,$$

so $k \geq \left(\frac{t}{2}+1\right)q+1$. It follows that $\chi_S(H) \geq \left(\frac{t}{2}+1\right)q+1$, and consequently $\chi_S(D(1,t)) \geq \left(\frac{t}{2}+1\right)q+1$. $\square$

The next corollary follows immediately from Theorem 5.5 and Theorem 5.6.

Corollary 5.7. *Let $t > 1$ be an even integer and $s \geq \max(t-2, \frac{t}{2})$ be an integer, and let $S = (1, s, s, \ldots)$ and $t(s+1-\frac{t}{2}) + \frac{t}{2} = (t+1)q+r$ with $q, r \in \mathbb{N}$ and $0 \leq r < t+1$. Then:*

(a) *If r is even and $r \notin \{0, t\}$, then*

$$\left(\frac{t}{2}+1\right)q + \frac{r}{2} + 1 \leq \chi_S(D(1,t)) \leq \left(\frac{t}{2}+1\right)q + r + 1.$$

(b) *If* $r = 0$, *then*
$$\chi_S(D(1,t)) = \left(\frac{t}{2}+1\right)q+1.$$
In other words, if $s = (t+1)a + t - 1$ *with* $a \in \mathbb{N}$, *then*
$$\chi_S(D(1,t)) = \left(\frac{t}{2}+1\right)q+1.$$

We conclude this paper with several illustrative computations of $\chi_S(D(1,t))$ obtained by applying our corollaries. These examples illustrate the exactness of our results and their behavior for different values of S and t.

Example 5.8. (a) For $S = (1,3,3,\ldots)$ and $t = 4$, Corollary 5.7 gives $\chi_S(D(1,4)) = 7$.

(b) For $S = (1,4,4,\ldots)$ and $t = 2$, Corollary 5.7 gives $\chi_S(D(1,2)) = 7$.

(c) For $S = (2,3,3,\ldots)$ (i.e., $(s, s+1, s+1,\ldots)$ with $s = 2$), Corollary 3.3 yields $\chi_S(D(1,2)) = 7$ and $\chi_S(D(1,3)) = 8$.

(c) For $S = (1,3,3,\ldots)$ and $t \in \{5,3\}$, Corollary 5.4 gives $\chi_S(D(1,5)) = 5$ and $\chi_S(D(1,3)) = 5$.

(d) For $S = (1,7,7,\ldots)$ and $t \in \{9,7,5,3\}$, Corollary 5.4 gives $\chi_S(D(1,9)) = 17$, $\chi_S(D(1,7)) = 17$, $\chi_S(D(1,5)) = 15$, and $\chi_S(D(1,3)) = 11$.

References

[1] B. Benmedjdoub, I. Bouchemakh, and É. Sopena. 2-distance colorings of integer distance graphs. *Discussiones Mathematicae Graph Theory*, 39:589–603, 2019. https://doi.org/10.7151/dmgt.2040.

[2] B. Brešar, J. Ferme, S. Klavžar, and D. Rall. A survey on packing colorings. *Discussiones Mathematicae Graph Theory*, 2021. https://doi.org/10.7151/dmgt.2320. in press.

[3] B. Brešar, S. Klavžar, and D. Rall. On the packing chromatic number of cartesian products, hexagonal lattice, and trees. *Discrete Applied Mathematics*, 155:2303–2311, 2007. https://doi.org/10.1016/j.dam.2007.06.008.

[4] B. Brešar, J. Ferme, and K. Kamenická. S-packing colorings of distance graphs $G(Z, \{2,t\})$. *Discrete Applied Mathematics*, 298:143–154, 2021. https://doi.org/10.1016/j.dam.2021.04.001.

[5] J. Ekstein, P. Holub, and B. Lidický. Packing chromatic number of distance graphs. *Discrete Applied Mathematics*, 160:518–524, 2012. https://doi.org/10.1016/j.dam.2011.11.022.

[6] J. Ekstein, P. Holub, and O. Togni. The packing coloring of distance graphs $D(k,t)$. *Discrete Applied Mathematics*, 167:100–106, 2014. https://doi.org/10.1016/j.dam.2013.10.036.

[7] W. Goddard, S. Hedetniemi, S. Hedetniemi, J. Harris, and D. Rall. Broadcast chromatic numbers of graphs. *Ars Combinatoria*, 86:33–49, 2008.

[8] W. Goddard and H. Xu. The S-packing chromatic number of a graph. *Discussiones Mathematicae Graph Theory*, 32:795–806, 2012. https://doi.org/10.7151/dmgt.1642.

[9] P. Holub and J. Hofman. On S-packing colourings of distance graphs $D(1, t)$ and $D(1, 2, t)$. *Applied Mathematics and Computation*, 447, 2023. https://doi.org/10.1016/j.amc.2023.127855.

[10] F. Kramer and H. Kramer. A survey on the distance-colouring of graphs. *Discrete Mathematics*, 308:422–426, 2008. https://doi.org/10.1016/j.disc.2006.11.059.

[11] O. Togni. On packing colorings of distance graphs. *Discrete Applied Mathematics*, 167:280–289, 2014. https://doi.org/10.1016/j.dam.2013.10.026.

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