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On z-cycle factorizations with two associate classes where z is in $\{4, 4a\}$ with even parameters

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ABSTRACT

Let $K = K(a, p; \lambda_1, \lambda_2)$ be the multigraph with: the number of vertices in each part equal to a; the number of parts equal to p; the number of edges joining any two vertices of the same part equal to λ_1; and the number of edges joining any two vertices of different parts equal to λ_2. The existence of C_4-factorizations of K has been settled when a is even; when $a \equiv 1 \pmod 4$ with one exception; and for very few cases when $a \equiv 3 \pmod 4$. The existence of C_z-factorizations of K has been settled when $a \equiv 1 \pmod z$ and λ_1 is even; when $a \equiv 0 \pmod z$; and when $z = 2a$ where both a and λ_1 is even. In this paper, we give a construction for C_z-factorizations of K for $z \in \{4, 4a\}$ when a is even.

Keywords: cycle, factorization, two associate classes

1. Introduction

Let $K = K(a, p; \lambda_1, \lambda_2)$ denote the graph formed from p vertex-disjoint copies of the multigraph $\lambda_1 K_a$ by joining each pair of vertices in different copies with λ_2 edges. The vertex set, $V(K)$, is always chosen to be $\mathbb{Z}_a \times \mathbb{Z}_p$, with parts $\mathbb{Z}_a \times \{j\}$ for each $j \in \mathbb{Z}_p$; naturally, each part induces a copy of $\lambda_1 K_a$.We say the vertex (i, j) is on *level* i and in *part* j. An edge is said to be a *mixed edge* if it joins vertices in different parts, and is said to be a *pure edge* (in part j) if it joins two vertices in the jth part.

Let C_z denote a cycle of length z. A C_z-factorization is a 2-factorization such that each component of each 2-factor is a cycle of length z; each 2-factor of a C_z-factorization is known as a C_z-factor. C_z-factorizations are also known as *resolvable C_z-decompositions*.

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A $C_{\{z_1,z_2,\ldots,z_k\}}$-factorization is a 2-factorization such that each 2-factor is a C_w-factor where $w \in \{z_1, z_2, \ldots, z_k\}$.

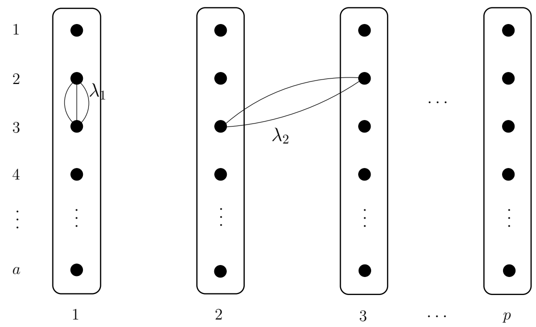

Fig. 1. $K = K(a, p; \lambda_1, \lambda_2)$

There has been considerable interest recently in C_z-decompositions of various graphs, such as complete graphs and complete mutipartite graphs. In the resolvable case, these results are collectively known as addressing the Oberwolfach problem. More recently, the existence problem for C_z-decompositions of K for $z \in \{3, 4\}$ has been solved [3, 4, 5]. Such decompositions are known as *C_z-group-divisible designs with two associate classes*, following the notation of Bose and Shimamoto who considered the existence problem for K_z-group divisible designs. The reason for this name is that the structure can be thought of as partitioning ap symbols, or vertices, into p sets of size a in such a way that symbols that are in the same set in the partition occur together in λ_1 blocks, and are known as *first associates*, whereas symbols that are in different sets in the partition occur together in λ_2 blocks, and are known as *second associates* [2].

C_z-factorizations of K have also been of interest[3]. Recently the existence of a C_4-factorization of K has been completely settled when a is even [1] and when $a \equiv 1 \pmod 4$ with one exception [10, 11]. Some work has also been doen for the case where $a \equiv 3 \pmod 4$ [6]. A general construction for C_z-factorizations of K when z is even, $a \equiv 1 \pmod z$, and λ_1 is even; when $a \equiv 0 \pmod z$ [12]; and when $z = 2a$ where a and λ_1 are even [8]. In this paper, we give a construction for C_z-factorizations of K for $z \in \{4, 4a\}$ when all parameters are even.

Lemma 1.1. *Let $z = 4a$ where a is even. If there exists a C_z-factorization of $K\,(a, p; \lambda_1, \lambda_2)$, then:*

(a) $p \equiv 0$ (*mod* 4),
(b) λ_1 is even, and
(c) $\lambda_2 > 0$.

Proof. Since the number of z-cycles in each C_z-factor is the number of vertices divided by z, z must divide ap, and since $a = z/4$, $p \equiv 0 \pmod 4$.

Each vertex is joined with λ_1 edges to each of the $(a-1)$ other vertices in its own part and with λ_2 edges to each of the $a(p-1)$ vertices in the other parts; so the degree of each vertex is:

$$d_K(v) = \lambda_1(a-1) + \lambda_2 a(p-1).$$

Clearly, since K has a C_z-factorization, it is regular of even degree. The second term is even since a is even. The first term must therefore be even, so since $(a-1)$ is odd, λ_1 must be even. Since $a < z$, each C_z-factor must contain mixed edges; hence $\lambda_2 > 0$. □

Lemma 1.2. *Let $z = 4a$ where a is even. If there exists a C_z-factorization of $K(a, p; \lambda_1, \lambda_2)$, then $\lambda_1 \le \lambda_2 a(p-1)$.*

Proof. Since $a < z$, each C_z-factor contains at most $(a-1)$ pure edges in each part. So each C_z-factor contains at most $(a-1)p$ pure edges. Since there are $\lambda_1\binom{a}{2}p$ pure edges, the number of C_z-factors in any C_z-factorization is at least:

$$\frac{\lambda_1\binom{a}{2}p}{(a-1)p} = \frac{\lambda_1 a}{2}.$$

Each C_z-factor has ap edges, of which at most $(a-1)p = ap - p$ are pure, so there are at least p mixed edges in any C_z-factor. Then the number of mixed edges in any C_z-factorization is at least:

$$\frac{\lambda_1 ap}{2}.$$

Therefore, this number must be at most the number of mixed edges, $\lambda_2\binom{p}{2}a^2$, in K:

$$\frac{\lambda_1 ap}{2} \le \lambda_2\binom{p}{2}a^2,$$

so

$$\lambda_1 \le \lambda_2 a(p-1).$$

□

Lemma 1.3. *Let a be even. There exists a cyclical decomposition of K_a into edge-disjoint Hamiltonian paths such that the ends of the paths are vertices i and $i + a/2$ for $i \in \mathbb{Z}_{a/2}$.*

Proof. Let $i \in \mathbb{Z}_{a/2}$. The ith such Hamiltonian path is

$$h_i = (i, i+1, i+(a-1), i+2, i+(a-2), \dots, i+(a/2-1), i+(a/2+1), i+(a/2)).$$

See Figure 2.

Note that

$$K_a = \bigcup_{i\in\mathbb{Z}_{a/2}} h_i,$$

and the ends of the Hamiltonian paths are always i and $i + a/2 \pmod a$. Let

$$H_a = \{h_i | i \in \mathbb{Z}_{a/2}\}.$$

□

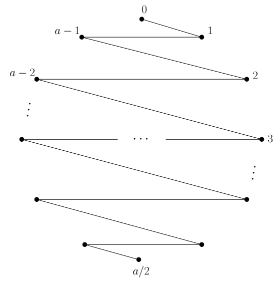

Fig. 2. Hamiltonian path of K_a

Theorem 1.4. [7] *Suppose z is even and $p \equiv 0 \pmod{z}$. C_z-factorizations of λK_p exist for all even λ.*

Theorem 1.5. [9] *Suppose $z > 2$. There exists a C_z-factorization of $K(a, p; 0, 1)$ if and only if $K \neq K(6, 2; 0, 1)$ where $z = 6$.*

Theorem 1.6. [1] *Let a be even. There exists a C_4-factorization of $K(a, p; \lambda_1, \lambda_2)$.*

2. z is $4a$ and λ_2 is even

When $z = 4a$, there are two cases to consider with respect to the upper bound on λ_1:

(a) $\lambda_1 = 0$ or $\lambda_2 a(p - 1)$,

(b) $\lambda_1 < \lambda_2 a(p - 1)$.

If λ_1 is equal to zero or its upper bound, then all of the factors in the factorization of K will be C_{4a}-factors; however, if λ_1 is greater than zero but less than its upper bound, then some of the factors in the factorization of K will be C_{4a}-factors while others will be C_4-factors. Specifically, any factor that uses pure edges will be a C_{4a}-factor while any factor consisting entirely of mixed edges will be a C_4-factor.

Theorem 2.1. *Let $z = 4a$ where $a \geq 2$ and $\lambda_2 > 0$ are even. There exists a $C_{\{4,4a\}}$-factorization of $K = K(a, p; \lambda_1, \lambda_2)$ if and only if*

(a) *$p \equiv 0 \pmod{4}$,*

(b) *λ_1 is even, and*

(c) *$\lambda_1 \leq \lambda_2 a(p - 1)$.*

Proof. The necessity of these conditions follows from Lemmas 1.1 and 1.2. So now assume that K satisfies conditions (1–3). If $\lambda_1 = 0$, then the required factorization is given by Theorem 1.5. So we may also assume that $\lambda_1 > 0$.

Given part size a, there are a mixed differences, 0, 1, ... $a - 1$, between the levels of the vertices in each part. Given two parts, m and n, an edge of mixed difference 0 would join the vertex on level ℓ in part m to the vertex on level ℓ in part n. An edge of mixed difference d would join a vertex on level ℓ in part m to the vertex on level $(\ell + d) \pmod{a}$ in part n.

Using Theorem 1.4, let

$$\pi = \{\pi_s \mid s \in \mathbb{Z}_{\lambda_2(p-1)/2}\},$$

be the s^{th} C_4-factor of a C_4-factorization of $\lambda_2 K_p$. For each $s \in \mathbb{Z}_{\lambda_2(p-1)/2}$, $d \in \mathbb{Z}_a$, and $\ell \in \mathbb{Z}_a$, let

$$M(s,d,\ell) = \{((\ell,c_1),(\ell+d,c_2),(\ell,c_3),(\ell+d,c_4)) \mid (c_1,c_2,c_3,c_4) \in \pi, c_1 < c_2,c_3,c_4\},$$

be a 4-cycle consisting entirely of mixed edges. See Figure 3 for an example.

Then for each $s \in \mathbb{Z}_{\lambda_2(p-1)/2}$ and for each $d \in \mathbb{Z}_a$, define the following C_4-factor of $K(a,p;0,\lambda_2)$ that consists entirely of mixed edges:

$$M(s,d) = \bigcup_{\ell \in \mathbb{Z}_a} M(s,d,\ell).$$

Define

$$M^+(s,d,\ell) = \{((\ell,c_1),(\ell+d,c_2)),((\ell,c_3),(\ell+d,c_4)) \mid (c_1,c_2,c_3,c_4) \in \pi, c_1 < c_2,c_3,c_4\},$$

and

$$M^-(s,d,\ell) = \{((\ell,c_1),(\ell+d,c_4)),((\ell+d,c_2),(\ell,c_3)) \mid (c_1,c_2,c_3,c_4) \in \pi, c_1 < c_2,c_3,c_4\},$$

such that

$$M(s,d,\ell) = M^+(s,d,\ell) \cup M^-(s,d,\ell),$$

and

$$M(s,d) = \bigcup_{\ell \in \mathbb{Z}_a} M^+(s,d,\ell) \cup \bigcup_{\ell \in \mathbb{Z}_a} M^-(s,d,\ell).$$

See Figure 4 for an example of $M^+(s,d,\ell)$ and $M^-(s,d,\ell)$.

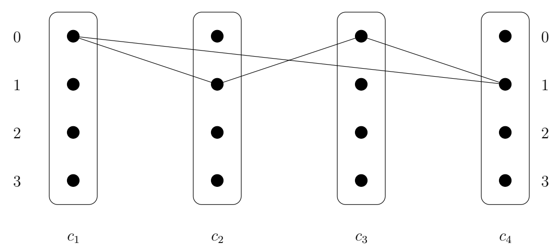

Fig. 3. An example of $M(s,d,\ell)$

Notice that these C_4-factors can be used to produce a C_4-factorization of $K(a,p;0,\lambda_2)$, namely:

$$\bigcup_{s \in \mathbb{Z}_{\lambda_2(p-1)/2}} \bigcup_{d \in \mathbb{Z}_a} M(s,d).$$

However, we have pure edges to use too, which is accomplished by spreading the edges of the 4-cycles in $M(s,d)$ among a C_{4a}-factors p edges at a time. Each such C_{4a}-factor contains p mixed edges of $M(s,d)$ for some $d \in \mathbb{Z}_a$ together with a Hamiltonian path in

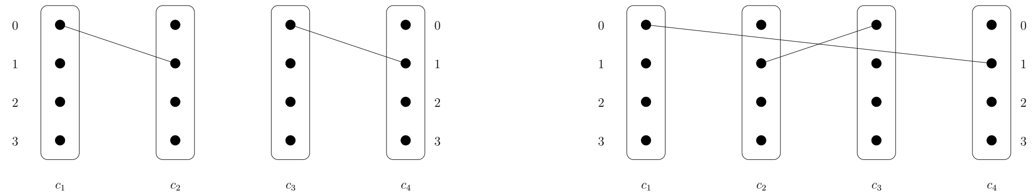

Fig. 4. An example of $M^+(s,d,\ell)$ and $M^-(s,d,\ell)$

each part. More specifically, for each $\ell \in \mathbb{Z}_a$ and $k \in \mathbb{Z}_p$, using Lemma 1.3, let $h_\ell(k)$ be the Hamiltonian path of a cyclical, edge-disjoint Hamiltonian path decomposition of K_a on the vertex set $\mathbb{Z}_a \times \{k\}$ where the ends of the path are ℓ and $\ell + a/2 \pmod a$.

For $s \in \mathbb{Z}_{\lambda_2(p-1)/2}$, $d \in \mathbb{Z}_a$, and $\ell \in \mathbb{Z}_a$, let

$$P(s,d,\ell) = \{h_\ell(c_1) \cup h_{\ell+d}(c_2) \cup h_\ell(c_3) \cup h_{\ell+d}(c_4) \cup M^+(s,d,\ell) \cup M^-(s,d,\ell+a/2) \mid (c_1,c_2,c_3,c_4) \in \pi_s, c_1 < c_2, c_3, c_4\},$$

be such a C_{4a}-factor of K; see Figure 5 for an example. Notice that

$$P(s,d) = \bigcup_{\ell \in \mathbb{Z}_a} P(s,d,\ell),$$

contains

(a) each pure edge in each part exactly twice, and

(b) precisely the mixed edges in $M(s,d)$.

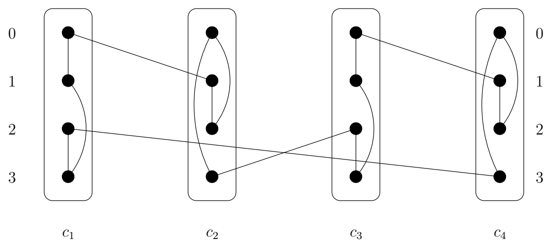

Fig. 5. An example of $P(s,d,\ell)$

Let $S = \{(s,d) \mid s \in \mathbb{Z}_{\lambda_2(p-1)}, d \in \mathbb{Z}_a\}$. Let $S_1 \subseteq S$ have size $\frac{\lambda_1}{2}$. Notice that by condition 3. of the theorem, $\lambda_1 \le \lambda_2 a(p-1)$, so $|S_1| = \frac{\lambda_1}{2} \le \frac{\lambda_2 a(p-1)}{2} = |S|$, so such a set $|S_1|$ exists. Then

$$\bigcup_{(s,d)\in S_1} P(s,d),$$

is a set of $\frac{\lambda_1 a}{2}$ C_{4a}-factors that contains each pure edge $2|S_1| = \lambda_1$ times by (a), and uses precisely the mixed edges in

$$\bigcup_{(s,d)\in S_1} M(s,d),$$

by (b). Therefore, the required $C_{\{4,4a\}}$-factorization of K is defined by

$$P = \left(\bigcup_{(s,d)\in S_1} P(s,d) \right) \cup \left(\bigcup_{(s,d)\in S\setminus S_1} M(s,d) \right).$$

Notice that

$$\begin{aligned} |P| &= a|S_1|+|S \setminus S_1| \\ &= \frac{\lambda_1 a}{2} + \frac{\lambda_2 a(p-1)}{2} - \frac{\lambda_1}{2} \\ &= \frac{\lambda_1(a-1)}{2} + \frac{\lambda_2 a(p-1)}{2}, \end{aligned}$$

as required. □

References

[1] E. Billington and C. Rodger. Resolvable 4-cycle group divisible designs with two associate classes: part size even. *Discrete Mathematics*, 308:303–307, 2008. https://doi.org/10.1016/j.disc.2006.11.043.

[2] R. Bose and T. Shimamoto. Classification and analysis of partially balanced incomplete block designs with two associate classes. *Journal of the American Statistical Association*, 47:151–184, 1952. https://doi.org/10.1080/01621459.1952.10501161.

[3] H. Fu and C. Rodger. Group divisible designs with two associate classes: $n = 2$ or $m = 2$. *Journal of Combinatorial Theory, Series A*, 83:94–117, 1998. https://doi.org/10.1006/jcta.1998.2868.

[4] H. Fu and C. Rodger. 4-cycle group-divisible designs with two associate classes. *Combinatorics, Probability and Computing*, 10:317–343, 2001. https://doi.org/10.1017/S0963548301004734.

[5] H. Fu, C. Rodger, and D. Sarvate. The existence of group divisible designs with first and second associates, having block size 3. *Ars Combinatoria*, 54:33–50, 2000.

[6] C. Goss, M. Tiemeyer, and R. Waller. On C_4-factorizations with two associate classes where $a \equiv 3 \pmod 4$ and small. *Congressus Numerantium*, 219:97–128, 2014.

[7] P. Gvozdjak. On the oberwolfach problem for complete multigraphs. *Discrete Mathematics*, 173:61–69, 1997. https://doi.org/10.1016/S0012-365X(96)00096-9.

[8] J. Lambert and M. Tiemeyer. On z-cycle factorizations with two associate classes where z is $2a$ and a is even. *Electronic Journal of Graph Theory & Applications*, 11(2):447–454, 2023. https://dx.doi.org/10.5614/ejgta.2023.11.2.9.

[9] J. Liu. A generalization of the oberwolfach problem and C_t-factorizations of complete equipartite grpahs. *Journal of Combinatorial Designs*, 8:42–49, 2000. https://doi.org/10.1002/(SICI)1520-6610(2000)8:1%3C42::AID-JCD6%3E3.0.CO;2-R.

[10] C. Rodger and M. Tiemeyer. C_4-factorizations with two associate classes. *Australasian Journal of Combinatorics*, 40:217–228, 2008.

[11] C. Rodger and M. Tiemeyer. C_4-factorizations with two associate classes, λ_1 is odd. *Australasian Journal of Combinatorics*, 50:259–278, 2011.

[12] M. Tiemeyer. On z-cycle factorizations with two associate classes where z is even and a is 0 or 1 mod z. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 105:11–19, 2018.

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