

The $\{2\}$ -domination in rooted product graphs

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ABSTRACT

A $\{2\}$ -dominating function ($\{2\}$ DF) on a graph $G = (V(G), E(G))$ is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that $f(N[v]) \geq 2$ for every $v \in V(G)$, where $N[v]$ is the closed neighbourhood of v . The $\{2\}$ -domination number of G is the minimum weight $\omega(f) = \sum_{v \in V(G)} f(v)$ among all $\{2\}$ -dominating functions on G . In this article, we prove that if G and H are graphs with no isolated vertex, then for any vertex $v \in V(H)$ there are six closed formulas for the $\{2\}$ -domination number of the rooted product graph $G \circ_v H$. We also characterize the graph G and H that satisfy each of these formulas.

Keywords: $\{2\}$ -domination, quasi $\{2\}$ -dominating pair, rooted product graph

1. Introduction

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. The *order* $|V(G)|$ of G is denoted by $n(G)$. The *open neighborhood* of $v \in V(G)$ is $N(v) = \{w \in V(G) : vw \in E(G)\}$ and its *closed neighborhood* is $N[v] = N(v) \cup \{v\}$. When expressing the given graph G , $N(v)$ is denoted as $N_G(v)$. The $\{2\}$ -domination in graphs has been studied in [1, 2, 6, 8, 11]. A $\{2\}$ -dominating function ($\{2\}$ DF) on a graph G is a function $f : V(G) \rightarrow \{0, 1, 2\}$ such that $f(N[v]) \geq 2$ for every $v \in V(G)$. A $\{2\}$ DF f gives an ordered partition $f(A_0, A_1, A_2)$, where $A_i := \{x \in V(G) : f(x) = i\}$. The $\{2\}$ -domination number of G , denoted by $\gamma_{\{2\}}(G)$, is the minimum weight $\omega(f) := \sum_{v \in V(G)} f(v)$ among all $\{2\}$ -dominating functions on G . A $\gamma_{\{2\}}(G)$ -function is a $\{2\}$ DF of G with the weight $\gamma_{\{2\}}(G)$.

In [3], Godsil and McKay introduced the concept of rooted product graph. Given a

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graph G of order $n(G)$ and a graph H with root vertex v , the rooted product graph $G \circ_v H$ is the graph obtained from G and H by taking one copy of G and $n(G)$ copies of H and identifying the i th vertex of G with the root vertex v in the i th copy of H for every $i \in \{1, \dots, n\}$. For each $x \in V(G)$, the copy of $G \circ_v H$ containing x is denoted by H_x , and for every function f on $G \circ_v H$, the restriction of f to $V(H_x)$ and $V(H_x) \setminus \{x\}$ is denoted by f_x and f_x^- , respectively.

Various domination parameters on rooted product graphs have been studied in [4, 5, 7, 9, 10, 12]. In this article, we study the $\{2\}$ -domination number of rooted product graphs following the methodology of [9].

In the rest of this section, we present some definition and basic results. Given a graph G , a leaf of G is a vertex of degree one. A support vertex is a vertex adjacent to a leaf. We denote the sets of leaves and support vertices by $\mathcal{L}(H)$ and $\mathcal{S}(H)$, respectively. For a subset S of $V(G)$, $G - S$ denotes the subgraph of G induced by $V(G) \setminus S$.

Theorem 1.1. *Let G be a graph with no isolated vertex. Then $\gamma(G) + 1 \leq \gamma_{\{2\}}(G) \leq 2\gamma(G)$.*

Now, we introduce a new domination parameter which is expressed as one of possible values in our main theorem (See Theorem 2.5). An ordered pair (K, ϕ) is a quasi $\{2\}$ -dominating pair of G if $K \subseteq V(G)$ and ϕ is a $\{2\}$ DF of $G - K$. The quasi $\{2\}$ -domination number of G is defined to be $\gamma_{q\{2\}}(G) := \min\{|K| + \omega(\phi) : K \subseteq V(G) \text{ and } \phi \text{ is a } \{2\}\text{DF of } G - K\}$. A $\gamma_{q\{2\}}(G)$ -pair is a quasi $\{2\}$ -dominating pair (K, ϕ) which satisfies the condition $\gamma_{q\{2\}}(G) = |K| + \omega(\phi)$.

2. $\{2\}$ -domination in rooted product graphs

In this section, firstly we show that for any vertex $v \in V(H)$ there are six possible expressions, in terms of domination parameters of the factor graphs, for the $\{2\}$ -domination number of the rooted product graph $G \circ_v H$. Secondly, we characterize the graphs G and H that satisfy each of these expressions

2.1. Six types of $\{2\}$ -domination number

Proposition 2.1. *The following statements hold for graphs G and H with no isolated vertex, and any $v \in V(H)$.*

- (i) $\gamma_{\{2\}}(G \circ_v H) \leq n(G)\gamma_{\{2\}}(H)$.
- (ii) If $v \in \mathcal{S}(H)$, then $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$.
- (iii) If $v \in V(H) \setminus \mathcal{S}(H)$, then $\gamma_{\{2\}}(G \circ_v H) \leq \gamma_{\{2\}}(G) + n(G)\gamma_{\{2\}}(H - v)$.

Proof. (i) Let f be a $\gamma_{\{2\}}(H)$ -function, and let $g : V(G \circ_v H) \rightarrow \{0, 1, 2\}$ be a function such that g_x is the function on H_x induced by f for each $x \in V(G)$. Then it is easy to see that g is a $\{2\}$ DF of $G \circ_v H$. This implies that $\gamma_{\{2\}}(G \circ_v H) \leq \omega(g) = \sum_{x \in V(G)} \omega(f_x) = n(G)\gamma_{\{2\}}(H)$.

(ii) Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function, and let $x \in V(G)$. Assume that $v \in \mathcal{S}(H)$. Since $f_x(N[y]) \geq 2$ for each $y \in \mathcal{L}(H_x) \cap N(x)$, f_x is a $\{2\}$ DF of H_x . This

implies that $\omega(f_x) \geq \gamma_{\{2\}}(H)$. Thus, $\gamma_{\{2\}}(G \circ_v H) = \omega(f) = \sum_{x \in V(G)} \omega(f_x) \geq n(G)\gamma_{\{2\}}(H)$.

By (i), the equality follows.

(iii) Assume that $v \in V(H) \setminus \mathcal{S}(H)$. Let f be a $\gamma_{\{2\}}(H - v)$ -function, and let g be a function on $(G \circ_v H) \setminus V(G)$ such that $g|_{H_x - x}$ is the function on $H_x - x$ induced by f for each $x \in V(G)$. For any $\gamma_{\{2\}}(G)$ -function h , it follows that $g \cup h$ is a $\{2\}$ DF of $G \circ_v H$. Thus, $\gamma_{\{2\}}(G \circ_v H) \leq \omega(g \cup h) = \gamma_{\{2\}}(G) + n(G)\gamma_{\{2\}}(H - v)$. \square

By Proposition 2.1(ii), we consider rooted product graphs where the root vertex is not a support vertex.

Lemma 2.2. *Let H be a graph with no isolated vertex. If $v \in V(H) \setminus \mathcal{S}(H)$, then $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H) - 2$. Furthermore, if $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$, then $N(v) \cap (A_1 \cup A_2) = \emptyset$ for every $\gamma_{\{2\}}(H - v)$ -function $f(A_0, A_1, A_2)$.*

Proof. Let $v \in V(H) \setminus \mathcal{S}(H)$, and let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(H - v)$ -function. For every $u \in N(v)$, the function $g(A_0, A_1 \cup \{v, u\}, A_2)$ is a $\{2\}$ DF of H . Thus, $\gamma_{\{2\}}(H) - 2 \leq \omega(g) - 2 \leq \omega(f) = \gamma_{\{2\}}(H - v)$.

Assume that $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$. If there exists a vertex $u \in (A_1 \cup A_2) \cap N(v)$, then $h(A_0, A_1 \cup \{v\}, A_2)$ is a $\{2\}$ DF of H , a contradiction. Thus, $N(v) \cap (A_1 \cup A_2) = \emptyset$. \square

Lemma 2.3. *If f is a $\gamma_{\{2\}}(G \circ_v H)$ -function and $x \in V(G)$, then $\omega(f_x) \geq \gamma_{\{2\}}(H) - 2$. Furthermore, if $\omega(f_x) = \gamma_{\{2\}}(H) - 2$, then $f(N_H[x]) = 0$.*

Proof. Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function and $x \in V(G)$. Define a function f' on H_x by $f'(x) = 2$ and $f'(v) = f_x^-(v)$ for $v \in V(H_x) \setminus \{x\}$. Then f' is a $\{2\}$ DF of H_x . Thus, $\gamma_{\{2\}}(H) - 2 = \gamma_{\{2\}}(H_x) - 2 \leq \omega(f') - 2 \leq \omega(f_x)$.

Let x be a vertex of $V(G)$ such that $\omega(f_x) = \gamma_{\{2\}}(H) - 2$. Suppose that there exists $y \in V(G)$ such that $y \in N_{H_x}[x] \cap (A_1 \cup A_2)$. If $f(y) = 2$, then clearly f_x is a $\gamma_{\{2\}}(H_x)$ -function, a contradiction. If $f(y) = 1$, then the function replaced the value of $f(y)$ by 2 is a $\{2\}$ DF of H_x with the weight $\gamma_{\{2\}}(H) - 1$, a contradiction. Thus, we have $f(N_H[x]) = 0$. \square

For every $\gamma_{\{2\}}(G \circ_v H)$ -function f , define the following sets

$$\mathcal{A}_f = \{x \in V(G) : \omega(f_x) \geq \gamma_{\{2\}}(H)\},$$

$$\mathcal{B}_f = \{x \in V(G) : \omega(f_x) = \gamma_{\{2\}}(H) - 1\},$$

and

$$\mathcal{C}_f = \{x \in V(G) : \omega(f_x) = \gamma_{\{2\}}(H) - 2\}.$$

Lemma 2.4. *Given a $\gamma_{\{2\}}(G \circ_v H)$ -function $f(A_0, A_1, A_2)$ with $\mathcal{B}_f \neq \emptyset$ and $\mathcal{C}_f = \emptyset$, the following statements hold.*

- (i) If $\mathcal{B}_f \cap (A_1 \cup A_2) \neq \emptyset$, then $\gamma_{\{2\}}(G \circ_v H) = n(G)(\gamma_{\{2\}}(H) - 1)$.
(ii) If $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$, then $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$, and $\gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1) \leq \gamma_{\{2\}}(G \circ_v H) \leq \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$.

Proof. (i) Let $z \in \mathcal{B}_f \cap (A_1 \cup A_2)$, and let g be a function on $G \circ_v H$ such that g_x is the function on H_x induced by f_z for each $x \in V(G)$. Then it is easy to see that g is a $\{2\}$ DF of $G \circ_v H$. This implies that $\gamma_{\{2\}}(G \circ_v H) \leq \omega(g) = \sum_{x \in V(G)} \omega(g_x) = n(G)(\gamma_{\{2\}}(H) - 1)$. Since $\mathcal{C}_f = \emptyset$, we deduce that $\gamma_{\{2\}}(G \circ_v H) = \omega(f) \geq n(G)(\gamma_{\{2\}}(H) - 1)$. The equality follows.

(ii) Let $z \in \mathcal{B}_f$. Since $z \notin (A_1 \cup A_2)$, we deduce that $z \notin \mathcal{S}(H_z)$, which implies that f_z is a $\{2\}$ DF of $H_z - z$. Thus, $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H_z - z) \leq \omega(f_z) = \gamma_{\{2\}}(H) - 1$. By Lemma 2.2, it follows from the above inequality that $\gamma_{\{2\}}(H - v) \in \{\gamma_{\{2\}}(H) - 1, \gamma_{\{2\}}(H) - 2\}$.

If $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$, then by Proposition 2.1(iii) and $\mathcal{C}_f = \emptyset$, we have that $n(G)(\gamma_{\{2\}}(H) - 1) \leq \gamma_{\{2\}}(G \circ_v H) \leq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$, which implies that $\gamma_{\{2\}}(G \circ_v H) = n(G)(\gamma_{\{2\}}(H) - 1)$ and so $\mathcal{B}_f = V(G)$. Since $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$, this implies that f_x is a $\{2\}$ DF of H with the weight $\gamma_{\{2\}}(H) - 1$, a contradiction. Thus, $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$.

Now, we prove the upper bound of $\gamma_{\{2\}}(G \circ_v H)$. Let (K, ϕ) be a $\gamma_{q\{2\}}(G)$ -pair, h a $\gamma_{\{2\}}(H - v)$ -function and g a $\gamma_{\{2\}}(H)$ -function. We construct a $\{2\}$ DF f of $G \circ_v H$ as follows. If $x \in K$, then f_x is equal to g . If $x \in V(G) \setminus K$, then f_x^- is equal to h . The $f|_{V(G) \setminus K}$ is defined to be the $\gamma_{\{2\}}(G - K)$ -function ϕ . Then it is easy to see that f is a $\{2\}$ DF of $G \circ_v H$, which implies that

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) \leq \omega(f) &= |K|\gamma_{\{2\}}(H) + \omega(\phi) + (n(G) - |K|)(\gamma_{\{2\}}(H) - 1) \\ &= \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

We prove the lower bound of $\gamma_{\{2\}}(G \circ_v H)$. Note that $f_x(N[x]) \leq 1$ for each $x \in \mathcal{B}_f$, which implies that \mathcal{A}_f is a dominating set of G . Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &= \sum_{x \in \mathcal{A}_f} \omega(f_x) + \sum_{x \in \mathcal{B}_f} \omega(f_x) \\ &\geq |\mathcal{A}_f|\gamma_{\{2\}}(H) + |\mathcal{B}_f|(\gamma_{\{2\}}(H) - 1) \\ &\geq |\mathcal{A}_f| + n(G)(\gamma_{\{2\}}(H) - 1) \\ &\geq \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

□

Theorem 2.5. *Let G and H be graphs with no isolated vertex. If $v \in V(H)$, then $\gamma_{\{2\}}(G \circ_v H) \in \{n(G)\gamma_{\{2\}}(H), n(G)(\gamma_{\{2\}}(H) - 1), \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1), \gamma_I(G) + n(G)(\gamma_{\{2\}}(H) - 1), \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1), \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)\}$.*

Proof. Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function, and let $\mathcal{A}_f, \mathcal{B}_f$ and \mathcal{C}_f be the subsets of $V(G)$ defined above. We consider the following cases.

Case 1. $\mathcal{B}_f \cup \mathcal{C}_f = \emptyset$. Then $\mathcal{A}_f = V(G)$ and so $\gamma_{\{2\}}(G \circ_v H) \geq n(G)\gamma_{\{2\}}(H)$. By Proposition 2.1(i), we have that $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$.

Case 2. $\mathcal{B}_f \neq \emptyset$ and $\mathcal{C}_f = \emptyset$. If $\mathcal{B}_f \cap (A_1 \cup A_2) \neq \emptyset$, then it follows from Lemma 2.4(i) that $\gamma_{\{2\}}(G \circ_v H) = n(G)(\gamma_{\{2\}}(H) - 1)$.

From now on, assume that $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$ in Case 2. Observe that if $x \in \mathcal{B}_f$, then $f(M) \leq 1$, where $M := N_{H_x}(x) \cap (A_1 \cup A_2)$. Otherwise, f_x is a $\{2\}$ DF of H_x with the weight less than $\gamma_{\{2\}}(H)$, a contradiction.

Now we define the following subsets. Let $\mathcal{B}'_f = \{x \in \mathcal{B}_f : |N_{H_x}(x) \cap A_1| = 1\}$ and $\mathcal{B}''_f = \{x \in \mathcal{B}_f : |N_{H_x}(x) \cap A_1| = 0\}$, and let $\mathcal{A}'_f = \{x \in \mathcal{A}_f : \omega(f_x) = \gamma_{\{2\}}(H)\}$ and $\mathcal{A}''_f = \{x \in \mathcal{A}_f : \omega(f_x) \geq \gamma_{\{2\}}(H) + 1\}$. Note that $\mathcal{A}''_f \subseteq (A_1 \cup A_2)$. We consider the following three subcases in Case 2.

Subcase 2.1. $\mathcal{B}'_f \neq \emptyset$.

Let $y \in \mathcal{B}'_f$. Define a function $g : V(G \circ_v H) \rightarrow \{0, 1, 2\}$ by $g_x^- = f_y^-$ for each $x \in V(G)$ and $g|_{V(G)}$ is a $\{0, 1\}$ -function such that the inverse image of 1 is a minimum dominating set of G . Then g is a $\{2\}$ DF of $G \circ_v H$. It follows that $\omega(g) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1) \geq \gamma_{\{2\}}(G \circ_v H)$. By the lower bound of Lemma 2.4(ii), we have $\gamma_{\{2\}}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$.

Subcase 2.2. $\mathcal{B}'_f = \emptyset$ and there exists $z \in \mathcal{A}'_f$ such that f_z is a $\gamma_{\{2\}}(H_z)$ -function with $f(z) > 0$.

Subcase 2.2.1. There exists no $z \in \mathcal{A}'_f$ such that f_z is a $\gamma_{\{2\}}(H_z)$ -function with $f(z) = 2$. This implies that $\omega(f_u) = \gamma_{\{2\}}(H) + 1$ for $u \in \mathcal{A}''_f$.

To construct a $\{2\}$ DF of $G \circ_v H$, let $v \in \mathcal{B}_f$, and let $y \in \mathcal{A}'_f$ such that f_y is a $\gamma_{\{2\}}(H_y)$ -function with $f(y) = 1$. Let $s \in \mathcal{A}''_f$, and let (B_0, B_1, B_2) be a $\gamma_I(G)$ -function. Without loss of generality, we can assume that $f(s) = 2$. Define a function g on $G \circ_v H$ by $g_x = f_y$ for $x \in B_1$, $g_x = f_s$ for $x \in B_2$ and $g_x = f_v$ for $x \in B_0$. Then g is a $\{2\}$ DF of $G \circ_v H$. Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &\leq \sum_{x \in B_0} \omega(g_x) + \sum_{x \in B_1} \omega(g_x) + \sum_{x \in B_2} \omega(g_x) \\ &= |B_0|(\gamma_{\{2\}}(H) - 1) + |B_1|\gamma_{\{2\}}(H) + |B_2|(\gamma_{\{2\}}(H) + 1) \\ &= |B_1| + 2|B_2| + n(G)(\gamma_{\{2\}}(H) - 1) \\ &= \gamma_I(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

Since $\mathcal{B}_f = \mathcal{B}''_f$, it follows that $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$. This condition and $\mathcal{C}_f = \emptyset$ imply that $f|_{V(G)}$ is an IDF of G . So, $\gamma_I(G) \leq \omega(f|_{V(G)}) \leq |\mathcal{A}'_f| + 2|\mathcal{A}''_f|$. Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &= \sum_{x \in \mathcal{A}'_f} \omega(f_x) + \sum_{x \in \mathcal{A}''_f} \omega(f_x) + \sum_{x \in \mathcal{B}_f} \omega(f_x) \\ &\geq |\mathcal{A}'_f|\gamma_{\{2\}}(H) + |\mathcal{A}''_f|(\gamma_{\{2\}}(H) + 1) + |\mathcal{B}_f|(\gamma_{\{2\}}(H) - 1) \\ &= |\mathcal{A}'_f| + 2|\mathcal{A}''_f| + n(G)(\gamma_{\{2\}}(H) - 1) \\ &\geq \gamma_I(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

It follows from the above inequalities that $\gamma_{\{2\}}(G \circ_v H) = \gamma_I(G) + n(G)(\gamma_{\{2\}}(H) - 1)$.

Subcase 2.2.2. There exists $z \in \mathcal{A}'_f$ such that f_z is a $\gamma_{\{2\}}(H_z)$ -function with $f(z) = 2$.

Without loss of generality, we can assume that $f(x) = 2$ for $x \in \mathcal{A}_f$. Let $y \in \mathcal{B}_f''$.

Define a function g by $g_x = f_z$ for $x \in X$, where X is a minimum dominating set of G , and $g_x = f_y$ for $x \in V(G) \setminus X$. Then g is a $\{2\}$ DF of $G \circ_v H$. Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &\leq \sum_{x \in X} \omega(g_x) + \sum_{x \in V(G) \setminus X} \omega(g_x) \\ &= |X|(\gamma_{\{2\}}(H)) + |V(G) \setminus X|(\gamma_{\{2\}}(H) - 1) \\ &= |X| + n(G)(\gamma_{\{2\}}(H) - 1) \\ &= \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

Since $\mathcal{B}_f = \mathcal{B}_f''$, it follows that $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$. This condition and $\mathcal{C}_f = \emptyset$ imply that \mathcal{A}_f is a dominating set of G . So, $\gamma(G) \leq |\mathcal{A}_f|$. Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &= \sum_{x \in \mathcal{A}_f} \omega(f_x) + \sum_{x \in \mathcal{B}_f} \omega(f_x) \\ &\geq |\mathcal{A}_f|(\gamma_{\{2\}}(H)) + |\mathcal{B}_f|(\gamma_{\{2\}}(H) - 1) \\ &= |\mathcal{A}_f| + n(G)(\gamma_{\{2\}}(H) - 1) \\ &\geq \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

It follows from the above inequalities that $\gamma_{\{2\}}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$.

Subcase 2.3. $\mathcal{B}_f' = \emptyset$, and

(1) $\mathcal{A}_f' = \emptyset$ or (2) for any $x \in \mathcal{A}_f'$, either f_x is not a $\gamma_{\{2\}}(H_x)$ -function or $x \in A_0$.

Note that the condition (2) means that $\mathcal{A}_f' \neq \emptyset$ and every $x \in \mathcal{A}_f'$ satisfies one of the following statements.

- (a) f_x is a $\gamma_{\{2\}}(H_x)$ -function such that $x \in A_0$.
- (b) f_x is not a $\{2\}$ DF of H_x and $x \in A_1$.
- (c) f_x is not a $\{2\}$ DF of H_x and $x \in A_0$.

Since (c) can be replaced by (a), we only assume the cases of (a) and (b).

Let $\mathcal{A}'_{f,0} = \{x \in \mathcal{A}_f' \mid x \text{ satisfies the condition (a)}\}$. Define a function $h : V(G) \setminus \mathcal{A}'_{f,0} \rightarrow \{0, 1\}$ by $h(x) = 0$ for $x \in \mathcal{B}_f$, $h(x) = 1$ for $\mathcal{A}_f' \setminus \mathcal{A}'_{f,0}$, $h(x) = 1$ for every $x \in \mathcal{A}'_{f,0}$ with $N(x) \cap (A_1 \cup A_2) \neq \emptyset$ and $h(x) = 2$ for every $x \in \mathcal{A}'_{f,0}$ with $N(x) \cap (A_1 \cup A_2) = \emptyset$. Then it is easy to see that h is a $\{2\}$ DF of $G - \mathcal{A}'_{f,0}$. This implies that $(\mathcal{A}'_{f,0}, h)$ is a quasi $\{2\}$ -dominating pair of G . So, $\gamma_{q\{2\}}(G) \leq |\mathcal{A}'_{f,0}| + \omega(h) \leq |\mathcal{A}'_{f,0}| + (|\mathcal{A}_f' \setminus \mathcal{A}'_{f,0}| + 2|\mathcal{A}'_{f,0}|) = |\mathcal{A}_f'| + 2|\mathcal{A}''_f|$. Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &= \sum_{x \in \mathcal{A}''_f} \omega(f_x) + \sum_{x \in \mathcal{A}'_f} \omega(f_x) + \sum_{x \in \mathcal{B}_f} \omega(f_x) \\ &\geq |\mathcal{A}''_f|(\gamma_{\{2\}}(H) + 1) + |\mathcal{A}'_f|\gamma_{\{2\}}(H) + |\mathcal{B}_f|(\gamma_{\{2\}}(H) - 1) \\ &= 2|\mathcal{A}''_f| + |\mathcal{A}'_f| + n(G)(\gamma_{\{2\}}(H) - 1) \\ &\geq \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

It follows from Lemma 2.4(ii) that $\gamma_{\{2\}}(G \circ_v H) = \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$.

Case 3. $\mathcal{C}_f \neq \emptyset$. Let $x \in \mathcal{C}_f$. By Lemma 2.3, $f(N_H[x]) = 0$. This implies that $v \notin \mathcal{S}(H)$. So, f_x^- is a $\{2\}$ DF of $H_x - x$. Thus, $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H_x - x) \leq \omega(f_x^-) = \gamma_{\{2\}}(H) - 2$. It follows from Lemma 2.2 that $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$.

Now, to prove that $\omega(f) \geq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$, we show that $g(\mathcal{C}_f, \mathcal{B}_f, \mathcal{A}_f)$ is a $\{2\}$ DF of G . If $x \in \mathcal{C}_f$, then $f_x(N[x]) = 0$. So, $f(N(x) \cap V(G)) \geq 2$, which implies that $g(N[x]) \geq 2$. If $x \in \mathcal{A}_f$, then clearly $g(N[x]) \geq 2$. Assume that $x \in \mathcal{B}_f$. Note that $f(x) \neq 2$ for each $x \in \mathcal{B}_f$. Then either $f_x(N(x)) = 0$ or $f_x(N(x)) = 1$ and $f_x(x) = 0$, since f_x is not a $\{2\}$ DF of H_x . In either case, $N(x) \cap (\mathcal{A}_f \cup \mathcal{B}_f) \neq \emptyset$ and so $g(N[x]) \geq 2$. Thus,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &= \sum_{x \in \mathcal{A}_f} \omega(f_x) + \sum_{x \in \mathcal{B}_f} \omega(f_x) + \sum_{x \in \mathcal{C}_f} \omega(f_x) \\ &\geq |\mathcal{A}_f| \gamma_{\{2\}}(H) + |\mathcal{B}_f| (\gamma_{\{2\}}(H) - 1) + |\mathcal{C}_f| (\gamma_{\{2\}}(H) - 2) \\ &\geq 2|\mathcal{A}_f| + |\mathcal{B}_f| + n(G)(\gamma_{\{2\}}(H) - 2) \\ &\geq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2). \end{aligned}$$

By Proposition 2.1(iii) and the above fact that $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$, we have $\gamma_{\{2\}}(G \circ_v H) = \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$. □

2.2. Characterization of six types

Lemma 2.6. *Let f be a $\gamma_{\{2\}}(G \circ_v H)$ -function and assume that $\gamma_{\{2\}}(G) < n(G)$. Then $\mathcal{C}_f \neq \emptyset$ if and only if $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$.*

Proof. Let $x \in \mathcal{C}_f$. Then $f(N_H[x]) = 0$ by Lemma 2.3. This implies that f_x is a $\{2\}$ DF of $H_x - x$, since $x \notin \mathcal{S}(H_x)$. Thus, $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H_x - x) \leq \omega(f_x) = \gamma_{\{2\}}(H) - 2$. By Lemma 2.2, the equality follows.

If $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$, then it follows from Proposition 2.1(iii) that $\omega(f) \leq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$. Since $\gamma_{\{2\}}(G) < n(G)$, it follows from Lemma 2.3 that $\mathcal{C}_f \neq \emptyset$. □

Theorem 2.7. *Assume that $\gamma_{\{2\}}(G) < n(G)$. Then the following statements are equivalent.*

- (i) $\gamma_{\{2\}}(G \circ_v H) = \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$.
- (ii) $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$.

Proof. Assume that (ii) holds. Let f be a $\gamma_{\{2\}}(G \circ_v H)$ -function. By Lemma 2.6, (ii) is equivalent to $\mathcal{C}_f \neq \emptyset$. By Case 3 of the proof of Theorem 2.5, (i) holds.

Conversely, assume that (i) holds. Then $\omega(f) = \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2) < n(G) + n(G)(\gamma_{\{2\}}(H) - 2) = n(G)(\gamma_{\{2\}}(H) - 1)$, which implies that $\mathcal{C}_f \neq \emptyset$. Thus, it follows from Lemma 2.6 that (ii) holds. □

Theorem 2.8. *Let G and H be graphs with no isolated vertex and $v \in V(H)$. Then $\gamma_{\{2\}}(G \circ_v H) = n(G)(\gamma_{\{2\}}(H) - 1)$ if and only if one of the following conditions holds.*

(i) $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$ and $\gamma_{\{2\}}(G) = n(G)$.

(ii) $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H) - 1$ and there exists a $\{2\}$ DF $h(B_0, B_1, B_2)$ on $H - N[v]$ with the weight $\gamma_{\{2\}}(H) - 2$ such that $B_1 \cup B_2$ is a dominating set of $H - v$.

Proof. Assume that $\gamma_{\{2\}}(G \circ_v H) = n(G)(\gamma_{\{2\}}(H) - 1)$. Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function. By Proposition 2.1(ii), we deduce that $v \in V(H) \setminus \mathcal{S}(H)$. It follows from Lemma 2.2 that $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H) - 2$. We consider the following cases.

Case 1. $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$. By Proposition 2.1(iii), $\omega(f) \leq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$. This implies that $\gamma_{\{2\}}(G) = n(G)$. Thus, (i) holds.

Case 2. $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H) - 1$. If $\mathcal{C}_f \neq \emptyset$, then it follows from Lemma 2.3 that $f(N_H[x]) = 0$ for $x \in \mathcal{C}_f$. This implies that $v \notin \mathcal{S}(H)$. So, f_x^- is a $\{2\}$ DF of $H_x - x$ and $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H_x - x) \leq \omega(f_x^-) = \gamma_{\{2\}}(H) - 2$. It follows from Lemma 2.2 that $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 2$, a contradiction. Thus, $\mathcal{C}_f = \emptyset$, which implies that $\mathcal{B}_f = V(G)$. By Lemma 2.4, there exists $x \in V(G) \cap A_1$. If $N_H(x) \cap (A_1 \cup A_2) \neq \emptyset$, then f_x is a $\{2\}$ DF of H_x , a contradiction. Thus, $N_H(x) \subseteq A_0$ and $f|_{V(H_x) \setminus N[x]}$ is a $\{2\}$ DF on $H - N[x]$ with the weight $\gamma_{\{2\}}(H) - 2$, and $(V(H_x) \setminus \{x\}) \cap (A_1 \cup A_2)$ is a dominating set of $H_x - x$. Thus, (ii) holds.

Conversely, assume that one of the conditions (i) and (ii) holds. First, suppose that (i) holds. By Proposition 2.1(iii) and the first condition of (i), $\gamma_{\{2\}}(G \circ_v H) \leq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$. By Theorem 2.5 and $\gamma_{\{2\}}(G) = n(G)$, $\gamma_{\{2\}}(G \circ_v H) = \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2) = n(G)(\gamma_{\{2\}}(H) - 1)$.

Second, suppose that (ii) holds. Let $h(B_0, B_1, B_2)$ be a $\{2\}$ DF on $H - N[v]$ with the weight $\gamma_{\{2\}}(H) - 2$. Define g on $G \circ_v H$ by $g_x = (V(H_x) \setminus (B_1 \cup \{x\} \cup B_2), B_1 \cup \{x\}, B_2)$ for each $x \in V(G)$. Then g is a $\{2\}$ DF of $G \circ_v H$ and so $\gamma_{\{2\}}(G \circ_v H) \leq \omega(g) = n(G)(\omega(h) + 1) = n(G)(\gamma_{\{2\}}(H) - 1)$. By Theorem 2.5, $\gamma_{\{2\}}(G \circ_v H) \in \{n(G)(\gamma_{\{2\}}(H) - 1), \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)\}$. If $\gamma_{\{2\}}(G) = n(G)$, then we are done. If $\gamma_{\{2\}}(G) < n(G)$, then it follows from Theorem 2.7 and the first condition of (ii) that $\gamma_{\{2\}}(G \circ_v H) \neq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$. Thus, $\gamma_{\{2\}}(G \circ_v H) = n(G)(\gamma_{\{2\}}(H) - 1)$. \square

Proposition 2.9. *Let G be a graph such that $\gamma_{\{2\}}(G) < n(G)$. Let H be a graph with no isolated vertex and $v \in V(H) \setminus \mathcal{S}(H)$. If $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H)$, then $\gamma_{\{2\}}(G \circ_v H) \in \{n(G)(\gamma_{\{2\}}(H) - 1), n(G)\gamma_{\{2\}}(H)\}$.*

Proof. Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function. It follows from Theorem 2.5 that $\omega(f) \leq n(G)\gamma_{\{2\}}(H)$. If $\omega(f) = n(G)\gamma_{\{2\}}(H)$, then we are done. Suppose that $\omega(f) < n(G)\gamma_{\{2\}}(H)$. Then $\mathcal{C}_f = \emptyset$ by Lemma 2.6. If $\mathcal{B}_f = \emptyset$, then $V(G) = \mathcal{A}_f$, a contradiction to $\omega(f) < n(G)\gamma_{\{2\}}(H)$. Thus, $\mathcal{B}_f \neq \emptyset$. By Lemma 2.4 and $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H)$, $\mathcal{B}_f \cap (A_1 \cup A_2) \neq \emptyset$ and so $\omega(f) = n(G)(\gamma_{\{2\}}(H) - 1)$. \square

Proposition 2.10. *Let G be a graph such that $\gamma_{\{2\}}(G) < n(G)$. Let H be a graph with*

no isolated vertex and $v \in V(H) \setminus \mathcal{S}(H)$. If $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H)$ and $g(v) = 0$ for any $\gamma_{\{2\}}(H)$ -function g , then $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$.

Proof. By Proposition 2.9, $\gamma_{\{2\}}(G \circ_v H)$ is either $n(G)\gamma_{\{2\}}(H)$ or $n(G)(\gamma_{\{2\}}(H) - 1)$. Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function, and suppose that $\omega(f) = n(G)(\gamma_{\{2\}}(H) - 1)$. Since $\gamma_{\{2\}}(G) < n(G)$, if $\mathcal{C}_f \neq \emptyset$, then by Lemma 2.6 we can construct a $\{2\}$ DF of $G \circ_v H$ with the weight $\gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 2)$ less than $n(G)(\gamma_{\{2\}}(H) - 1)$, a contradiction. Thus, $\mathcal{C}_f = \emptyset$, which implies that $\mathcal{B}_f \neq \emptyset$. By Lemma 2.4 and $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H)$, let $x \in \mathcal{B}_f \cap A_1$. Define a function $g : V(H_x) \rightarrow \{0, 1, 2\}$ by $g(x) = 2$ and $g(y) = f(y)$ for $y \in V(H_x) \setminus \{x\}$. Then g is a $\gamma_{\{2\}}(H)$ -function satisfying $g(x) = 2$, a contradiction. Thus, $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$. \square

Theorem 2.11. *Let G be a graph such that $\gamma_{\{2\}}(G) < n(G)$. Let H be a graph with no isolated vertex and $v \in V(H)$. Then $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$ if and only if either $v \in \mathcal{S}(H)$ or the following conditions holds.*

- (i) $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H)$.
- (ii) *If $H - N[v]$ has no isolated vertex, then there exists no $\{2\}$ DF on $H - N[v]$ with the weight $\gamma_{\{2\}}(H) - 2$.*

Proof. Assume that $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$. If $v \in \mathcal{S}(H)$, then we are done. Now, assume that $v \in V(H) \setminus \mathcal{S}(H)$. By Lemma 2.2 and Theorem 2.7, $\gamma_{\{2\}}(H - v) \geq \gamma_{\{2\}}(H) - 1$.

If $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$, then $\gamma_{\{2\}}(G \circ_v H) \leq \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1) < n(G)\gamma_{\{2\}}(H)$ by Proposition 2.1(iii), a contradiction. Thus, (i) holds.

Assume that $H - N[v]$ has no isolated vertex. Suppose that there exists a $\{2\}$ DF h on $H - N[v]$ with the weight $\gamma_{\{2\}}(H) - 2$. Define a function $f : V(G \circ_v H) \rightarrow \{0, 1, 2\}$ by $f|_{V(G)} = 1$ and f_x^- is induced by h . It is easy to see that f is a $\{2\}$ DF of $G \circ_v H$ with the weight $n(G)(\gamma_{\{2\}}(H) - 1)$, a contradiction. Thus, (ii) holds.

Conversely, assume that either $v \in \mathcal{S}(H)$ or (i) and (ii) hold. If $v \in \mathcal{S}(H)$, then $\gamma_{\{2\}}(G \circ_v H) = n(G)\gamma_{\{2\}}(H)$ by Proposition 2.1(ii). Now, assume that (i) and (ii) hold, and let f be a $\gamma_{\{2\}}(G \circ_v H)$ -function. Since (ii) contradicts Theorem 2.8(ii), we have $\omega(f) \neq n(G)(\gamma_{\{2\}}(H) - 1)$. By (i) and Proposition 2.9, we have $\omega(f) = n(G)\gamma_{\{2\}}(H)$. \square

Theorem 2.12. *Let G and H be graphs with no isolated vertex and $v \in V(H)$. Let f be a $\gamma_{\{2\}}(G \circ_v H)$ -function. Then $\gamma_{\{2\}}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$ if and only if*

- (i) $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$ and one of the following conditions holds.
 - (a) *For some $x \in V(G)$, there exists a $\gamma_{\{2\}}(H)$ -function f_x such that $f_x(v) = 2$.*
 - (b) *For some $x \in V(G)$, there exists a $\gamma_{\{2\}}(H - v)$ -function f_x^- such that $f_x^-(N(v)) = 1$.*
- (ii) *There exists no $\{2\}$ DF $h(B_0, B_1, B_2)$ on $H - N[v]$ with the weight $\gamma_{\{2\}}(H) - 2$ such that $B_1 \cup B_2$ is a dominating set of $H - v$.*

Proof. Assume that $\gamma_{\{2\}}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. Observe that (ii) follows as a consequence of Theorems 2.7 and 2.8

Now, we prove (i). Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function. By the Case 3 of the proof of Theorem 2.5, we deduce that $\mathcal{C}_f = \emptyset$. Since $\gamma(G) < n(G)$, we have $\mathcal{B}_f \neq \emptyset$. By Lemma 2.4, $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$ and $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$. Thus, the first equality of (i) follows.

Let $x \in \mathcal{B}_f$ such that $f(N(x))$ is maximum. We consider the following cases.

Case 1. $f(N(x)) = 0$. By the fact that $f(N(x))$ is maximum, $f(N(y)) = 0$ for every $y \in \mathcal{B}_f$. This implies that $\gamma(G) \leq |\mathcal{A}_f|$. But,

$$\begin{aligned} \gamma_{\{2\}}(G \circ_v H) &= \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1) \\ &= \sum_{x \in \mathcal{A}_f} \omega(f_x) + \sum_{x \in \mathcal{B}_f} \omega(f_x) \\ &\geq |\mathcal{A}_f| \gamma_{\{2\}}(H) + |\mathcal{B}_f| (\gamma_{\{2\}}(H) - 1) \\ &= |\mathcal{A}_f| + n(G)(\gamma_{\{2\}}(H) - 1). \end{aligned}$$

Thus, $|\mathcal{A}_f| \leq \gamma(G)$ and so $|\mathcal{A}_f| = \gamma(G)$. By Theorem 1.1, there exists a vertex $z \in \mathcal{A}_f$ such that $f(z) = 2$, i.e., (a) follows.

Case 2. $f(N(x)) \geq 1$. This case implies that f_x^- is a $\gamma_{\{2\}}(H - v)$ -function such that $f_x^-(N(v)) = 1$, i.e., (b) follows

Conversely, assume that (i) and (ii) hold. By Theorems 2.5, 2.7 and 2.8, we have $\gamma_{\{2\}}(G \circ_v H) \geq \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. Let D be a $\gamma(G)$ -set.

First, assume that (a) holds. Let $x \in V(G)$ such that f_x be a $\gamma_{\{2\}}(H)$ -function such that $f_x(v) = 2$. Let h be the function extended from a $\gamma_{\{2\}}(H - v)$ -function by assigning 0 at v . Define a function $g : V(G \circ_v H) \rightarrow \{0, 1, 2\}$ by $g_y = f_x$ for $y \in D$ and $g_y = h$ for $y \in V(G) \setminus D$. Then g is a $\{2\}$ DF of $G \circ_v H$ and so $\gamma_{\{2\}}(G \circ_v H) \leq \omega(g) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. By Theorems 2.5 and 2.7, we have $\gamma_{\{2\}}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$.

Second, assume that (b) holds. Let $x \in V(G)$ such that f_x^- is a $\gamma_{\{2\}}(H - v)$ -function with $f_x^-(N(v)) = 1$. Define a function $g : V(G \circ_v H) \rightarrow \{0, 1, 2\}$ by $g_y^- = f_x^-$ for each $y \in V(G)$ and $g|_{V(G)}$ is a $\{0, 1\}$ -function such that $g^{-1}(1) = D$. Then g is a $\{2\}$ DF of $G \circ_v H$ and so $\gamma_{\{2\}}(G \circ_v H) \leq \omega(g) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. By Theorems 2.5 and 2.7, we have $\gamma_{\{2\}}(G \circ_v H) = \gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. \square

Theorem 2.13. *Let G be a graph such that $\gamma(G) < \gamma_{q\{2\}}(G) < \gamma_{\{2\}}(G) \leq n(G)$. Let H be a graph with no isolated vertex and $v \in V(H)$. Let f be a $\gamma_{\{2\}}(G \circ_v H)$ -function. Then $\gamma_{\{2\}}(G \circ_v H) = \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$ if and only if*

- (i) $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$.
- (ii) For every $x \in V(G)$, f_x^- is a $\gamma_{\{2\}}(H - v)$ -function such that $f_x^-(N(v)) = 0$.

Proof. Assume that $\gamma_{\{2\}}(G \circ_v H) = \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. Let $f(A_0, A_1, A_2)$ be a $\gamma_{\{2\}}(G \circ_v H)$ -function. By the Case 3 of the proof of Theorem 2.5, we deduce that $\mathcal{C}_f = \emptyset$. As $\gamma_{q\{2\}}(G) < n(G)$, it follows that $\mathcal{B}_f \neq \emptyset$. By Lemma 2.4, $\mathcal{B}_f \cap (A_1 \cup A_2) = \emptyset$ and $\gamma_{\{2\}}(H - v) = \gamma_{\{2\}}(H) - 1$, i.e., (i) follows. Note that $N_H(x) \cap (A_1 \cup A_2) = \emptyset$ for every

$x \in \mathcal{B}_f$, otherwise we can construct a $\{2\}$ DF of $G \circ_v H$ with the weight $n(G)(\gamma_{\{2\}}(H) - 1)$, a contradiction.

Suppose that f_x^- is a $\gamma_{\{2\}}(H - v)$ -function such that $f_x^-(N(v)) = 1$ for some $x \in V(G)$. Then we can construct a $\{2\}$ DF of $G \circ_v H$ with the weight $\gamma(G) + n(G)(\gamma_{\{2\}}(H) - 1)$, a contradiction. Thus, (ii) follows.

Conversely, assume that (i) and (ii) hold. By Theorems 2.5, 2.7, 2.8 and 2.12, we deduce that $\gamma_{\{2\}}(G \circ_v H) \geq \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. For a $\gamma_{q\{2\}}(G)$ -pair (K, ϕ) , we construct a $\{2\}$ DF g of $G \circ_v H$ with the weight $\gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$ as follows.

Let h be a $\gamma_{\{2\}}(H - v)$ -function, and let i be a $\gamma_{\{2\}}(H)$ -function. If $x \in K$, then $g_x = i$. If $x \in V(G) \setminus K$, then g_x is defined as the function extended from h by $\phi(x)$. Since

$$\begin{aligned} \omega(g) &= |K|\gamma_{\{2\}}(H) + \omega(\phi) + (n(G) - |K|)(\gamma_{\{2\}}(H) - 1) \\ &= |K| + \omega(\phi) + n(G)(\gamma_{\{2\}}(H) - 1) \\ &= \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1), \end{aligned}$$

we have $\gamma_{\{2\}}(G \circ_v H) = \gamma_{q\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$. \square

As proved in Theorem 2.5, there are six closed formulas for $G \circ_v H$. Five cases were characterized in Theorems 2.7, 2.8, 2.11, 2.12 and 2.13. Thus, we can characterize the case of $\gamma_{\{2\}}(G \circ_v H) = \gamma_{\{2\}}(G) + n(G)(\gamma_{\{2\}}(H) - 1)$ by eliminating the previous ones from the family of all graphs G and H with no isolated vertices and roots v of H .

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