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Toughness and (a,b)-parity factors in graphs

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ABSTRACT

In this paper, we consider (a, b)- parity factors in graphs and obtain a toughness condition for the existence of (a, b)-parity factors. Furthermore, we show that the result is sharp in some sense.

Keywords: toughness; (g, f)-parity factor; (a, b)-parity factor

1. Background and the main results

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. Given a vertex $x \in V(G)$, let $N_G(x)$ denote the set of vertices adjacent to x in G and $d_G(x) = |N_G(x)|$. A subset I of $V(G)$ is an independent set of G if no two elements of I are adjacent in G and a subset C of $V(G)$ is a covering set if every edge of G has at least one end in C. For other terminologies and notations, we refer the reader to [2].

Given two positive integer functions g, f with $g(x) \equiv f(x) (\mod 2)$, we say that G has a (g, f)-parity factor if there is a spanning subgraph H of G such that $g(x) \leq d_H(x) \leq f(x)$ and $d_H(x) \equiv f(x) \mod (2)$for every $x \in V(G)$. Let $a \leq b$ be two integers and $a \equiv b(\mod 2)$. If $g(x) = a$ and $f(x) = b$ for all $x \in V(G)$, then a (g, f)-parity factor is called an (a, b)-parity factor.

Lovász gave a characterization of graphs having (g, f)-parity factors. Amahashi [1] obtained a condition for a graph to have $(1, k)$-odd factors, which was generalized to $(1, f)$-odd factors by Cui and Kano [7].

Theorem 1.1. *A graph G has a (g, f)-parity factor if and only if for any two disjoint*

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subsets D, S of $V(G)$,

$$f(D) - g(S) + d_{G-D}(S) - q(D,S) \geq 0,$$

where $q(D,S)$ denotes the number of components in $G - D - S$, such that $g(V(C)) + e_G(V(C), S) \equiv 1(\mod 2)$, where the component C is called g-odd components.

Theorem 1.2. *A graph G has a $(1,k)$-odd factor if and only if for any subset of $V(G)$,*

$$C_o(G - S) \leq k|S|,$$

where $C_o(G - S)$ denotes the number of odd components of $G - S$.

Let G be a non-complete graph and t be a real number. We say G is t-tough, if $|S| \geq t\omega(G - S)$, for any vertex-cutset S of G, where $\omega(G - S) > 1$ is the number of components in $G - S$. The largest t such that G is t-tough is called the toughness of G and is denoted by $t(G)$. That is,

$$t(G) = \min \left\{ \frac{|S|}{\omega(G - S)} | S \subset V(G), \omega(G - S) \geq 2 \right\}.$$

If $G \simeq K_n$, $t(G)$ is defined as ∞.

Chvátal introduced the concept of toughness in [3] and made the following conjecture.

Conjecture 1.3. *Let G be a graph and k a positive integer such that $k|V(G)|$ is even and G is k-tough. Then G has a k-factor.*

In [4] it was proved that the conjecture is true. Katernis [5] presented two theorems which implied the truth of Chvátal's conjecture.

Theorem 1.4. *Let G be a graph and a,b two positive integers. Suppose that*

$$t(G) \geq \begin{cases} \frac{(b+a)^2 + 2(b-a)}{4a}, & \text{if } b \equiv a(\mod 2), \\ \frac{(b+a)^2 + 2(b-a) + 1}{4a}, & \text{otherwise.} \end{cases}$$

If f is a positive integer function such that $f(V(G))$ is even and $a \leq f(x) \leq b$ for all $x \in V(G)$, then G has an f-factor.

Theorem 1.5. *Let G be a graph and a,b two positive integers with $b \geq a$. If $t(G) \geq (a-1) + \frac{a}{b}$ and $a|V(G)|$ is even when $a = b$, then G has an (a,b)-factor.*

In [6], they obtained a degree condition for the existence of (a, b)-parity factors in graphs. In this paper we give a sufficient condition for a graph to have an (a, b)-parity factor in terms of toughness of G. Our main result strengthens Katernis's results in some sense.

Theorem 1.6. *Let G be a connected graph. If b is an odd integer and $|V(G)|$ is even, $t(G) \geq \frac{1}{b}$, then G has a $(1,b)$-odd factor.*

Theorem 1.7. *Let G be a connected graph, a and b two integers with $b \geq a \geq 2$ and $a \equiv b(\mod 2)$. Suppose that $a|V(G)|$ is even and $t(G) \geq a - 1 + \frac{a}{b}$, then G has an (a,b)-parity factor.*

For the proof of main theorem we need the following lemmas in [3] and [5] respectively.

Lemma 1.8. *If a graph G is not complete, then $t(G) \leq \frac{1}{2}\delta(G)$*

Lemma 1.9. *Let H be a graph and $S_1, ..., S_{a-1}$ be a partion of the vertices of H such that if $x \in S_j$ then $D_H(x) \leq j$.(We allow $S_j = \emptyset$.) Then there exists a covering set C of H and an independent set I, such that*

$$\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1} j(a-j)i_j,$$

where $|I \cap S_j| = i_j and |C \cap S_j| = c_j$, for every $j = 1, ..., a-1$.

The result in Theorem 1.6 is sharp. We consider $G_1 = K_m \vee (mb+1)K_1$, where "$\vee$" means join and m is an arbitrary positive integer. It is easy to fnd out that $t(G) = \frac{m}{mb+1} < \frac{1}{b}$ and $b|S| < C_0(G-S)$ if $S = V(K_m)$. By Theorem 1.2 G_1 has no $(1,b)$-odd factor.

The assumption $|V(G)|$ even is necessary. For example, $K_{1,2n}$ has no $(1,b)$-odd factor.

To see the result in Theorem 1.7 is sharp even a is even. We construct the following graph G: $V(G) = A \cup B \cup C$ where A, B, C are disjoint with $|A| = (nb+1)(a-1)$, $|B| = nb+1$ and $|C| = na$. A is isomorphic to $(nb+1)K_{a-1}$, and $B = (nb+1)K_1$, while C is a clique of G. Other edges in G are all edges between every pair K_1 of A and K_{a-1} of B respectively, and all the edges between B and C. Let $X = A \cup C$. Then $|X| = (nb+1)(a-1) + na$ and $\omega(G-X) = nb+1$. This follows that

$$t(G) \leq \frac{|X|}{\omega(G-X)} = a - 1 + a\frac{n}{nb+1} < a - 1 + \frac{a}{b}.$$

If we set $D = C$ and $S = A$, $b|D| - a|S| + d_{G-D}(S) - q(D,S) = -a \leq -2$, where $q(D,S) = nb+1$. Clearly G has no (a,b)-parity factor.

2. Proof of main results

Proof.
Suppose to the contary, G has no $(1,b)$-factor. There exists a subset S of $V(G)$ such that $C_o(G-S) > b|S|$, where $C_o(G-S)$ denoted the odd components of $G-S$. Obviously $S \neq \emptyset$. Since $|V(G)|$ is even and $S = \emptyset$, $C_o(G-S) = C_o(G) = 0 > 0$, a contradiction. Therefore $t(G) \leq \frac{|S|}{\omega(G-S)} \leq \frac{|S|}{C_o(G-S)} < \frac{|S|}{b|S|} = \frac{1}{b}$, contradicts to the assumption. □

Proof. Suppose that G has no (a,b)-parity factor, there exist two disjoint subsets $D, S \subseteq V(G)$ such that

$$b|D|-a|S|+d_{G-D}(S)-q(D,S)\leq -1. \tag{1}$$

Obviously $D\cup S\neq\emptyset$. Otherwise $q(D,S)=0$, since $a|V(G)|$ is even. Thus $0\leq -1$ a contradiction.

Clearly $S\neq\emptyset$.

If $S=\emptyset$, then $q(D,\emptyset)\geq b|D|+1\geq 2$. By the definition of $t(G)$, we have $t(G)\leq \frac{|D|}{\omega(G-D}\leq\frac{|D|}{b|D|+1}<\frac{1}{b}$, contradicts to $t(G)\geq a-1+\frac{a}{b}$.

Let $\Delta=\max\{d_{G-D}(x)|x\in S\}$ and $S_j=\{x\in S|d_{G-D}(x)=j\}$, $|S_j|=s_j$, $0\leq j\leq\Delta$. Set $H=G[S_1\cup S_2\cup ...\cup S_{a-1}]$. Since for every $x\in S_j$, $d_H(x)\leq j$, by Lemma 9 we can find a covering set C' and an independent set I' of H, such that

$$\sum_{j=1}^{a-1}(a-j)c'_j\leq\sum_{j=1}^{a-1}j(a-j)i'_j,$$

where $|I'\cap S_j|=i'_j$ and $|C'\cap S_j|=c'_j$ for every $j=1,2,...,a-1$. We may assume that I' is a maximal independent set of H. We choose a maximal independent set I of $G[S]-S_0$, such that $I'\subseteq I$. Putting $C=(S-S_0)\setminus I$ we have $C'\subseteq C$, by the maximality of I'. If we put $I_j=I\cap S_j$, $|I_j|=i_j$, $C_j=C\cap S_j$ and $|C_j|=c_j$, then for $1\leq j\leq a-1$,

$$i_j=i'_j,\ c_j=c'_j.$$

Now Let Ω be the set of components of $W=G-D-S$ and set $Y=\{H\in\Omega|e(x,I)=0,\ \text{for}\ x\in V(H)\}$, $X_1=\{H\in\Omega|e(x,I)\geq 2,\ \text{for}\ x\in V(H)\}$, $X_2=\Omega\setminus(X_1\cup Y)$. Let $y=|Y|$, $x_1=|X_1|$ and $x_2=|X_2|$. Suppose that $X_2=\{H_1,...,H_{x_2}\}$ and choose $z_i\in V(H_i)$ such that $e(z_i,I)=1$ for every $1\leq i\leq x_2$. Let $U=D\cup C\cup((N(I)\cap V(W)))\setminus\{z_1,...z_{x_2}\})$

$$|U|\leq|D|+\sum_{j=1}^{\Delta}ji_j+s_0-(x_1+x_2),$$

and

$$\omega(G-U)\geq\sum_{j=1}^{\Delta}i_j+s_0+y.$$

Let $t=t(G)$. We have $d_{G-D}(x)\leq a$ for some $x\in S$. Otherwise, $d_{G-D}(x)\geq a+1$ for every $x\in S$. Then $q(D,S)\geq b|D|-a|S|+d_{G-D}(S)+1\geq b|D|+|S|+1$ and $t<1<a-1+\frac{a}{b}$, a contradiction.

Claim 1. $|U|\geq t\omega(G-U)$, if $\omega(G-U)=1$.

In fact, by Lemma 1.8, $2t\leq d(x)\leq d_{G-D}(x)+|D|$ for all $x\in S$. Thus choosing $x\in S$ with $d_{G-D}(x)\leq a$ we get $|D|\geq 2t-a$ and $|D|\geq t-1\geq 1$ since $t\geq a-1$. In the case $C\neq\emptyset$, $|U|\geq|D|+|C|\geq t$. Otherwise, we get $C_1\cup\ldots\cup C_{\Delta-1}=\emptyset$, then $S_1\cup\ldots\cup S_{\Delta-1}$ is an independent set. Since $1=\omega(G-U)\geq\sum_{j=1}^{\Delta}i_j+s_0\geq 1$, we have $s_0=1$ or $\sum_{j=1}^{\Delta}i_j=1$. Obviously, if $s_0=1$, $G-S_0-D\neq\emptyset$. Otherwise, $0>b|D|-a|S_0|+d_{G-D}(S_0)\geq b|D|-a>0$, a contradiction. Thus $|D|\geq t\omega(G-D)\geq 2t$, since $\omega(G-D)\geq\omega(G-D-S_0)+1\geq 2$. Suppose that $\sum_{j=1}^{\Delta}i_j=1$, set $i_1=1$, $S_1=\{v\}$ and $N_{G-D}(v)=\{u\}$. If $N_G(u)-(D\cup S_1)\neq\emptyset$, then $t \le \frac{|D\cup u|}{\omega(G-(D\cup u))}$ and $|D| \ge 2t-1$. Otherwise, there exists $x \in D$ but $x \notin N_G(u)$ or $D \subseteq N_G(u)$, we have $t \le \frac{|(D\setminus x)\cup S_1|}{\omega(G-((D\setminus x)\cup S_1))}$ or $d_G(u) = |D|+1 \ge 2t$ respectively. In all cases, $|D| \ge t$, therefore $|U| \ge |D| \ge t\omega(G-U)$.

If $\omega(G-U) > 1$, then $|U| \ge t\omega(G-U)$ for G is t-tough. Therefore,

$$s_0 + |D| + \sum_{j=1}^{\Delta} j i_j - (x_1+x_2) \ge t\left(\sum_{j=1}^{\Delta} i_j + y + s_0\right).$$

And since $\omega(G-D-S) \ge q(D,S) > b|D| - a|S| + d_{G-D}(S)$ we have

$$\begin{aligned}
|D| &\ge \sum_{j=1}^{\Delta}(t-j)i_j + (ty + x_1 + x_2) + ts_0 - s_0 \\
&\ge \omega(G-D-S) + \sum_{j=1}^{\Delta}(t-j)i_j(t-j)i_j + ts_0 - s_0 \\
&> b|D| - a|S| + d_{G-D}(S) + \sum_{j=1}^{\Delta}(t-j)i_j(t-j)i_j + ts_0 - s_0 \\
&= b|D| - \sum_{j=1}^{\Delta}(a-j)(i_j+c_j)\sum_{j=1}^{\Delta}(t-j)i_j(t-j)i_j + ts_0 - as_0 - s_0 \\
&\ge b|D| + \sum_{j=1}^{\Delta}(t-a)i_j + \sum_{j=1}^{\Delta}(j-a)c_j + ts_0 - as_0 - s_0 \\
&\ge b|D| + \sum_{j=1}^{a-1}(t-a)i_j + \sum_{j=1}^{a-1}(j-a)c_j + (t-a-1)s_0.
\end{aligned}$$

$$(b-1)|D| \le \sum_{j=1}^{a-1}(a-t)i_j + \sum_{j=1}^{a-1}(a-j)c_j + (a-t+1)s_0. \tag{2}$$

Claim 2. $|D \cup N(I')| \ge t\omega(G-(D\cup N(I')))$.

If $\omega(G-(D\cup N(I'))) \ge 2$, it holds obviously.

If $\omega(G-(D\cup N(I'))) = 1$, we have $I' \ne \emptyset$, otherwise $0 = \sum_{j=1}^{a-1} j(a-j)i_j \ge \sum_{j=1}^{a-1}(a-j)c_j \ge 0$, and $C_1 \cup ... \cup C_{a-1} = \emptyset$. $H = \emptyset$, and $S \ne \emptyset$, so $|S_0| = 1$ or $|S_j| = 1$ for $a+1 \le j \le \Delta$. Both cases contradict to (1). Thus $|I'| = 1$.set $I' = \{x\}$, then $|D \cup N(x)| \ge d(x) \ge 2t > t\omega(G-D\cup N(x))$.

Hence

$$|D| + \sum_{j=1}^{a-1} j i_j \ge t\left(\sum_{j=1}^{a-1} i_j + s_0\right),$$

and

$$(b-1)|D| \ge \sum_{j=1}^{a-1}(b-1)(t-j)i_j + (b-1)ts_0. \tag{3}$$

So from (2) and (3) we have

$$\sum_{j=1}^{a-1}(a-t)i_j + \sum_{j=1}^{a-1}(a-j)c_j + (a-t)s_0 \geq \sum_{j=1}^{a-1}(b-1)(t-j)i_j + (b-1)ts_0,$$

and

$$\sum_{j=1}^{a-1}(a-t)i_j + \sum_{j=1}^{a-1}j(a-j)i_j \geq \sum_{j=1}^{a-1}(b-1)(t-j)i_j + (bt-t-a+t)s_0,$$

since $\sum_{j=1}^{a-1}(a-j)c_j \leq \sum_{j=1}^{a-1}j(a-j)i_j$. As $bt-t-a+t > 0$, we have $a-t+j(a-j) \geq (b-1)(t-j)$ and $t < \frac{a+j^2-aj-bj+j}{b}$ for all j, $1 \leq j \leq a-1$, which contradicts $t \geq a-1+\frac{a}{b}$. □

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