Changing and unchanging of the connected domination number of a graph

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ABSTRACT

Given a connected graph $G = (V, E)$ of order $n \geq 2$ and two distinct vertices $u, v \in V(G)$, consider two operations on G: the *k-multisubdivision* and the *k-path addition*. Let $msd_{\gamma_c}(G)$ and $pa_{\gamma_c}(G)$ denote, respectively, the connected domination multisubdivision and path addition numbers of G. In both operations, k represents the number of vertices added to $V(G)$, resulting in a new graph denoted by $G_{u,v,k}$. We prove that $\gamma_c(G) \leq \gamma_c(G_{u,v,k})$ for $k = msd_{\gamma_c}(G) \in \{1, 2, 3\}$ in the case of k-multisubdivision, where $uv \in E(G)$. Additionally, we show that $\gamma_c(G) - 2 \leq \gamma_c(G_{u,v,k})$ for $k = pa_{\gamma_c}(G) \in \{0, 1, 2, 3\}$ in the case of k-path addition, where $uv \notin E(G)$, and provide both necessary and sufficient conditions under which these inequalities hold.

Keywords: domination, connected domination number, multisubdivision number, path addition number

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1. Introduction

Let $G = (V, E)$ be a finite, simple, and connected graph with a vertex-set $V(G)$ of order $|V| = n$ and an edge-set $E(G)$ of size $|E| = m$. For any nonempty subset $S \subset V(G)$, let $G[S]$ denote the subgraph of G induced by S. For any vertex v of G, $N(v)$ denotes the set of all neighbors of v in G and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *private neighborhood* of a vertex $v \in S$ with respect to S is the set $pn[v, S] = \{u \in V(G) : N[u] \cap S = \{v\}\}$. Each vertex in $pn[v, S]$ is called a *private neighbor* of v with respect to S. Let $\Delta(G)$ (respectively, $\delta(G)$) denote the *maximum* (respectively, *minimum*

degree in G. Throughout this paper, the notations P_n, C_n, and K_n always denote a path, a cycle, and a complete graph of order n, respectively. By $P_{u,v}(G)$ we denote a path P_n with end vertices u and v. For further basic notations and graph theory terminology, we refer the reader to the book of Berge [4].

Throughout the history of graph theory, understanding how domination parameter and its variants behave under specific structural modifications has been a central topic of investigation. In particular, considerable attention has been devoted to studying how operations such as *subdividing an edge* or *adding a path* between two distinct vertices affect the values of these parameters. These operations can be viewed as local structural transformations in a network, introducing intermediate or linking vertices while preserving the essential topology of the graph. Analyzing whether a domination parameter remains invariant (*unchanging*) or increases or decreases (*changing*) under these modifications provides deep insight into the stability and sensitivity of dominating sets. For an explicit introduction to the theory of domination, we refer the reader to the book by Haynes et al. [16]. The effect of edge subdivision on domination-type parameters in graphs was first explored by Haynes et al. [14, 15] for the classical domination and independence parameters, where the original problem leading to this study was introduced by Arumugam in a private communication to Haynes [14, Ref. 1]. In the subsequent work on domination parameters, new studies were introduced by Bhattacharya and Vijayakumar [5], Favaron et al. [8], and more recently, Samodivkin [20] presented an explicit study of the behavior of the domination number when a path P_n is added to a graph. Following these papers, the focus shifted towards the total domination variant, where many authors investigated its behaviour. See, for example, the works of Haynes et al. [17, 18], Favaron et al. [11, 9], Avella-Alaminos et al. [3], and Çiftçi and Aytaç [7]. Several papers have also examined related topics in Roman domination, such as those by Khodkar et al. [19] and Atapour et al. [2, 1].

Despite this extensive research, the connected domination parameter still lacks explicit studies concerning the effects of the multisubdivision and path addition operations. A review of the literature suggests that the first and only paper addressing the impact of edge subdivision on the connected domination number is due to Favaron et al. [10], where the authors investigated the connected domination subdivision number, denoted by $sd_{\gamma_c}(G)$ (each edge subdivided at most once). They established upper bounds for this parameter and characterized those graphs for which the inequality $\gamma_c(G) + sd_{\gamma_c}(G) \leq n-1$ holds, where $\gamma_c(G)$ denotes the connected domination number of G. For additional results and discussions related to the connected domination parameter, the reader is referred to Chellali et al. [6]. Consequently, there remains a clear gap in the literature regarding how these two operations affect the connected domination number and under which structural conditions this parameter remains invariant or changes. The goal of the present paper is therefore to investigate the effect of edge multisubdivision and path addition on the connected domination number by providing necessary and sufficient conditions for each possible change value.

A *connected dominating set* of a connected graph G is a subset $D \subseteq V(G)$ such that every vertex of $V(G) \setminus D$ is adjacent to at least one vertex in D, and the induced subgraph

$G[D]$ is connected. The *connected domination number* of G, denoted by $\gamma_c(G)$, is the minimum cardinality of a connected dominating set in G. A connected dominating set of G with cardinality $\gamma_c(G)$ is called a γ_c-*set* of G.

Let G be a connected graph and let uv be an edge of G. By *multisubdividing* the edge uv, we mean forming a graph H from G by adding k new vertices $x_1, x_2, \ldots, x_k$ and replacing the edge uv with the path $ux_1, x_1x_2, \ldots, x_kv$. Formally,

$$V(H) = V(G) \cup \{x_1, x_2, \ldots, x_k\} \quad \text{and} \quad E(H) = (E(G) \setminus \{uv\}) \cup \{ux_1, x_1x_2, \ldots, x_kv\}.$$

The *path addition* operation is defined as the disjoint union of a connected graph G with a path P_k, where the end vertices of this path are identified with two distinct vertices of G. Formally, let u and v be distinct vertices of G and let $P_k : x_1, x_2, \ldots, x_k$ be a path of length $k \geq 0$. The graph H is obtained from the disjoint union of G and P_k by identifying u with one end vertex of P_k and v with the other. Note that in a path addition, when u and v are nonadjacent and $k = 0$, the operation simply corresponds to adding the edge uv to G. The resulting graph from either operation is denoted by $G_{u,v,k}$, where k is the number of newly added vertices to the original graph G.

Fricke et al. [12] introduced the concept of γ_c-*good* and γ_c-*bad* vertices in graphs. A vertex v of a graph G is said to be γ_c-*good* if it belongs to some γ_c-set of G, and γ_c-*bad* if it belongs to no γ_c-set of G.

Let $msd_{\gamma_c}(u,v)$ and $pa_{\gamma_c}(u,v)$ denote the minimum integer k such that $\gamma_c(G) \neq \gamma_c(G_{u,v,k})$ under the multisubdivision and path addition operations, respectively. For every connected graph G of order $n \geq 2$, we define the *minimum e-multisubdivision number* (respectively, the *minimum e-path addition number*) with respect to connected domination, denoted by $emsd_{\gamma_c}(G)$ (respectively, $epa_{\gamma_c}(G)$), as follows:

$$emsd_{\gamma_c}(G) = \min\{\, msd_{\gamma_c}(u,v) \mid u, v \in V(G), \ uv \in E(G) \,\},$$
$$epa_{\gamma_c}(G) = \min\{\, pa_{\gamma_c}(u,v) \mid u, v \in V(G), \ uv \notin E(G) \,\}.$$

Similarly, the *maximum e-multisubdivision number* (respectively, the *maximum e-path addition number*) with respect to connected domination, denoted by $Emsd_{\gamma_c}(G)$ (respectively, $Epa_{\gamma_c}(G)$), is defined as

$$Emsd_{\gamma_c}(G) = \max\{\, msd_{\gamma_c}(u,v) \mid u, v \in V(G), \ uv \in E(G) \,\},$$
$$Epa_{\gamma_c}(G) = \max\{\, pa_{\gamma_c}(u,v) \mid u, v \in V(G), \ uv \notin E(G) \,\}.$$

If G is a complete or an edgeless graph, then the above numbers are not defined, and we write

$$epa_{\gamma_c}(G) = Epa_{\gamma_c}(G) = \infty, \quad \text{and} \quad emsd_{\gamma_c}(G) = Emsd_{\gamma_c}(G) = \infty,$$

respectively.

Throughout this paper, we assume that all graphs under consideration are connected. We end this section with some results and definitions which will be useful in proving our main results.

Definition 1.1. Let $G = (V, E)$ be a connected graph. A vertex $u \in V$ is called a *cut-vertex* of G if the removal of u disconnects the graph G.

Lemma 1.2. *Let G be a connected graph and let D be a γ_c-set of G. A vertex $u \in D$ if and only if one of the following holds:*
- *u has at least one* private neighbor *in $V \setminus D$ with respect to D,*
- *u is a* cut-vertex *of the induced subgraph $G[D]$.*

Proof. $\Rightarrow$ Let D be a γ_c-set of G with $|D| = \gamma_c(G)$, and let $u \in D$. Suppose, to the contrary, that u has no private neighbor in $V \setminus D$ and that u is not a cut-vertex of $G[D]$. Then every vertex in $N(u)$ is either adjacent to another vertex of D or belongs to D entirely, implying that $D - \{u\}$ is still a connected dominating set of G, contradicting the minimality of D. Hence, u must have at least one private neighbor in $V \setminus D$ or u is a cut-vertex of $G[D]$.

$\Leftarrow$ If u has at least one private neighbor in $V \setminus D$, then clearly $u \in D$, since otherwise that neighbor would not be dominated by D. On the other hand, if u is a cut-vertex of $G[D]$, then $G[D - \{u\}]$ is disconnected, implying that $D - \{u\}$ cannot be a connected dominating set of G. Therefore, u must belong to the γ_c-set D of G. $\square$

The concept of a cut-vertex was first formalized by Whitney [21] in his study of non-separable graphs, and later popularized under its modern terminology by Harary [13].

Remark 1.3. A cut-vertex v may or may not have a private neighbor.

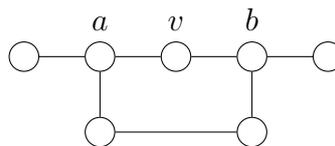**Remark 1.4.** A cut-vertex in $G[D]$ can only lie on a path of $G[D]$.

Fig. 1. An illustration of Lemma 1.2, where $D = \{a, v, b\}$. Each of a and b has a private neighbor, while v is a cut-vertex in $G[D]$

The following Lemma describes all possible membership configurations of two adjacent vertices in a γ_c-set of a connected graph G.

Lemma 1.5. *Let G be a connected graph and let u, v be adjacent vertices of G. Fix a γ_c-set D of G. Exactly one of the following cases occurs:*
    (A) *$u, v \in D$, and the edge uv is contained in a cycle of $G[D]$.*
    (B) *$u, v \in D$, and the edge uv lies on a path of $G[D]$.*
    (C) *$u \in D$, $v \notin D$, u is γ_c-good, v is γ_c-bad, and v is a private neighbor of u with respect to D (i.e. $N(v) \cap D = \{u\}$).*
    (D) *$u \in D$, $v \notin D$, u is γ_c-good, v is γ_c-bad, and v has at least two neighbors in D (i.e. $|N(v) \cap D| \geq 2$).*

(E) $u \in D$, $v \notin D$, and both u and v are γ_c-good, with v not being a private neighbor of u with respect to D (i.e. $|N(v) \cap D| \geq 2$, but v belongs to another γ_c-set).

(F) $u, v \notin D$ (both u and v are γ_c-bad).

Proof. Let G be connected, let D be a γ_c-set of G, and let u, v be adjacent vertices of G. Each of u and v either belongs to D or does not. Then we have:

Case 1. $u, v \in D$. Then $uv \in G[D]$. If uv belongs to a cycle of $G[D]$, we obtain case (A). Otherwise, uv lies on a path of $G[D]$, which corresponds to case (B).

Case 2. Without loss of generality, $u \in D$ and $v \notin D$. Since D is a minimal connected dominating set, every vertex outside D is dominated by at least one vertex of D, hence $N(v) \cap D \neq \emptyset$. Two situations arise depending on the status of v:

• If v is a γ_c-bad vertex, then it belongs to no γ_c-set of G. By definition of domination, we have $|N(v) \cap D| \geq 1$. If $|N(v) \cap D| = 1$, then v is a private neighbor of u in D, giving case (C). If $|N(v) \cap D| \geq 2$, we obtain case (D).

• If v is a γ_c-good vertex, then it belongs to some other γ_c-set of G. Therefore, $|N(v) \cap D| \geq 2$. Otherwise, if $|N(v) \cap D| = 1$, v would be a private neighbor of its unique dominator u in D, which would prevent v from belonging to any γ_c-set. This yields case (E).

Case 3. $u, v \notin D$. If this holds for every γ_c-set D of G, then both vertices are γ_c-bad, which corresponds to case (F).

Since the above cases are mutually exclusive and cover all possible membership configurations of u and v with respect to D, exactly one of (A)–(F) must hold. $\square$

Lemma 1.6. *Let G be a connected graph of order $n \geq 2$, $uv \in E(G)$ and let $G_{u,v,k}$ be obtained from G by multisubdividing uv with $k \geq 1$ new vertices. Let D be a γ_c-set of G with $u, v \in D$. If the edge uv belongs to some path of $G[D]$, then*

$$\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1.$$

Proof. We have that $\gamma_c(G) = |D|$. Since $u, v \in D$ and uv is a bridge of $G[D]$, the removal of the edge uv disconnects $G[D]$ into at least two components. In $G_{u,v,k}$ every vertex of $V(G)$ has a neighbour in D by definition of D. However, the edge uv is replaced by the path $u, x_1, \ldots, x_k, v$. Because uv was a bridge of $G[D]$, there is no u, v-path inside $G[D]$ that avoids the edge uv. Consequently the induced subgraph $G_{u,v,k}[D]$ is disconnected. Any connected dominating set of $G_{u,v,k}$ must induce a connected subgraph and must dominate all vertices. Thus, one must include k vertices from the path joining u and v in $G_{u,v,k}$ for $k \geq 1$. Thus any γ_c-set of $G_{u,v,k}$ has size at least $|D|+1$, so $\gamma_c(G_{u,v,k}) \geq |D|+1 = \gamma_c(G) + 1$. $\square$

Lemma 1.7. *Let G be a connected graph of order $n \geq 2$, $uv \in E(G)$ and let $G_{u,v,k}$ be obtained from G by multisubdividing uv with $k \geq 1$ new vertices. Let D be a γ_c-set of G with $u, v \in D$. If the edge uv belongs to some cycle of $G[D]$, then*

$$\gamma_c(G_{u,v,k}) = \gamma_c(G) \text{ for } k = 1, 2 \text{ and } \gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1 \text{ for } k \geq 3.$$

Proof. Let M be a γ_c-set of G such that $u, v \in M$ and the edge uv belongs to a cycle in $G[M]$. In the graph $G_{u,v,2}$, every vertex of $V(G)$ remains dominated by M. Moreover, since x_1 is dominated by u and x_2 is dominated by v, the set M is also a γ_c-set of $G_{u,v,2}$ (by the same argument as in the case $k = 1$). For $k \geq 3$, we add exactly $k - 2$ new vertices to the set M so that $G[M]$ remains connected. Consequently, for $k \geq 3$, we obtain $\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1$. $\hfill\square$

Lemma 1.8. *Let G be a connected graph of order $n \geq 2$, let D be a γ_c-set of G with $u \in D$, and let v be a γ_c-bad vertex of G. Let $G_{u,v,k}$ be the graph obtained from G by multisubdividing the edge uv with $k \geq 1$ new vertices. If $v \in pn[u, D]$, then*

$$\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1.$$

Proof. Let $\gamma_c(G) = |D|$ and assume $v \in pn[u, D]$, so $N(v) \cap D = \{u\}$. In G the vertex v is dominated only by $u \in D$. After multisubdivision the edge uv is replaced by the path $u, x_1 \cdots x_k, v$, so u is no longer adjacent to v in $G_{u,v,k}$. Consequently, v is not dominated by D in $G_{u,v,k}$ anymore. Then, any connected dominating set of $G_{u,v,k}$ must therefore contain k vertices from the set $\{x_1, \ldots, x_k\}$, because x_k dominates v in $G_{u,v,k}$. Therefore, every connected dominating set of $G_{u,v,k}$ has size at least $|D|+1 = \gamma_c(G) + 1$. That is, $\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1$. $\hfill\square$

The remainder of this paper is organized as follows. In Section 2, we prove that $msd_{\gamma_c}(G) \in \{1, 2, 3\}$ for every edge $uv \in E(G)$. In Section 3, we show that $pa_{\gamma_c}(G) \in \{0, 1, 2, 3\}$ for every pair of nonadjacent vertices $u, v \notin E(G)$.

2. The effect of multisubdivision on the connected domination number

In this section, we denote by $G_{u,v,k}$ the graph obtained by multisubdivising the edge uv. The aim is to prove that $msd_{\gamma_c}(G) \in \{1, 2, 3\}$ by providing necessary and sufficient conditions, where $uv \in E(G)$ for any connected graph G of order $n \geq 2$.

Theorem 2.1. *Let u and v be adjacent vertices of a graph G. Then $\gamma_c(G) \leq \gamma_c(G_{u,v,1}) \leq \gamma_c(G) + 1$. Moreover,*

(I) $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ *if and only if one of the following holds:*

(a) *both u and v are included in every γ_c-set of G and the edge uv belongs to a cycle in the subgraph induced by these γ_c-sets,*

(b) *u and v are γ_c-good and γ_c-bad vertices of G, respectively. u is included in every γ_c-set of G, where the vertex v is not a private neighbor of u with respect to the γ_c-set,*

(c) *both u and v are γ_c-good vertices of G and every γ_c-set contains only one of them. Without loss of generality, v is dominated by at least another vertex besides u,*

(II) $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ *if and only if one of the following is true:*

(d) *both u and v are included in every γ_c-set of G and the edge uv belongs to no cycle in the subgraph induced by these γ_c-sets,*

(e) u is a γ_c-good vertex, v is a γ_c-bad vertex, and every γ_c-set of G contains u, such that v is a private neighbor of u with respect to these γ_c-sets,

(f) both u and v are γ_c-bad vertices of G.

Proof. $(a) \Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ with $\gamma_c(G) = \mid D \mid$ and $u, v \in D$. By (A) of Lemma 1.5, the edge uv belongs to a cycle of $G[D]$. Suppose, to the contrary, that the edge uv belongs to no cycle in $G[D]$. Now, according to Lemma 1.6, we have $\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1$ for $k \geq 1$, a contradiction. Then, the edge uv must belongs to a cycle in $G[D]$.

$(a) \Leftarrow$ The rest follows by Lemma 1.7.

$(b) \Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds and D is a γ_c-set of G such that $u \in D$ and v is a γ_c-bad vertex of G. Suppose, to the contrary, that v is a private neighbor of u with respect to D, then following Lemma 1.8, we obtain that $\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1$ for $k \geq 1$, a contradiction. Thus $\mid N(v) \cap D \mid \geq 2$.

$(b) \Leftarrow$ Let M be a γ_c-set of G with $u, w \in M$ and v is a γ_c-bad vertex of G such that v is adjacent to both vertices u and w in the set M. In the graph $G_{u,v,1}$, we have that $x_1 \in pn[u, M]$ and $v \in pn[w, M]$. Thus M is a γ_c-set of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \gamma_c(G)$.

$(c) \Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds and D is a γ_c-set of G such that both u and v are γ_c-good vertices of G. Now, without loss of generality, assume that $u \in D$ and $v \in pn[u, D]$. Then, in this case, v cannot be γ_c-good vertex of G, thus it is a γ_c-bad vertex and according Lemma 1.8, $\gamma_c(G_{u,v,k}) \geq \gamma_c(G) + 1$ for $k \geq 1$, a contradiction. Then, v is a γ_c-good vertex and $\mid N(v) \cap D \mid \geq 2$.

$(c) \Leftarrow$ Let M be a γ_c-set of G and both u and v are γ_c-good vertices of G. Without loss of generality, let $u \in M$ and $\mid N(v) \cap M \mid \geq 2$, then in the graph $G_{u,v,1}$, the vertex x_1 is dominated by u and v is dominated by some other vertices in $M - \{u\}$, which means that M is a γ_c-set of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \gamma_c(G)$.

$(d) \Rightarrow$ Let D be a γ_c-set of G with $u, v \in D$. Since the edge uv is included in a path of $G[D]$, then Lemma 1.6 is fullfiled and $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ for $k = 1$.

$(d) \Leftarrow$ Let M be a γ_c-set of G with $u, v \in M$ and the edge uv belongs to a path in $G[M]$. In the graph $G_{u,v,1}$, it is obvious that $M \cup \{x_1\}$ is a γ_c-set of $G_{u,v,1}$. Hence $\gamma_c(G_{u,v,k}) = \mid M \cup \{x_1\} \mid = \gamma_c(G) + 1$.

$(e) \Rightarrow$ This case follows from Lemma 1.8 for $k = 1$ such that $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$.

$(e) \Leftarrow$ Let M be a γ_c-set of G with $u \in M$, $v \notin M$, v is a γ_c-bad vertex, and $v \in pn[u, M]$. In the graph $G_{u,v,1}$, the set M dominates all the vertices of the set $V(G_{u,v,1}) - \{v\}$. Therefore, $M \cup \{x_1\}$ is a γ_c-set of $G_{u,v,1}$ such that $\gamma_c(G_{u,v,1}) = \mid M \cup \{x_1\} \mid = \gamma_c(G) + 1$.

$(f) \Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ holds and D is a γ_c-set of $G_{u,v,1}$. Without loss of generality, the set $D - \{u\}$ is a γ_c-set of G since $x_1 \in pn[u, D]$ such that $\mid D - \{u\} \mid = \gamma_c(G)$. Thus $\gamma_c(G_{u,v,1}) = \mid D \mid = \gamma_c(G) + 1$.

$(f) \Leftarrow$ Let both u and v be γ_c-bad vertices of G. Let M be a γ_c-set of G and U a γ_c-set of $G_{u,v,1}$. The set M dominates all the vertices of the set $V(G_{u,v,1}) - \{x_1\}$, hence one of u and v must be included in U. Then, $U = M \cup \{u\}$ or $U = M \cup \{v\}$ are both γ_c-sets of $G_{u,v,1}$ and $\gamma_c(G_{u,v,1}) = \mid U \mid = \gamma_c(G) + 1$. $\qquad \square$

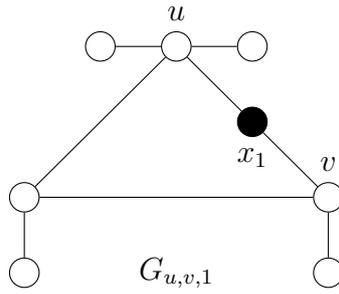The following Figures 2–9 illustrate all the possible cases described in Theorem 2.1.

Fig. 2. Case (a) where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$, corresponding to the situation $|pn[u, D]| \geq 2$ and $|pn[v, D]| \geq 2$
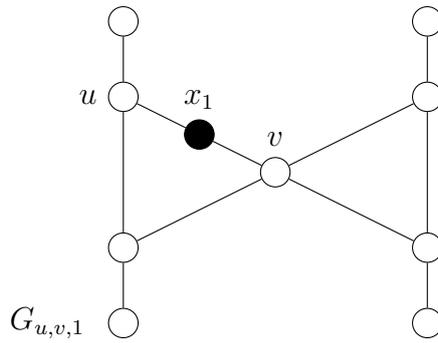
Fig. 3. Case (a) where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 5$, corresponding to $|pn[u, D]| \geq 2$ and $pn[v, D] = \{v\}$, i.e., v is a cut-vertex of $G[D]$
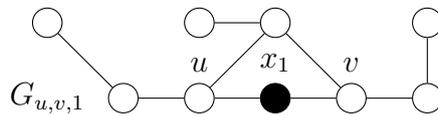
Fig. 4. Case (a), where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 6$, where both u and v being cut-vertices of $G[D]$
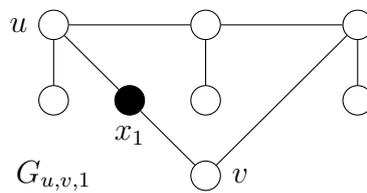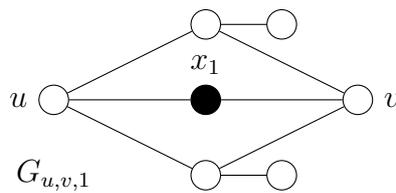
Fig. 5. Case (b), where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$

Fig. 6. Case (c), where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$
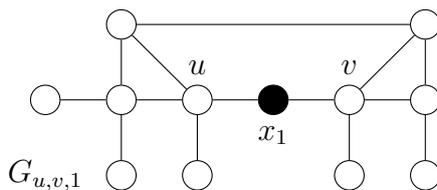
Fig. 7. Case (d), where $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,1}) = 5$
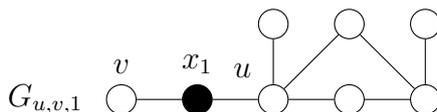
Fig. 8. Case (e), where $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,1}) = 4$
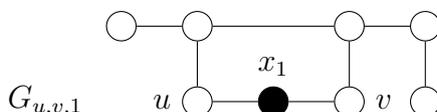
Fig. 9. Case (f), where $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,1}) = 4$

Theorem 2.2. *Let u and v be adjacent vertices of a graph G. Then $\gamma_c(G) \leq \gamma_c(G_{u,v,2}) \leq \gamma_c(G) + 2$. Moreover,*

(I) $\gamma_c(G_{u,v,2}) = \gamma_c(G)$ if and only if both u and v are included in every γ_c-set of G and the edge uv belongs to a cycle in the subgraph induced by these γ_c-sets,

(II) $\gamma_c(G_{u,v,2}) = \gamma_c(G) + 1$ if and only if u is a γ_c-good vertex and v is either γ_c-good or γ_c-bad in G. In the case where both u and v are γ_c-good vertices, every γ_c-set of G contains exactly one of them. Without loss of generality, v is dominated by at least two vertices.

(III) $\gamma_c(G_{u,v,2}) = \gamma_c(G) + 2$ if and only if one of the following holds:

(a) u is a γ_c-good vertex, v is a γ_c-bad vertex, and every γ_c-set of G contains u, such that v is a private neighbor of u with respect to these γ_c-sets,

(b) both u and v are in every γ_c-set of G and the edge uv belongs to no cycle in the subgraph induced by these γ_c-sets,

(c) both u and v are γ_c-bad vertices of G.

Proof. (I) $\Rightarrow$ Suppose that $\gamma_c(G_{u,v,2}) = \gamma_c(G)$ holds. Let D be a γ_c-set of G such that $u, v \in D$, and assume that the edge uv belongs to a path in $G[D]$. Then Lemma 1.6 applies, implying that $\gamma_c(G_{u,v,k}) > \gamma_c(G)$ for all $k \geq 1$, a contradiction.

(I) $\Leftarrow$ The converse follows from Lemma 1.7.

(II) $\Rightarrow$ Suppose that $\gamma_c(G_{u,v,2}) = \gamma_c(G) + 1$ holds, and let D be a γ_c-set of G. According to parts (b) and (c) of Theorem 2.1, in the graph $G_{u,v,2}$ and without loss of generality, suppose $u \in D$ and $v \notin D$, we obtain that $D \cup \{x_1\}$ and $D \cup \{v\}$ are both γ_c-sets of $G_{u,v,2}$ such that $\gamma_c(G_{u,v,2}) = \mid D \cup \{x_1\} \mid = \mid D \cup \{v\} \mid = \gamma_c(G) + 1$.

(II) $\Leftarrow$ Let M be a γ_c-set of G. Without loss of generality, assume that $u \in M$, $v \notin M$, and $\mid N(v) \cap M \mid \geq 2$. In the graph $G_{u,v,2}$, the vertex u dominates x_1, while v is

dominated by some vertices in $M \setminus \{u\}$. Hence, all vertices of M are adjacent to every vertex of $V(G_{u,v,2}) \setminus \{x_2\}$. Consequently, $M \cup \{x_1\}$ and $M \cup \{v\}$ are both γ_c-sets of $G_{u,v,2}$, and $\gamma_c(G_{u,v,2}) = \mid M \cup \{x_1\} \mid = \gamma_c(G) + 1$.

$(a) \Rightarrow$ This direction follows from Lemma 1.8 for $k = 2$, which gives $\gamma_c(G_{u,v,2}) = \gamma_c(G) + 2$.

$(a) \Leftarrow$ Let M be a γ_c-set of G such that $u \in M$, $v \notin M$, and $v \in pn[u, M]$. Since v is adjacent only to u in M, it follows that the set $M \cup \{x_1, x_2\}$ is a γ_c-set of $G_{u,v,2}$, and $\gamma_c(G_{u,v,2}) = \mid M \cup \{x_1, x_2\} \mid = \gamma_c(G) + 2$.

$(b) \Rightarrow$ This follows from Lemma 1.6 for $k = 2$, and we obtain $\gamma_c(G_{u,v,2}) = \mid D \cup \{x_1, x_2\} \mid = \gamma_c(G) + 2$.

$(b) \Leftarrow$ Let M be a γ_c-set of G with $u, v \in M$ and the edge uv belongs to a path in $G[M]$. Now, by (d) of Theorem 2.1 and since $N(x_1) = \{u, x_2\}$ and $N(x_2) = \{x_1, v\}$, we arrive at the fact that $M \cup \{x_1, x_2\}$ is a γ_c-set of $G_{u,v,2}$ in which $\gamma_c(G_{u,v,2}) = \mid M \cup \{x_1, x_2\} \mid = \gamma_c(G) + 2$.

$(c) \Rightarrow$ Suppose that $\gamma_c(G_{u,v,2}) = \gamma_c(G) + 2$ holds, and let D be a γ_c-set of $G_{u,v,1}$. According to (f) of Theorem 2.1, we have $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$. Hence, $\gamma_c(G) \in \{\mid D - \{u\} \mid, \mid D - \{v\} \mid\}$. That is, both u and v are γ_c-bad vertices of G. Now, without loss of generality, let $u \in D$ and $v \notin D$. Then D dominates all vertices of $V(G_{u,v,2}) - \{x_2\}$. Therefore, one of x_1 and v must be included in any γ_c-set of $G_{u,v,2}$, and $\gamma_c(G_{u,v,2}) = \mid D \cup \{x_1\} \mid = \mid D \cup \{v\} \mid = \gamma_c(G) + 2$.

$(c) \Leftarrow$ Let both u and v be γ_c-bad vertices of G. Since each of them is adjacent to some vertex of the γ_c-set M of G, in the graph $G_{u,v,2}$, the set M dominates $V(G_{u,v,2}) - \{x_1, x_2\}$. This implies that $M \cup \{u, x_1\}$ and $M \cup \{v, x_2\}$ are both γ_c-sets of $G_{u,v,2}$, and $\gamma_c(G_{u,v,2}) = \mid M \cup \{u, x_1\} \mid = \gamma_c(G) + 2$. $\square$

The following Figures 10–15 illustrate all the possible cases described in Theorem 2.2.
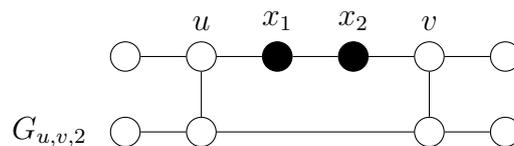

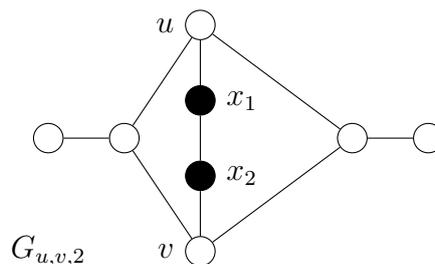**Fig. 10.** Case *(I)*, where $\gamma_c(G) = \gamma_c(G_{u,v,2}) = 4$

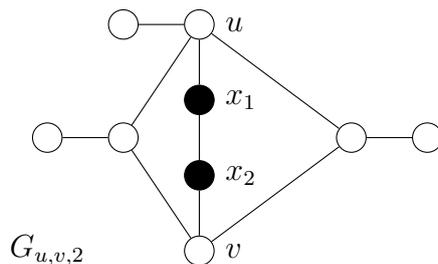**Fig. 11.** Case *(II)*, where $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,2}) = 4$, and both u and v are γ_c-good vertices of G

Fig. 12. Case *(II)*, where $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,2}) = 4$, and u and v are respectively γ_c-good and γ_c-bad vertices of G
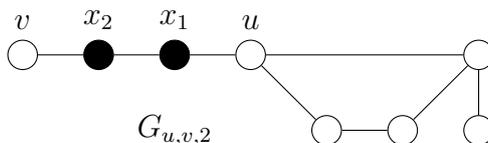

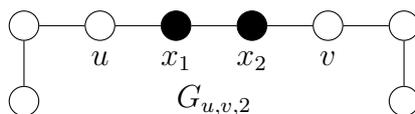**Fig. 13.** Case (a), where $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 4$, and u and v are respectively γ_c-good and γ_c-bad vertices of G

Fig. 14. Case (b), $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,2}) = 6$
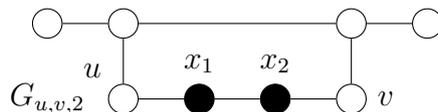
Fig. 15. Case (c), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 4$

Theorem 2.3. *Let u and v be adjacent vertices of a graph G. Then $\gamma_c(G_{u,v,3}) \geq \gamma_c(G)+1$.*

Proof. From Lemma 1.5, we have exactly six possible cases. By Lemmas 1.6, 1.7, and 1.8, we know that for $k \geq 3$, $\gamma_c(G_{u,v,k}) \geq \gamma_c(G)+1 > \gamma_c(G)$. On the other hand, if $u \in D$, $v \notin D$ and $\mid N(v) \cap D \mid \geq 2$, then $D \cup \{v, x_1\}$, $D \cup \{x_1, x_2\}$, and $D \cup \{v, x_3\}$ are all γ_c-sets of $G_{u,v,3}$ such that $\gamma_c(G_{u,v,3}) = \mid D \cup \{v, x_1\} \mid = \gamma_c(G) + 2 > \gamma_c(G)$. Finally, if both u and v are γ_c-bad vertices of G, then $D \cup \{u, v, x_1\}$ is one of many other γ_c-sets of $G_{u,v,3}$ such that $\gamma_c(G_{u,v,3}) = \mid D \cup \{u, v, x_3\} \mid = \gamma_c(G) + 3 > \gamma_c(G)$. In conclusion, in each case, at least three vertices among u, v, x_1, x_2, and x_3 are included in any γ_c-set of $G_{u,v,3}$, which always leads to the fact that $\gamma_c(G_{u,v,3}) \geq \gamma_c(G) + 1$. $\square$

Corollary 2.4. *Let u and v be adjacent vertices of a graph G of order $n \geq 3$, then*
   (i) $Emsd_{\gamma_c}(G) = 1$ *if and only if $\gamma_c(G) = 1$.*

(ii) *If $\gamma_c(G) = 2$ and G contains a $\gamma_c - $ set such that $N(u) \cap N(v) = \emptyset$. Then, $Emsd_{\gamma_c}(G) = 1$.*

(iii) *If $d_G(u) = d_G(v) = 2$, then $Emsd_{\gamma_c}(G) = 1$.*

Proof. $(i) \Rightarrow$ Let $Esd_{\gamma_c}(G) = 1$ hold and $\gamma_c(G) = 1$. Then there exists a vertex $u \in V(G)$ such that $d_G(u) = n - 1$. Now, in the graph $G_{u,v,1}$, we know that $d_{G_{u,v,1}}(v) \leq n - 2$ for all vertices of $V(G_{u,v,1})$, which means that u is no longer a universal vertex in $G_{u,v,1}$. Therefore, any γ_c-set of $G_{u,v,1}$ must contain at least two vertices. Thus, $\gamma_c(G_{u,v,1}) \geq 2$.

(i) $\Leftarrow$ Let $\gamma_c(G) = 1$. It is obvious that for all $k \geq 1$, $\gamma_c(G_{u,v,k}) > \gamma_c(G)$.

(ii) Let $\gamma_c(G) = 2$ and D is a γ_c-set of G such that $N(u) \cap N(v) = \emptyset$. In the graph $G_{u,v,1}$, $D \cup \{x_1\}$ is a γ_c-set if $u, v \in D$. If $u, v \notin D$, then $D \cup \{u\}$ and $D \cup \{v\}$ are both γ_c-sets. If $u \in D$ and $v \notin D$, then at least one of the vertices v, x_1, or those in $N(u) \cup N(v)$ must be included in the γ_c-set of $G_{u,v,1}$. Hence, in every case, $\gamma_c(G_{u,v,1}) \geq 3 > \gamma_c(G)$.

(iii) Let D be a γ_c-set of G, and let u and v be adjacent vertices of degree 2 in the graph G. That is, the vertices u and v are part of a path P_k in G. Therefore, according to (d) of Theorem 2.1, $\gamma_c(G_{u,v,1}) > \gamma_c(G)$. $\qquad\square$

The following Figures 16–20 illustrate all the possible cases described in Corollary 2.4.
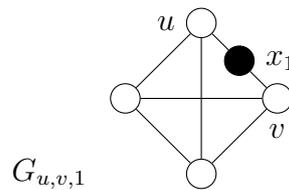

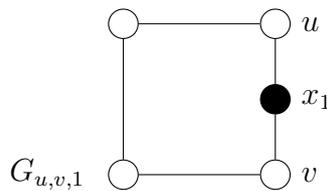**Fig. 16.** Case (i), where $\gamma_c(G) = 1 \neq \gamma_c(G_{u,v,1}) = 2$

Fig. 17. Case (ii), where $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,1}) = 3$
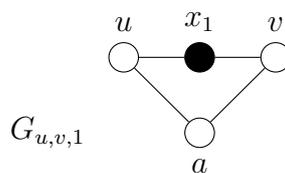

Fig. 18. Case (iii), where $\gamma_c(G) = 1 \neq \gamma_c(G_{u,v,1}) = 2$ and $a = b$
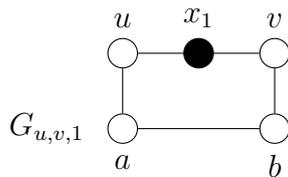
Fig. 19. Case (iii), where $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,1}) = 3$, and $a \neq b$ with $(a,b) \in E(G)$
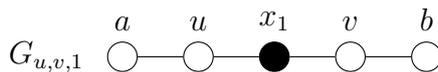

Fig. 20. Case (iii), where $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,1}) = 3$, and $a \neq b$ with $(a,b) \notin E(G)$

Corollary 2.5. *Let u and v be adjacent vertices of a graph G of order $n \geq 3$. Then,*
  (iv) $Emsd_{\gamma_c}(G) = 1$ for any complete graph K_n, star $K_{1,n-1}$, wheel W_n, or any connected graph with a universal vertex.
  (v) $Emsd_{\gamma_c}(G) = 1$ for any complete bipartite graph $K_{p,q}$ with $p, q \geq 2$.
  (vi) $Emsd_{\gamma_c}(G) = 1$ for any cycle C_n or path P_n.

Proof. Statements (iv), (v), and (vi) follow immediately from (i), (ii), and (iii) of Corollary 2.4, respectively. $\qquad\qquad\square$

Corollary 2.6. *Let T be a tree of order $n \geq 3$. Then, $Emsd_{\gamma_c}(T) = 1$.*

Proof. Since every tree T is acyclic, each edge $uv \in E(T)$ lies on a path in T. Let D and M be γ_c-sets of T and $T_{u,v,1}$, respectively. In the tree $T_{u,v,1}$, there exists at least one leaf vertex that is not dominated by any set of cardinality $\mid D \mid = \gamma_c(T)$. Therefore, $M = D \cup \{x_1\}$ is a γ_c-set of $T_{u,v,1}$ such that $\gamma_c(T_{u,v,1}) = \mid D \cup \{x_1\} \mid > \gamma_c(T)$. $\qquad\square$

3. The effect of path addition on the connected domination number

In this section, we denote by $G_{u,v,k}$ the graph obtained by the addition of a path P_k whose end vertices are two distinct vertices u and v of G. We shall prove that $pa_{\gamma_c}(G) \in \{0, 1, 2, 3\}$ by establishing necessary and sufficient conditions for $0 \leq epa_{\gamma_c}(G) \leq Epa_{\gamma_c}(G) \leq 3$, where $uv \notin E(G)$ for any connected graph G of order $n \geq 3$.

Theorem 3.1. *Let u and v be non-adjacent vertices, and t and w be adjacent vertices of a graph G of order $n \geq 6$, and let D be a γ_c-set of G with $u, t, w, v \in D$. If t and w are cut-vertices of $G[D]$ with no private neighbors with respect to D, then $\gamma_c(G_{u,v,0}) = \gamma_c(G) - 2$.*

Proof. Let D be a γ_c-set of G with $u, v, t, w \in D$, and let the path $P_{u,v}$ of $G[D]$ contains the adjacent vertices t and w (both are cut-vertices of $G[D]$). In the graph $G_{u,v,0}$ obtained

by adding the edge uv to G, a new cycle is formed. If at least one of them has a private neighbor in $V - D$, then it cannot be removed from the dominating set D. Now, Since the vertices of the set $D - \{t, w\}$ are adjacent to all vertices of $V(G_{u,v,0}) \setminus (D - \{t, w\})$, we arrive at $\gamma_c(G_{u,v,0}) = \mid D - \{w, t\} \mid = \gamma_c(G) - 2$.   □

The following Figure 21 illustrates the case described in Theorem 3.1.
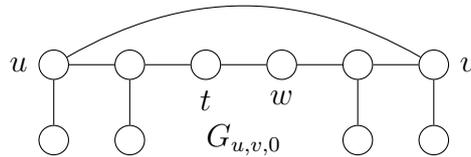

Fig. 21. An illustration for Theorem 3.1. The figure shows the graph $G_{u,v,0}$ obtained by adding the edge uv to G, where $\gamma_c(G) = 6 \neq \gamma_c(G_{u,v,0}) = 4$

Theorem 3.2. *Let u and v be non-adjacent vertices, and let D be a γ_c-set of G. If one of the following conditions holds:*

(a) $u, w \in D$, w is not a cut-vertex in the graph $G - \{v\}$, and v is the only private neighbor of w with respect to D,

(b) $u, v, w \in D$ and w is a cut-vertex of $G[D]$ with no private neighbors with respect to D, then $\gamma_c(G_{u,v,0}) = \gamma_c(G) - 1$.

Proof. (a) Let D be a γ_c-set of G with $u, w \in D$ such that $pn[w, D] = \{w, v\}$, where w is not a cut-vertex of $G[D]$ (according to Lemma 1.2, w has a private neighbor with respect to D). In the graph $G_{u,v,0}$, obtained by adding the edge uv to G, the vertex v becomes adjacent to both u and w. Since w is adjacent to at least one vertex in D, the set $D - \{w\}$ forms a γ_c-set of $G_{u,v,0}$, and therefore $\gamma_c(G_{u,v,0}) = \mid D - \{w\} \mid = \gamma_c(G) - 1$.

(b) This case follows from Theorem 3.1.   □

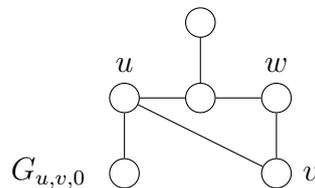The following Figures 22 and 23 illustrate all the possible cases described in Theorem 3.2.

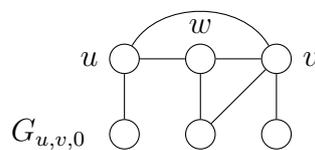**Fig. 22.** Case (a), where $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,0}) = 2$

Fig. 23. Case (b), where adding the edge uv to G yields $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,0}) = 2$

Theorem 3.3. *Let u and v be non adjacent vertices of a graph G of order $n \geq 3$, then $\gamma_c(G) \leq \gamma_c(G_{u,v,1}) \leq \gamma_c(G) + 1$. Moreover,*

(a) *$\gamma_c(G_{u,v,1}) = \gamma_c(G)$ if and only if at least one of u and v is included in every γ_c-set of G.*

(b) *$\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ if and only if both u and v are γ_c-bad vertices of G.*

Proof. (a) $\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G)$ holds, and let D be a γ_c-set of G. Suppose, to the contrary, that both u and v are γ_c-bad vertices of G. Then one of them must be included in any γ_c-set D' of $G_{u,v,1}$. That is, $\gamma_c(G_{u,v,1}) =\mid D' \mid=\mid D \cup \{u\} \mid=\mid D \cup \{v\} \mid= \gamma_c(G) + 1$, which contradicts the original hypothesis. Therefore, at least one u and v must be a γ_c-good vertex of G.

(a) $\Leftarrow$ Let M be a γ_c-set of G. First, let $u \in M$ and $v \notin$ is a γ_c-good or bad vertex of G. Then in $G_{u,v,1}$, since v is adjacent to some vertices of $M - \{u\}$ and $x_1 \in pn[u, M]$, then M is a γ_c-set of $G_{u,v,1}$, in which $\gamma_c(G_{u,v,1}) =\mid M \mid= \gamma_c(G)$. Now, if $u, v \in M$, it is obvious that $\gamma_c(G_{u,v,1}) = \gamma_c(G)$.

(b) $\Rightarrow$ Let $\gamma_c(G_{u,v,1}) = \gamma_c(G) + 1$ holds. Suppose, to the contrary, that at least one of u and v belongs to some γ_c-set of G. Then (a) of Theorem 3.3 is fullfiled and $\gamma_c(G_{u,v,1}) = \gamma_c(G)$, a contradiction. Then, both u and v are γ_c-bad vertices of G.

(b) $\Leftarrow$ Let M be a γ_c-set of G and both u and v are γ_c-bad vertices of G. In the graph $G_{u,v,1}$, we know that $N(x_1) = \{u, v\}$. Thus both the sets $M \cup \{u\}$ and $M \cup \{v\}$ are γ_c-sets of $G_{u,v,1}$ such that $\gamma_c(G_{u,v,1}) =\mid M \cup \{u\} \mid=\mid M \cup \{v\} \mid= \gamma_c(G) + 1$. $\qquad\square$

The following Figures 24–26 illustrate all the possible cases described in Theorem 3.3.
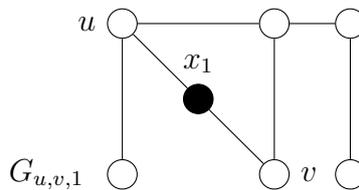
Fig. 24. Case (a), where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$. Here, u and v are respectively γ_c-good and γ_c-bad vertices of G
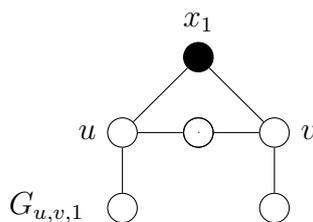

Fig. 25. Case (a), where $\gamma_c(G) = \gamma_c(G_{u,v,1}) = 3$. Here, both u and v are included in the γ_c-set of G
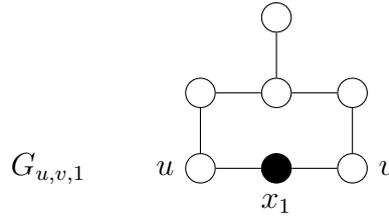
Fig. 26. Case (b), where $\gamma_c(G) = 3 \neq \gamma_c(G_{u,v,1}) = 4$

Theorem 3.4. *Let u and v be non adjacent vertices of a graph G of order $n \geq 3$, then $\gamma_c(G) \leq \gamma_c(G_{u,v,2}) \leq \gamma_c(G) + 2$. Moreover,*

(a) $\gamma_c(G_{u,v,2}) = \gamma_c(G)$ if and only if both u and v are included in every γ_c-set of G.

(b) $\gamma_c(G) + 1 \leq \gamma_c(G_{u,v,2}) \leq \gamma_c(G) + 2$ if and only if every γ_c-set of G contains at most one of u and v.

Proof. (a) $\Rightarrow$ This implication follows by applying the same reasoning as in (a) of Theorem 3.3, in the case where both u and v are included in every γ_c-set of G.

(a) $\Leftarrow$ Let M be a γ_c-set of G such that $u, v \in M$. In the graph $G_{u,v,2}$, we have $x_1 \in pn[u, M]$ and $x_2 \in pn[v, M]$. Hence, M remains a γ_c-set of $G_{u,v,2}$, and therefore $\gamma_c(G_{u,v,2}) = \gamma_c(G)$.

(b) $\Rightarrow$ Let $\gamma_c(G) + 1 \leq \gamma_c(G_{u,v,2}) \leq \gamma_c(G) + 2$ holds. Suppose, to the contrary, that $u, v \in D$, then (a) of Theorem 3.4 is fulfilled and $\gamma_c(G_{u,v,2}) = \gamma_c(G)$, a contradiction. Then every γ_c-set of G contains at most one of u and v.

(b) $\Leftarrow$ Let M be a γ_c-set of G. If $u \in M$ and $v \notin M$, then in $G_{u,v,2}$, x_2 is not dominated by any of the vertices of M. Thus $M \cup \{x_1\}$ and $M \cup \{v\}$ are both γ_c-sets of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid M \cup \{v\} \mid = \gamma_c(G) + 1$. Now, if $u, v \notin M$, then both x_1 and x_2 are not dominated by M. That is, $M \cup \{u, v\}$ is a γ_c-set of $G_{u,v,2}$ and $\gamma_c(G_{u,v,2}) = \mid M \cup \{u, v\} \mid = \gamma_c(G) + 2$. Hence, $\gamma_c(G) + 2 \geq \gamma_c(G_{u,v,2}) \geq \gamma_c(G) + 1$. $\qquad\square$

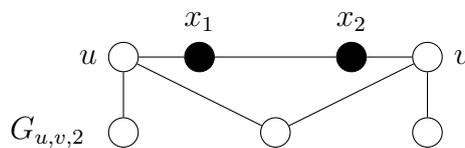The following Figures 27–29 illustrate all the possible cases described in Theorem 3.4.

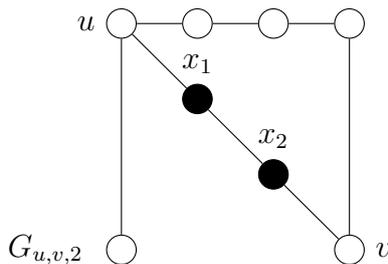**Fig. 27.** Case (a), where $\gamma_c(G) = \gamma_c(G_{u,v,2}) = 3$

Fig. 28. Case (*b*), where $\gamma_c(G) = 4 \neq \gamma_c(G_{u,v,2}) = 5$. Here, u and v are respectively γ_c-good and γ_c-bad vertices of G
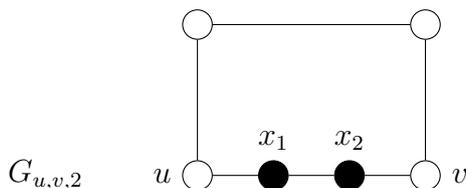

Fig. 29. Case (*b*), $\gamma_c(G) = 2 \neq \gamma_c(G_{u,v,2}) = 4$. Here, both u and v are γ_c-bad vertices of G

Theorem 3.5. *Let u and v be non adjacent vertices of a graph G of order $n \geq 3$, then $\gamma_c(G_{u,v,3}) \geq \gamma_c(G) + 1$.*

Proof. Let D be a γ_c-set of G. If $u, v \in D$, then in the graph $G_{u,v,3}$, the set D dominates all vertices in $V(G_{u,v,3}) - \{x_2\}$. Thus, both $D \cup \{x_1\}$ and $D \cup \{x_3\}$ are γ_c-sets of $G_{u,v,3}$, and consequently $\gamma_c(G_{u,v,3}) = \mid D \cup \{x_1\} \mid = \gamma_c(G) + 1$. Now, if $u \in D$ and $v \notin D$, then in $G_{u,v,3}$, the set D dominates all vertices in $V(G_{u,v,3}) - \{x_2, x_3\}$. Hence, the sets $D \cup \{x_1, v\}$, $D \cup \{x_1, x_2\}$, and $D \cup \{x_3, v\}$ are all γ_c-sets of $G_{u,v,3}$, and we have $\gamma_c(G_{u,v,3}) = \mid D \cup \{x_1, v\} \mid = \gamma_c(G) + 2$. Finally, if $u, v \notin D$, then D dominates all vertices in $V(G_{u,v,3}) - \{x_1, x_2, x_3\}$. Therefore, $D \cup \{u, x_1, v\}$ is a γ_c-set of $G_{u,v,3}$, and $\gamma_c(G_{u,v,3}) = \mid D \cup \{u, x_1, v\} \mid = \gamma_c(G) + 3$. Hence, $\gamma_c(G_{u,v,3}) \geq \gamma_c(G) + 1$. □

Corollary 3.6. *Let T be a tree of order $n \geq 4$ and let D be its γ_c-set. Let u and v be two non-adjacent vertices of T.*

(i) If u is either a root or a support vertex of T, w is a support vertex, and v is a leaf such that $N(v) = \{w\}$, then $\gamma_c(T_{u,v,0}) = \gamma_c(T) - 1$.

(ii) If both u and v are non-leaf vertices of T such that the path $P_{u,v}$ contains one or two adjacent support vertices having no neighbors in $V - D$, then $\gamma_c(T_{u,v,0}) \in \{\gamma_c(T) - 1, \gamma_c(T) - 2\}$, respectively.

(iii) If both u and v are leaves, then $Epa_{\gamma_c}(T) = 1$.

Proof. (i) Let D be a γ_c-set of T and suppose $u, w \in D$ with $pn[w, D] = \{w, v\}$. Since the support vertex w is adjacent to some vertices in D, in $T_{u,v,0}$ the vertex v becomes dominated by u. By Lemma 1.2, w is no longer a cut-vertex in $T_{u,v,0}$, and thus $D - \{w\}$ forms a γ_c-set of $T_{u,v,0}$ with $\gamma_c(T_{u,v,0}) = \mid D - \{w\} \mid = \gamma_c(T) - 1$.

(ii) This follows immediately from Theorems 3.1 and 3.2.

(iii) This follows directly from part (b) of Theorem 3.3. □

Corollary 3.7. *Let u and v be two non-adjacent vertices of a graph G containing a universal vertex. Then $Epa_{\gamma_c}(G) = 1$.*

Proof. Let $w \in V(G)$ be a universal vertex, so that $d_G(w) = n - 1$ and $\gamma_c(G) = 1$. In the graph $G_{u,v,1}$, for every vertex $t \in V(G_{u,v,1})$, we have $d_{G_{u,v,1}}(t) \leq n - 2$. Hence, $G_{u,v,1}$ contains no universal vertices, which implies that for any γ_c-set M of $G_{u,v,1}$, we have $\gamma_c(G_{u,v,1}) = \mid M \mid \geq 2 > \gamma_c(G) = 1$. Thus, $Epa_{\gamma_c}(G) = 1$. □

Corollary 3.8. *For any cycle C_n with $n \geq 4$, we have $Epa_{\gamma_c}(C_n) = 0$, and*

$$\gamma_c(C'_n) \in \{\gamma_c(C_n) - 1, \ \gamma_c(C_n) - 2\},$$

where $C'_n = (V(C_n), E(C_n) \cup \{uv\})$.

Proof. An immediate consequence of Theorems 3.1 and 3.2. □

4. Conclusion

The behavior of the parameter $\gamma_c(G)$ under the edge multisubdivision operation depends, in some cases, on the structure of the subgraph induced by the γ_c-sets, and in others, on the membership of the end vertices of the subdivided edge in the γ_c-sets of G. In general, seven possible distinct cases corresponding to this operation are presented in Lemma 1.5. Under the path addition operation, the behavior of $\gamma_c(G)$ primarily depends on the nature of the end vertices of the added path, particularly on whether they are γ_c-good or γ_c-bad vertices of G.

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Conflict of interest

The authors declare that they have no conflict of interest.

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