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Decomposing hypercubes into cycles: An approach to the oberwolfach problem

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ABSTRACT

Cartesian-product networks combine well-studied graphs to create new structures with inherited properties, making them valuable for interconnection networks and parallel algorithms. Cycle decompositions of these networks are crucial for fault tolerance and adaptive routing. In this paper, we address the hypercube version of the Oberwolfach problem, focusing on decompositions of Q_n into cycles of equal or unequal lengths. We present an algorithm that enumerates all possible cycle types in Q_n and determine which decompositions are feasible or infeasible for Q_4. Using an inductive approach, we extend these results to Q_n by leveraging distinct perfect matchings of Q_4, yielding a variety of cycle decompositions. Additionally, we present results on factorizations of Q_n when n is a power of 2. These findings enhance the understanding of cycle structures in hypercubes and their applications to interconnection networks.

Keywords: decomposition, cycles, factorization, hypercube

2020 Mathematics Subject Classification: 05C76, 05C38, 68R10.

1. Introduction

Ringel [9] posed the famous Oberwolfach problem on the cycle decomposition of complete graphs. At conferences held in Oberwolfach, it is customary for participants to dine together at circular tables of varying sizes, with assigned seating that rearranges participants from meal to meal. The Oberwolfach problem asks how to make a seating chart for a given set of tables so that all the tables are full at each meal and all pairs of conference participants are seated next to each other exactly once. Formulated as a problem in graph

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theory, the pairs of people sitting next to each other at a single meal can be represented as a disjoint union of cycles of specified lengths, with one cycle for each of the dining tables. This union of cycles is a 2-regular graph, and every 2-regular graph has this form. If G is this 2-regular graph with n vertices, the question is whether the complete graph K_n can be represented as an edge-disjoint union of copies of G. In order for a solution to exist, the total number of conference participants (equivalently, the total number of vertices of the given cycles) must be odd. The problem has, however, also been extended to even values of n by asking whether all the edges of the complete graph, except a perfect matching, can be covered by copies of the given 2-regular graph. Formally, the Oberwolfach problem $OP(m_1^{\alpha_1}, m_2^{\alpha_2}, \ldots, m_t^{\alpha_t})$ asks for the existence of a 2-factorization of a complete graph in which each 2-factor consists of exactly α_i cycles of length m_i, for $i = 1, 2, \ldots, t$. Much research has been done on this problem [1, 3, 2]. Gilbert [4] described cycle types of all lengths in Q_n for $n \leq 4$. Two paths in Q_n are said to have the *same type* if one path is obtained from the other by applying an automorphism. If no such automorphism exists between the two paths, then they are said to be distinct.

In this paper, we study the hypercube version of the Oberwolfach problem and obtain results related to cycle decompositions of hypercubes of even dimensions. We also develop an algorithm that generates all types of k-paths and hence k-cycles (since a cycle is a closed path) in Q_n, where $2 \leq k \leq 2^n$ and $n \geq 2$. Using the types of cycles, we give all possible cycle decompositions of Q_4. We develop a technique for extending the cycle decomposition of Q_4, together with a perfect matching, to the cycle decomposition of Q_6, Q_8, and, in general, Q_n. We use properties of Cartesian products to extend 2-factorizations of Q_4 to factorizations of Q_8, Q_{16}, and so on.

2. Algorithm for all the types of paths in Q_n

2.1. Notations and preliminaries

The vertices of Q_n are the binary n-tuples and two vertices are adjacent, if their corresponding n-tuples differ in exactly one co-ordinate. Let $e_i = (0, \ldots 0, \underbrace{1}_{i^{th} \text{ place}}, 0, \ldots 0)$ be the standard unit vector, $i \in \{1, 2, \ldots, n\}$. The direction of an edge $e = (u, v)$ in Q_n is i, if $u - v = e_i$ (subtraction is done modulo 2).

A path $P \equiv (u_1, u_2, \ldots, u_k)$ in Q_n can be expressed in terms of the initial vertex u_1 and the edge direction sequence $S = (i_1, i_2, \ldots, i_{k-1})$, where $i_j \in [n]$, such that for $j = 2, 3, \ldots, k$, $u_j = u_1 + \sum_{r=1}^{j-1} e_{i_r}$. Similarly, a cycle $C \equiv (u_1, u_2, \ldots, u_k, u_1)$ in Q_n can be expressed in terms of the initial vertex u_1 and edge direction sequence $S = (i_1, i_2, \ldots, i_k)$, where $i_j \in \{1, 2, \ldots, n\}$, such that for $j = 2, 3, \ldots, k$, $u_j = u_1 + \sum_{r=1}^{j-1} e_{i_r}$ and $u_k + e_{i_k} = u_1$. So the cycle $C \equiv ((0, 0), (0, 1), (1, 1), (1, 0), (0, 0))$ in Q_2 can be written as $C((0, 0), S)$, where $S = (1, 2, 1, 2)$.

In the following Figure 1, we have shown directions in hypercube Q_4.

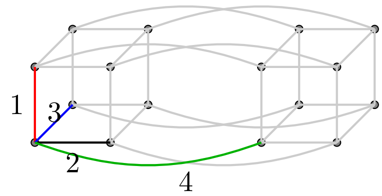


Fig. 1. Directions in Q_4

Remark 2.1. If the initial vertex of a path is *0*, then the path is completely determined by its edge direction sequence. So, for convenience, we will represent a path $P \equiv P(0, S)$, where $S = (i_1, i_2, \ldots, i_k)$, by $P = (i_1, i_2, \ldots, i_k)$ only.

The structure of $Aut(Q_n)$ is well known. It is discussed in various articles ([4, 6, 8, 10]). There are two types of automorphisms *complementation* and *permutation*. Complementation refers to interchanging 0 and 1 in certain co-ordinate positions in the n-tuple. So it basically corresponds to the power set of $\{1, 2, \ldots, n\}$. Suppose $A \subseteq \{1, 2, \ldots, n\}$, and $\sigma_A \in Aut(Q_n)$ is the corresponding complementation. Then $\sigma_A(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, where

$$y_i = \begin{cases} x_i, & \text{if } i \notin A; \\ 1 + x_i \ (mod\ 2), & \text{if } i \in A. \end{cases}$$

However, permutation corresponds to the symmetric group S_n. For $\theta \in S_n$, $\rho_\theta \in Aut(Q_n)$ is defined by $\rho_\theta(x_1, x_2, \ldots, x_n) = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$.

Every element of $Aut(Q_n)$ has a unique representation in the form $\rho_\theta o \sigma_A$. So $|Aut(Q_n)| = 2^n \cdot n!$.

As hypercube is vertex-transitive, in our algorithm the paths have initial vertex $0 = (0, 0, 0, \ldots, 0)$. So only the edge-direction sequence will determine the path. Moreover, we use only permutations from $Aut(Q_n)$ to check whether the types of a path are the same or distinct.

e.g.- (1) Consider paths $P(0, S)$ and $Q(0, S')$, where $S = (1, 2, 1)$ and $S' = (3, 4, 3)$ in Q_4. Then there exists a permutation $\theta = (1, 3)(2, 4)$, such that $\theta(P) = Q$. Therefore paths P and Q are of the same type.

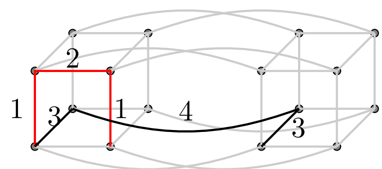


Fig. 2. Paths of same type in Q_4

(2) Suppose $S' = (1, 2, 3)$ in the above example, then there does not exist an automorhism which will map P on Q. So P and Q are of different types.

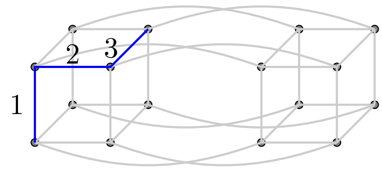


Fig. 3. Path of a type different than those in Figure 2

Remark 2.2. Note that any two paths of length k are isomorphic, graph theoretically. But they can be of different type, depending upon their direction sequences. (See the e.g.(1) and (2) above.)

The algorithm which we write, constructs the edge-direction sequences of all possible types of paths in Q_n of length k, where $n \geq 2$ and $1 \leq k \leq 2^n$. It also identifies the closed paths, i.e., cycles from the edge-direction sequences. For this purpose, we construct binary matrices with n rows and k columns, $1 \leq k \leq 2^n$. So if our path is $P_k \equiv (\mathit{0} = u_0, u_1, u_2, \ldots, u_k)$ of length k, then the matrix M_k corresponding to P_k will have n rows and k columns, where i^{th} column represents vertex u_i in the path, $i = 1, 2, \ldots, k$.

e.g.- Consider Q_4. Let $P = P_6$ be a path of length 6 in Q_4, starting with vertex $\mathit{0}$ and having an edge direction sequence $(i_1, i_2, \ldots, i_6) = (1, 2, 3, 4, 1, 3)$. Then the matrix corresponding to P will be as follows.

$$M_6 = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Observe that $e_{i_j} = u_j - u_{j-1}$, for $j = 1, 2, \ldots, 6$.

Recall that, in any path no two vertices are identical. As the columns of matrix under consideration correspond to vertices in a path, no two columns in the matrix are identical. Moreover, if the path is closed, i.e., if it is a cycle, then only the initial and final vertex are the same. So, here if a matrix corresponding to a path ends with $\mathit{0}$ vector, then we will get a cycle (as we are considering paths with initial vertex $\mathit{0}$). Conversely, if one has a cycle with initial vertex $\mathit{0}$, then its corresponding matrix ends with a column containing all zeros.

e.g.- Consider a cycle $C = (0, S)$, with $S = (1, 2, 3, 1, 2, 3)$, in Q_4. Then the corresponding matrix is

$$M_6 = \begin{array}{c} \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}.$$

So if a path is known, one can write a matrix corresponding to it. But then we have a question.

Question 2.3. *In the given hypercube Q_n, where $n \geq 2$, is it possible to find all types of paths of length k, where $1 \leq k \leq 2^n$, with the help of matrices?*

Answer to the above question is yes and is given by our algorithm in the following subsection.

2.2. Algorithm

We prove a lemma before writing the algorithm.

Lemma 2.4. *In a hypercube Q_n, $n \geq 2$, there exists only one type of path of length 2, viz. $P = (0, S)$, where $S = (1, 2)$ is the direction sequence.*

Proof. Let $Q = (v, S')$ with $S' = (i_1, i_2)$ be an arbitrary path of length 2 in Q_n. We claim that P and Q are of the same type, i.e., there exists an automorphism (say) $\phi \in Aut(Q_n)$, such that $\phi(P) = Q$.

If $P = Q$, there is nothing to prove.

Suppose $P \neq Q$. Then we have the following two cases.

Case 1. Suppose $v = 0$. Consider permutation $\phi = (1, i_1)(2, i_2)$. Then $\phi(P) = Q$.

Case 2. Suppose $v \neq 0$. So let the n-tupple corresponding to v has 1's in the places $i_1, i_2, \ldots, i_k$, where $V = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$. Now consider the complementation σ_V, which will take 0 to v. Now consider permutation $\theta = (1, i_1)(2, i_2)$. Then $\phi = \theta o \sigma_V$ is an automorphism of Q_n, such that $\phi(P) = Q$.

Hence the proof. □

Beacuse of the above lemma, for any path of length greater than 2, its edge direction sequence will have the first entry 1 and the second entry 2.

In our algorithm, we give a systematic way of generating paths of all types of a given length k in Q_n, using binary matrices of order $n \times k$, where $2 \leq k \leq 2^n$ and $n \geq 2$.

As all the paths generated in the algorithm have initial vertex 0, here onwards, we denote path by the edge-direction sequence only. So if $P \equiv P(0, S)$, where $S = (1, 2, 1)$ is the edge-direction sequence, then we denote P as just $P = (1, 2, 1)$.

Algorithm for constructing types of paths

Step 1. *Initialization:* We start with a binary matrix $M_2 = [u_1 = e_1 | u_2 = e_1 + e_2]_{n \times 2}$, which corresponds to the only type of path of length 2, $P_2 = (1, 2) = P$ (say). The idea is to extend P with the help of the matrix M_2. Note that the hypercube Q_n has the edges of directions from the set $[n]$. The path P has edges with directions 1 and 2. So we call $O = \{1, 2\}$ as the old set for P. However, edges with directions $3, \ldots, n$ are not present in P. So the set $N = \{3, \ldots, n\}$ is referred as the new set for P.

(We give the immediate step for understanding below. One can skip the same and

go to the Iteration step directly. Also note that, as there are no 3-cycles in Q_n, possibility for cycle is not discussed in Step 2.)

Step 2. Now for extending P, first consider the old set O corresponding to P. For every $i \in O$, construct a new matrix $M_3^i = [M_2|u_2 + e_i]$ of order $n \times 3$, by inserting a column at the end of M_2, which is a binary addition of the last column of M_2 and e_i. Now we consider only those M_3^i's which correspond to a path, i.e., no two columns of the matrix M_3^i are identical. We call those matrices as M_3^r, where $r \in O$ gives a valid path.

Once all the elements in the old set of P_2 are exhausted, we consider the new set N. In the new set of P, there are $n-2$ elements. Consider the smallest element of N (say) s. Then construct a matrix $M_3^s = [M_2|u_2 + e_s]$.

So we get matrices M_3^r's and a matrix M_3^s, which correspond to the valid path types of length 3. Also note that the old set and new set for paths corresponding M_3^r are same as that of P, viz. O and N. However, the old set and the new set for path corresponding to M_3^s are $O' = O \cup \{s\}$ and $N' = N \setminus \{s\}$, respectively.

Step 3. *Iteration: (i).* Suppose there are d_{k-1} types of path of length $k-1$, viz. $P_{k-1}^1, P_{k-1}^2, \ldots, P_{k-1}^{d_{k-1}}$, where $3 \leq k \leq 2^n$. Let $z_0 = 0$ and $1 \leq l \leq d_{k-1}$.

Consider P_{k-1}^l. For convenience, call it P^l. Let $M^l = [u_1|u_2|\ldots|u_{k-1}]$ be the corresponding matrix, $O^l = \{i_1, i_2, \ldots, i_j\}$, $j \leq k-1$ be the old set and $N^l = [n] \setminus O^l$ be the new set for P^l. For every $i_t \in O^l$, construct a new matrix $M^l(i_t) = [M^l|u_{k-1}+e_{i_t}]$ of order $n \times k$, by inserting a column at the end of M^l, which is a binary addition of the last column of M^l and e_{i_t}.

If any two columns in a matrix $M^l(i_t)$, for some i_t are identical, then the direction sequence corresponding to the matrix $M^l(i_t)$ does not represent a path. So it is not valid.

If the last column of a matrix $M^l(i_t)$, for some i_t is identical to *0* vector, then the direction sequence corresponding to the matrix $M^l(i_t)$ is a cycle. Moreover, it can not be extended further.

So we consider only those $M^l(i_t)$'s which correspond to a path, i.e., no two columns of the matrix $M^l(i_t)$ are identical for further expansion. Suppose there are $z_l - 1$ such matrices. So we rename those matrices as $M_{k_r}^l$, where $r \in \{z_{l-1}+1, z_{l-1}+2, \ldots, z_{l-1}+z_l-1\}$. Here l in the superscript stands for P^l, k in the subscript stands for the type of path of length k. Also note that the old set and the new set for types corresponding to $M_{k_r}^l$, where $r \in \{z_{l-1}+1, z_{l-1}+2, \ldots, z_{l-1}+z_l-1\}$, are same as that of P^l, i.e., O^l and N^l respectively.

Once all the elements in the old set of P^l are exhausted, we consider the new set N^l. In the new set of P^l, there are $n-j$ elements. Consider the smallest element of N^l (say) s. Then construct the matrix $M^l(s) = [M^l|u_{k-1} + e_s]$ and call it as $M_{k_{z_{l-1}+z_l}}^l$. Now the old set and the new set for path corresponding to $M_{k_{z_{l-1}+z_l}}^l$ are $O' = O^l \cup \{s\}$ and $N' = N^l \setminus \{s\}$, respectively.

So we get matrices $M^l_{k_r}$'s, where $z_{l-1}+1 \le r \le z_{l-1}+z_l$, which correspond to the valid path types of length k obtained from P^l. For convenience, we call these matrices as M^r_k, and corresponding paths as P^r_k, where $z_{l-1}+1 \le r \le z_{l-1}+z_l$.

(ii). Let $z_1 + z_2 + \ldots + z_{d_{k-1}} = d_k$.

Step 4. Repeat Step 3 for all the types of path of length k obtained in Step 3, till $k \le 2^n$.

Complexity analysis of the algorithm:

- Number of paths in Q_n: The number of simple paths in a hypercube grows exponentially. The number of Hamiltonian paths alone in Q_n is approximately $2^{n^2/2}$.
- Iterative Extensions: Each step extends all valid paths, leading to a combinatorial explosion.
- Checking for isomorphism: Determining unique paths up to isomorphism involves symmetry reductions, which is computationally expensive.

Given these factors, the worst-case time complexity is at least exponential, likely in the range of $O(2^{O(n^2)})$ due to the rapid growth in possible paths. The space complexity is also exponential since all valid paths must be stored.

Using this algorithm, a computer program was written to obtain all possible types of cycles in Q_n. We obtained types of cycles of all possible lengths in Q_4, using the program. But for the higher values of n, we need supercomputers to generate the types of cycles.

We give an illustration of the algorithm in Figure 4 for paths up to length 6 in Q_3. We have represented the algorithm as a rooted tree, with root $(1,2)$,, i.e., our initial path in the algorithm. The paths of length 3 constructed from $(1,2)$ are shown by the branches of the root, and we continue so on till we get paths of length 6. The paths shown in bold type are cycles.

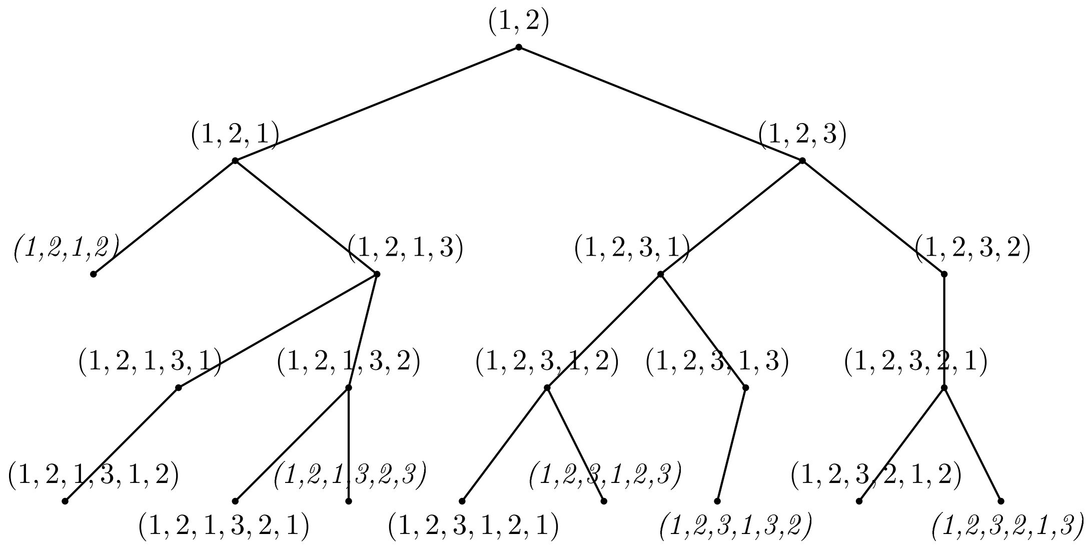


Fig. 4. Paths upto length 6 in Q_3

We also prove that any two types of a path of length k obtained from the algorithm are different, i.e., there does not exist an automorphism between any two types.

Theorem 2.5. *In a hypercube Q_n, $n \geq 2$, types of a path of length k obtained from the above algorithm are mutually distinct.*

Proof. We prove the result by induction on k.

For $k = 2$, we get path $(1, 2)$ by the algorithm. But that is the only type of path of length 2, by Lemma 2.4.

Suppose $k = 3$. Then by the algorithm, we get two types (say) $P_1 = (1, 2, 1)$ and $P_2 = (1, 2, 3)$. Suppose there exists an automorphism ϕ, such that $\phi(P_1) = P_2$. But then $\phi(1) = 1, \phi(2) = 2$ and $\phi(1) = 3$, which is a contradiction to ϕ is one-to-one, being an automorphism. So the two types P_1 and P_2 are distinct.

Assume the result for types of path of length $k - 1$.

Let P and Q be two types of path of length k generated by the algorithm. So suppose the type P is given by $(i_1, i_2, \ldots, i_{k-1}, i_k)$. Then the sequence $(i_1, i_2, \ldots, i_{k-1})$ corresponds to the type (say) P' of path of length $k - 1$, from which P is generated in the algorithm by adding a direction i_k. Similarly, let Q be given by $(j_1, j_2, \ldots, j_{k-1}, j_k)$, such that $Q' = (j_1, j_2, \ldots, j_{k-1})$ is the corresponding type of length $k - 1$.

If $P' = Q'$, then by construction in the algorithm, we must have $i_k \neq j_k$, as P and Q are not the same.

Suppose $P' \neq Q'$, i.e., P' and Q' are two distinct types of path of length $k-1$, generated by the algorithm. Now suppose if possible, there exists an automorphism ϕ such that $\phi(P) = Q$. But then one will get $\phi(P') = Q'$, a contradiction to the induction hypothesis. Therefore there does not exist an automorphism between P and Q.

□

We prove that, by using the algorithm, one gets all possible types of paths of length k, where $2 \leq k \leq 2^n$.

Theorem 2.6. *In a hypercube Q_n, $n \geq 2$, all the types of a path of length k, where $2 \leq k \leq 2^n$, are generated by the above algorithm.*

Proof. We prove the result by induction on k.

For $k = 2$, the type $(1, 2)$ is the initial path of the algorithm. But by Theorem 2.5, it is the only type for path of length 2.

Suppose $k = 3$. Then by the algorithm, we get two types (say) $P_1 = (1, 2, 1)$ and $P_2 = (1, 2, 3)$. We claim that these are the only two types of path of length 3. Suppose if possible, there exists another type (say) $Q = (i_1, i_2, i_3)$ of the path of length 3. Then we have the following cases.

Case 1. If $i_1 = i_2 = i_3$, then Q is not a path of length 3.

Case 2. Suppose any two among i_1, i_2, i_3 are same and the third is different. Then in order to form a path of length 3, the direction sequence must have equal directions at the first and third place. So we must have $i_1 = i_3$. So $Q = (i_1, i_2, i_1)$. But then there exists a permutation $\phi = (1, i_1)(2, i_2)$ such that $\phi(P_1) = Q$, a contradiction.

Case 3. Suppose i_1, i_2, i_3 are all distinct. Then there exists a permutation $\phi = (1, i_1)(2, i_2)(3, i_3)$ such that $\phi(P_2) = Q$, a contradiction.

Therefore P_1 and P_2 are the only types of path of length 3.

Assume the result for types of path of length $k-1$.

Now we prove the statement of the theorem for the types of path of length k. Suppose if possible there exists a type (say) $P = (i_1, i_2, \ldots, i_{k-1}, i_k)$ of path of length k, which is distinct from the types we get by the algorithm. But then the sequence $(i_1, i_2, \ldots, i_{k-1})$ corresponds to the type (say) P' of path of length $k-1$, from which P is generated, by adding a direction i_k. According to the induction hypothesis, all the types of path of length $k-1$ are generated in the algorithm. So there must be an automorphism, which takes P' to some type Q' of path of length $k-1$, generated by the algorithm. For convenience, we can consider $P' = Q'$.

Now one needs to think about i_k. If $i_k = i_j$ for some $j \in \{1, 2, k-1\}$, then i_k falls in the old set corresponding to P' in the algorithm. But then, in the algorithm, all the elements in the old set for P' are checked so as to get a valid type for path of length k from P'. So if P is a valid path type, then we are through. Suppose $i_k \neq i_j$, for all $j = 1, 2, \ldots, k-1$. Then i_k belongs to the new set corresponding to P'. In the algorithm, we consider only the smallest element from the new set corresponding to P' for constructing a path type of length 1 more than P'. Let s be the smallest element of the new set corresponding to P'. If $i_k = s$, we get P from the algorithm. If not, then there exists a permutation $\theta = (i_k, s)$, such that $\theta(P)$ is a type of path of length k generated by the algorithm. □

A path of length k in a hypercube needs at the most k directions, while a cycle of length k needs at the most $k/2$ directions. So we have the following remark.

Remark 2.7. Suppose Q_k is a subcube of Q_n, i.e., $k \leq n$. Then the types of paths upto length k in Q_k are the only types of paths upto length k in Q_n. Similarly, the types of cycles upto length $2k$ in Q_k are the only types of cycles upto length $2k$ in Q_n.

We are particularly interested in the types of a k-cycle in hypercube Q_n, where k is even and $4 \leq k \leq 2^n$. From the above algorithm, we get all types of cycles as well. In fact one can write a computer program to get all the types of a k-cycle in Q_n. We could generate all the types of k-cycles in Q_4 using that program, where k is even and $4 \leq k \leq 16$. In fact, for $n = 4$ itself, we got 392 types of cycles of length 14. One needs a super computer to generate the paths and cycles of Q_n, even if n is as small as 6.

In the next section, we discuss all possible cycle decompositions of Q_4.

3. Cycle decompositions of Q_4

Notation 3.1. We use ‘ $\sqcup$ ’ for the union of edge disjoint graphs and ‘ $\uplus$ ’ for the union of vertex disjoint graphs.

3.1. Possible/impossible cycle decompositions

Note that $|V(Q_4)| = 2^4 = 16$ and $|E(Q_4)| = 32$. We look at the problem of finding cycle decompositions of Q_4 in view of partitioning the number 32 as $k_1 + k_2 + \ldots + k_r$, where

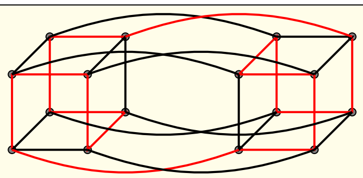
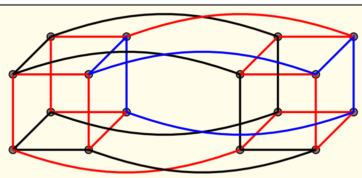
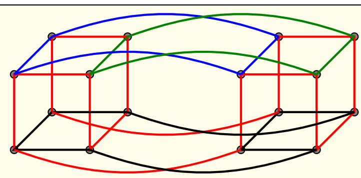
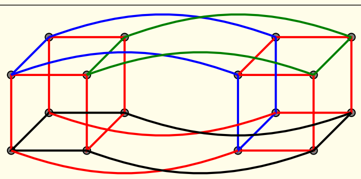
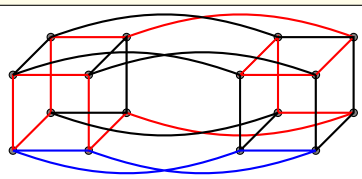
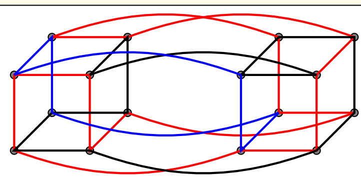
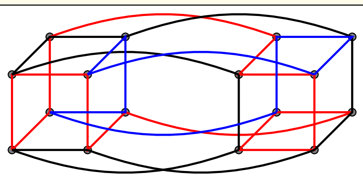
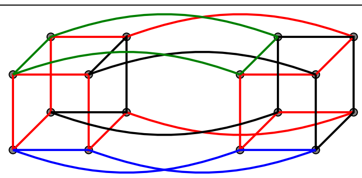
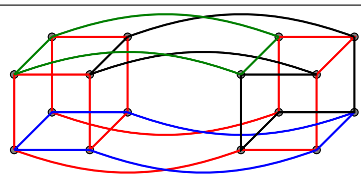
k_i's are all even and $4 \leq k_i \leq 16, \ i = 1, 2, \ldots, r$. Following is the list of such partitions.

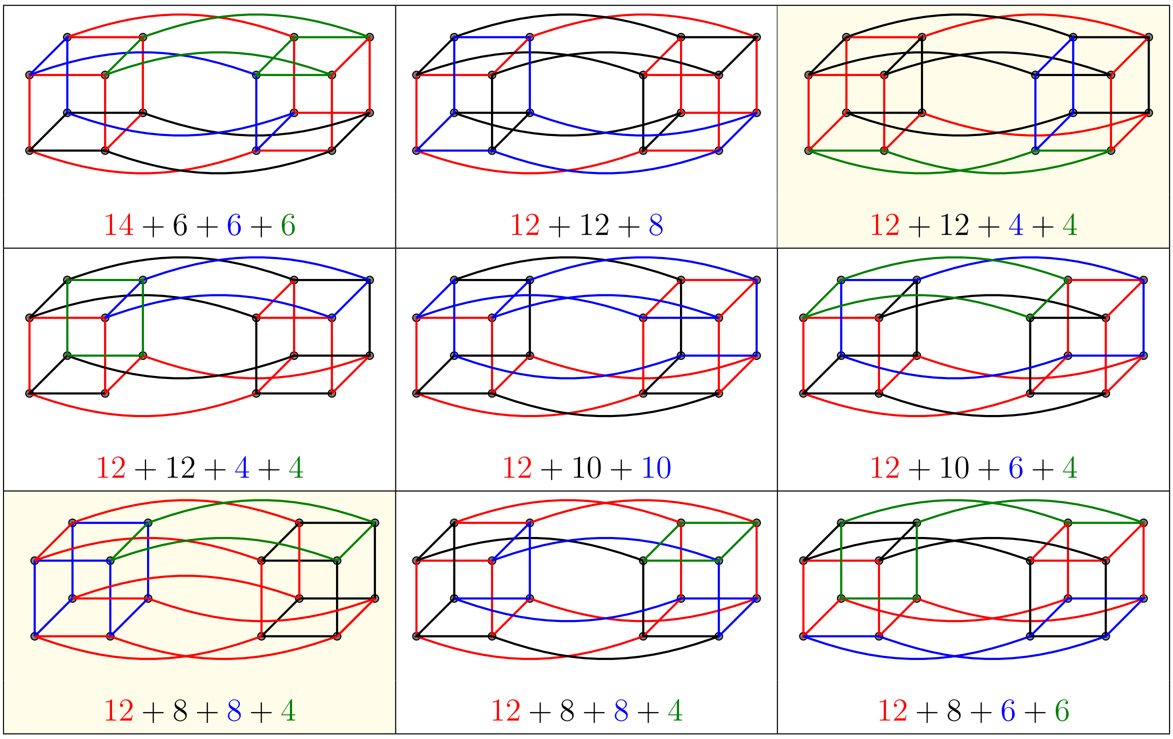
Table 1.

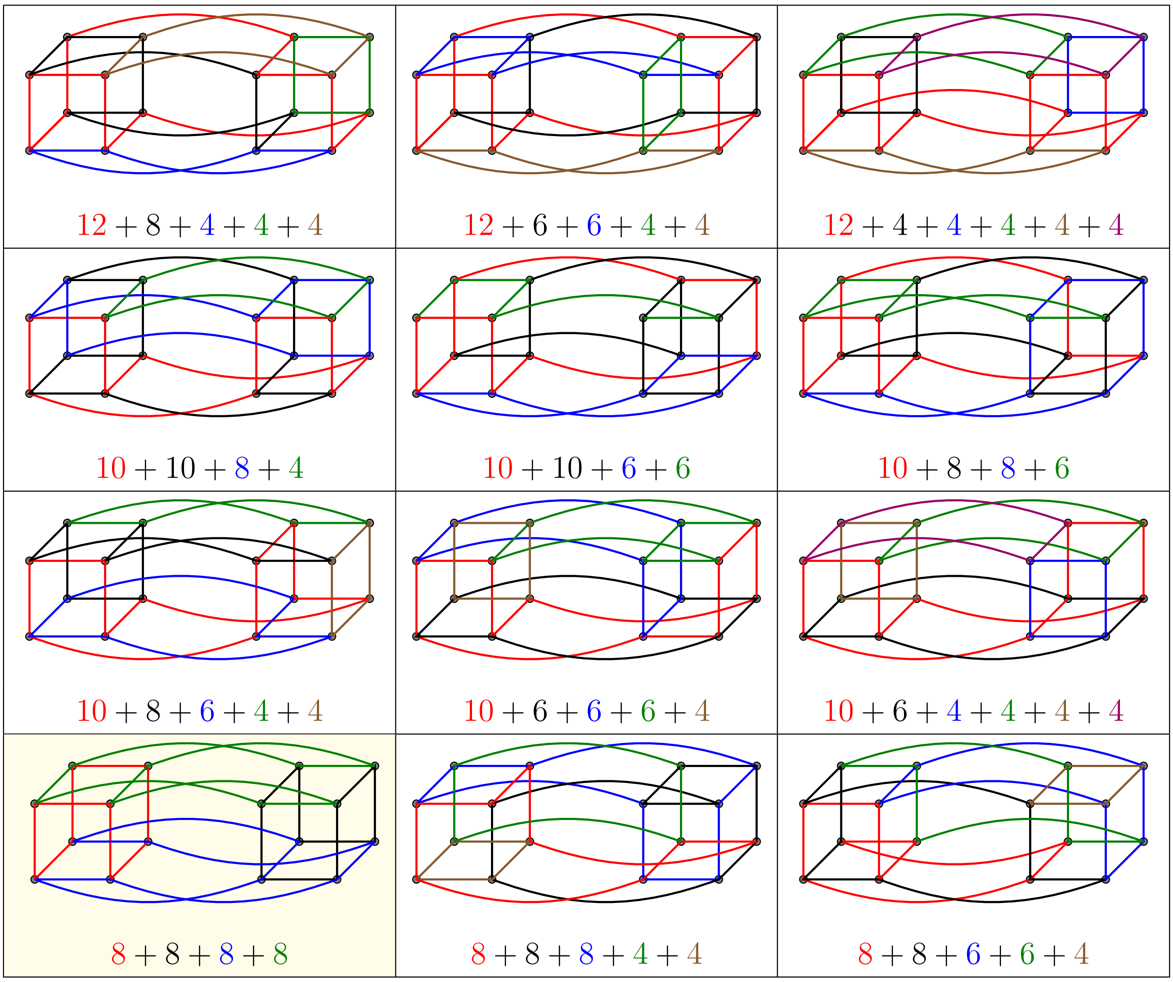
16+16	16+10+6	16+8+4+4	16+6+6+4
14+14+4	14+12+6	14+10+8	14+10+4+4
14+8+6+4	14+6+6+6	12+12+8	12+12+4+4
12+10+10	12+10+6+4	12+8+8+4	12+8+6+6
12+8+4+4+4	12+6+6+4+4	12+4+4+4+4+4	10+10+8+4
10+10+6+6	10+8+8+6	10+8+6+4+4	10+6+6+6+4
10+6+4+4+4+4	8+8+8+8	8+8+8+4+4	8+8+6+6+4
8+8+4+4+4+4	8+6+6+4+4+4	6+6+6+6+4+4	4+4+4+4+4+4+4+4
16+ 12+ 4	16+ 8+ 8	16+4+4+4+4	14+6+4+4+4
10+10+4+4+4	8+6+6+6+6	8+4+4+4+4+4+4	6+6+4+4+4+4+4

We observe that, there exists a cycle decomposition corresponding to the partitions with white background in Table 1. However, corresponding to the partitions with a pink background, there does not exist any cycle decomposition of Q_4, which we have verified with the help of the algorithm. In Table 2, possible cycle decompositions of Q_4 are drawn.

Table 2.

 $16 + 16$	 $16 + 10 + 6$	 $16 + 8 + 4 + 4$
 $16 + 6 + 6 + 4$	 $14 + 14 + 4$	 $14 + 12 + 6$
 $14 + 10 + 8$	 $14 + 10 + 4 + 4$	 $14 + 8 + 6 + 4$


14 + 6 + 6 + 6
12 + 12 + 8
12 + 12 + 4 + 4
12 + 12 + 4 + 4
12 + 10 + 10
12 + 10 + 6 + 4
12 + 8 + 8 + 4
12 + 8 + 8 + 4
12 + 8 + 6 + 6


12 + 8 + 4 + 4 + 4
12 + 6 + 6 + 4 + 4
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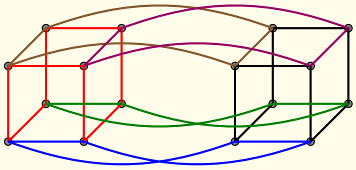
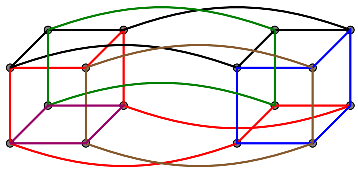
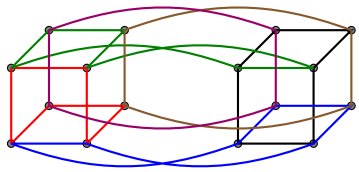
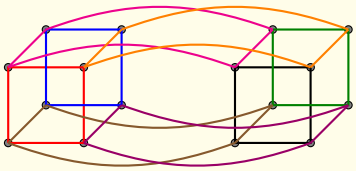
 $8+8+4+4+4+4$	 $8+6+6+4+4+4$	 $6+6+6+6+4+4$
 $4+4+4+4$ $+4+4+4+4$		

In the table, the 2-regular, spanning subgraphs of Q_4 are shown in the cells with yellow background. They are called as 2-factors. From these factorizations of Q_4 we can get factorizations of hypercubes of higher dimensions.

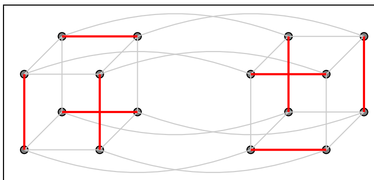
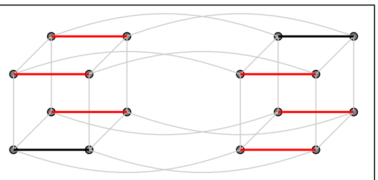
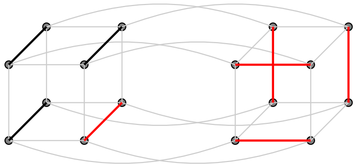
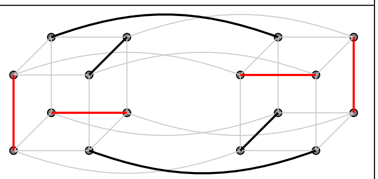
Now we discuss the cycle decomposition of Q_4 with perfect matchings, which also leads to the cycle decompositions of higher dimensional cubes.

3.2. *Cycle decompositions of Q_4 with perfect matchings*

Graham and Harary [5] proved that there are 272 different perfect matchings of Q_4. In each of the above possible cycle decompositions, we can have many perfect matchings that can be selected from the cycles in the decomposition.

For illustration, consider the first Hamiltonian decomposition of Q_4 in Table 2. We can choose different perfect matchings from the given cycle decomposition of Q_4 as shown in Table 3.

Table 3.

These decompositions will help us to get cycle decompositions of Q_6 as given in the next section.

4. Cycle decompositions of higher dimensional hypercubes

In this section, we find some cycle decompositions of $Q_6, Q_8, \dots$.

Theorem 4.1 (Induction step for Hypercubes)**.** *Let $n \geq 6$ be even. Suppose Q_{n-2} can be decomposed into $r(n-2)$-cycles $C_1, C_2, \dots, C_s$ where $s = 2^{n-r-3}$ such that each C_i contains r edges $e_{i1}, e_{i2}, \dots e_{ir}$, such that the set $M_i = \{e_{ij} : j = 1, 2, \dots, r;\ i = 1, 2, \dots, s\}$ forms a perfect matching in Q_{n-2}. Then Q_n can be decomposed into rn-cycles $Z_1, Z_2, \dots, Z_{4s}$ such that each Z_i contains r edges $f_{i1}, f_{i2}, \dots f_{ir}$, such that the set $\{f_{ij} : j = 1, 2, \dots, r;\ i = 1, 2, \dots, 4s\}$ forms a perfect matching in Q_n.*

Proof. Write Q_n as $Q_n = Q_{n-2} \square\, Q_2$. Let $H = Q_{n-2}$, $G = Q_n$ and Q_2 is nothing but a 4-cycle. Now Q_{n-2} is decomposed into $r(n-2)$-cycles, viz. $C_1, C_2, \dots, C_s$. Further, from each C_i, a set of r edges, viz. $M_i = \{e_{i1}, e_{i2}, \dots e_{ir}\}$ is selected such that $\bigcup_{i=1}^{s} M_i$ forms a perfect matching in Q_{n-2}. So Q_n has a decomposition into cycles $C_1^j, C_2^j, \dots, C_s^j$, $j = 1, 2, 3, 4$ of length $r(n-2) + 2r = rn$. Further, each C_i^j contains a set M_i^{j+1} of r edges such that $\bigcup_{j=1}^{4} \bigcup_{i=1}^{s} M_i^j$ is a perfect matching of Q_n (addition for j is taken modulo 4.) Rename cycle C_i^j as $Z_{(j-1)s+i}$ and edge $e_{it}^{j+1} \in M_i^{j+1}$ as $f_{js+i\ t}$ where $i = 1, 2, \dots, s$, $t = 1, 2, \dots, r$ and $j = 1, 2, 3, 4$. Hence the result. $\square$

We give illustration of the theorem below.

Illustration. Consider a hamiltonian decomposition of Q_4 into cycles C_1 (red colour) and C_2 (black colour) along with a perfect matching containing 4 edges each from both the cycles (shown by bold edges in Figure 5). Let the edges from cycle C_1 contributing to the perfect matching be $\{e_1 = (u_1, v_1), e_2 = (u_2, v_2), e_3 = (u_3, v_3), e_4 = (u_4, v_4)\}$ and those from C_2 be $\{e_5 = (u_5, v_5), e_6 = (u_6, v_6), e_7 = (u_7, v_7), e_8 = (u_8, v_8)\}$.

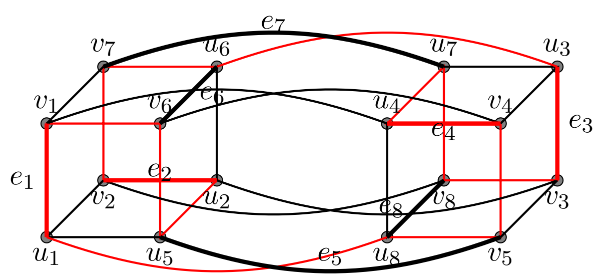


Fig. 5. Cycle decomposition of Q_4 with a perfect matching

We construct 24-cycles in Q_6 from the 16-cycles in the above decomposition and edges of perfect matching in Q_4. We know that $Q_6 = Q_4 \square Q_2$, where Q_2 is nothing but a 4-cycle. So Q_6 consists of four copies of Q_4 (say) $Q_4^{00}, Q_4^{01}, Q_4^{11}$ and Q_4^{10}, whose corresponding vertices are joined as shown by connections given in the following Figure 6.

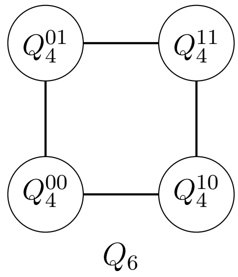


Fig. 6. Structure of Q_6

Let C_1^{ij} and C_2^{ij} be the 16-cycles in Q_4^{ij} and let u_k^{ij}, v_k^{ij} be the end vertices of the edges e_k^{ij} in the perfect matching of Q_4^{ij}, $i, j \in \{0,1\}$, $k \in \{1,2,\ldots,8\}$.

Now the 24-cycles which decompose Q_6 are as follows.

(a)

$$\begin{aligned} Z_1 =&[C_1^{00} \ \setminus \ \{e_1^{00}, e_2^{00}, e_3^{00}, e_4^{00}\}] \cup \{e_1^{01}, e_2^{01}, e_3^{01}, e_4^{01}, (u_1^{00}, u_1^{01}), (u_2^{00}, u_2^{01}), (u_3^{00}, u_3^{01}), \\ &(u_4^{00}, u_4^{01}), (v_1^{00}, v_1^{01}), (v_2^{00}, u_2^{01}), (v_3^{00}, v_3^{01}), (v_4^{00}, v_4^{01})\} \end{aligned}$$

(b)

$$\begin{aligned} Z_2 =&[C_1^{01} \setminus \{e_1^{01}, e_2^{01}, e_3^{01}, e_4^{01}\}] \cup \{e_1^{11}, e_2^{11}, e_3^{11}, e_4^{11}, (u_1^{01}, u_1^{11}), (u_2^{01}, u_2^{11}), (u_3^{01}, u_3^{11}), \\ &(u_4^{01}, u_4^{11}), (v_1^{01}, v_1^{11}), (v_2^{01}, u_2^{11}), (v_3^{01}, v_3^{11}), (v_4^{01}, v_4^{11})\} \end{aligned}$$

(c)

$$\begin{aligned} Z_3 =&[C_1^{11} \setminus \{e_1^{11}, e_2^{11}, e_3^{11}, e_4^{11}\}] \cup \{e_1^{10}, e_2^{10}, e_3^{10}, e_4^{10}, (u_1^{11}, u_1^{10}), (u_2^{11}, u_2^{10}), (u_3^{11}, u_3^{10}), \\ &(u_4^{11}, u_4^{10}), (v_1^{11}, v_1^{10}), (v_2^{11}, u_2^{10}), (v_3^{11}, v_3^{10}), (v_4^{11}, v_4^{10})\} \end{aligned}$$

(d)

$$\begin{aligned} Z_4 =&[C_1^{10} \setminus \{e_1^{10}, e_2^{10}, e_3^{10}, e_4^{10}\}] \cup \{e_1^{00}, e_2^{00}, e_3^{00}, e_4^{00}, (u_1^{10}, u_1^{00}), (u_2^{10}, u_2^{00}), (u_3^{10}, u_3^{00}), \\ &(u_4^{10}, u_4^{00}), (v_1^{10}, v_1^{00}), (v_2^{10}, u_2^{00}), (v_3^{10}, v_3^{00}), (v_4^{10}, v_4^{00})\} \end{aligned}$$

(e)

$$\begin{aligned} Z_5 =&[C_2^{00} \setminus \{e_5^{00}, e_6^{00}, e_7^{00}, e_8^{00}\}] \cup \{e_5^{01}, e_6^{01}, e_7^{01}, e_8^{01}, (u_5^{00}, u_5^{01}), (u_6^{00}, u_6^{01}), (u_7^{00}, u_7^{01}), \\ &(u_8^{00}, u_8^{01}), (v_5^{00}, v_5^{01}), (v_6^{00}, u_6^{01}), (v_7^{00}, v_7^{01}), (v_8^{00}, v_8^{01})\} \end{aligned}$$

(f)

$$\begin{aligned} Z_6 =&[C_2^{01} \setminus \{e_5^{01}, e_6^{01}, e_7^{01}, e_8^{01}\}] \cup \{e_5^{11}, e_6^{11}, e_7^{11}, e_8^{11}, (u_5^{01}, u_5^{11}), (u_6^{01}, u_6^{11}), (u_7^{01}, u_7^{11}), \\ &(u_8^{01}, u_8^{11}), (v_5^{01}, v_5^{11}), (v_6^{01}, u_6^{11}), (v_7^{01}, v_7^{11}), (v_8^{01}, v_8^{11})\} \end{aligned}$$

(g)

$$\begin{aligned} Z_7 =&[C_2^{11} \setminus \{e_5^{11}, e_6^{11}, e_7^{11}, e_8^{11}\}] \cup \{e_5^{10}, e_6^{10}, e_7^{10}, e_8^{10}, (u_5^{11}, u_5^{10}), (u_6^{11}, u_6^{10}), (u_7^{11}, u_7^{10}), \\ &(u_8^{11}, u_8^{10}), (v_5^{11}, v_5^{10}), (v_6^{11}, u_6^{10}), (v_7^{11}, v_7^{10}), (v_8^{11}, v_8^{10})\} \end{aligned}$$

(h)

$$Z_8 = [C_2^{10} \setminus \{e_5^{10}, e_6^{10}, e_7^{10}, e_8^{10}\}] \cup \{e_5^{00}, e_6^{00}, e_7^{00}, e_8^{00}, (u_5^{10}, u_5^{00}), (u_6^{10}, u_6^{00}), (u_7^{10}, u_7^{00}), (u_8^{10}, u_8^{00}), (v_5^{10}, v_5^{00}), (v_6^{10}, u_6^{00}), (v_7^{10}, v_7^{00}), (v_8^{10}, v_8^{00})\}$$

See Figure 7 for better understanding where we have shown the construction of cycle Z_1. One can construct the remaining cycles in the decomposition on the similar lines.

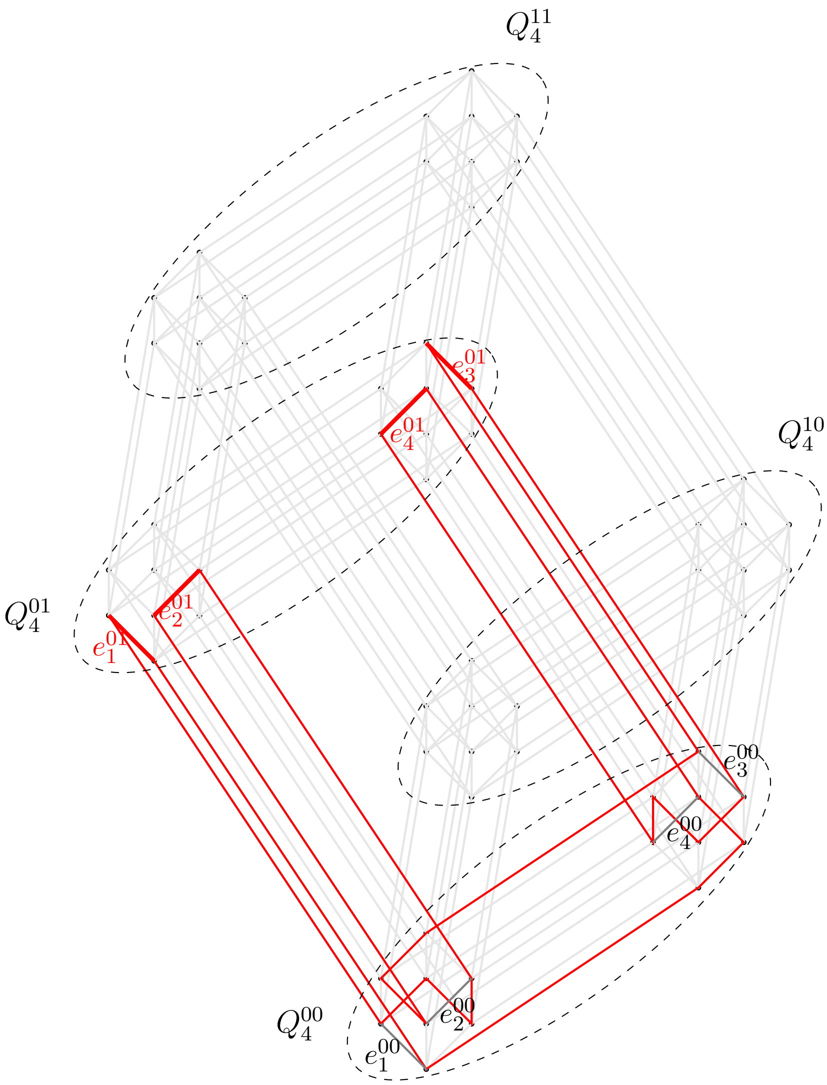


Fig. 7. Construction of a 24-cycle

Because cycles C_1 and C_2 of Q_4 are edge disjoint, it can be easily verified that the cycles $Z_1, Z_2, \ldots, Z_8$ are mutually edge disjoint. Moreover, all the cross edges between any two copies of Q_4 are exhausted, as they are taken along the edges of perfect matching in Q_4. Therefore we get edge decomposition of Q_6 into cycles of length 24.

Moreover as Q_6 contains four copies of Q_4 and perfect matchings in all of them are selected. Then $\bigcup_{i=1}^{8} M_i$ gives a perfect matching in Q_6, where M_i is the set of edges from Z_i, contributing to a perfect matching of Q_6. □

The following table gives the generalization of cycle decomposition of Q_4 with a perfect matching to the corresponding cycle decomposition of Q_n, with a perfect matching.

Table 4.

$Q_4 = C_1 \cup C_2$		Corresponding cycle decomposition of Q_6	Corresponding cycle decomposition of Q_8	Corresponding cycle decomposition of Q_n
m_1	m_2			
8	0	$4 \times C_{32} + 4 \times C_{16}$	$16 \times C_{48} + 16 \times C_{16}$	$2^{n-4} \times C_{8(n-2)} + 2^{n-4} \times C_{16}$
6	2	$4 \times C_{28} + 4 \times C_{20}$	$16 \times C_{40} + 16 \times C_{24}$	$2^{n-4} \times C_{6(n-2)+4} + 2^{n-4} \times C_{2(n-4)+16}$
5	3	$4 \times C_{26} + 4 \times C_{22}$	$16 \times C_{36} + 16 \times C_{28}$	$2^{n-4} \times C_{5(n-2)+6} + 2^{n-4} \times C_{3(n-4)+16}$
4	4	$8 \times C_{24}$	$32 \times C_{32}$	$2^{n-3} \times C_{4n}$

Theorem 4.2. *Let $n \geq 4$ be even. Then the hypercube Q_n can be decomposed into $4n$-cycles (say) $C_1, C_2, \ldots, C_r$, where $r = 2^{n-3}$ such that*

(I) *every C_i contains 4 edges $e_{i1}, e_{i2}, e_{i3}, e_{i4}$ such that $C_i \setminus \{e_{i1}, e_{i2}, e_{i3}, e_{i4}\}$ has 4 components each of which is a path of length $n-1$;*

(II) *$M = \{e_{ij} : j = 1, 2, 3, 4; i = 1, 2, \ldots, r\}$ forms a perfect matching in Q_n.*

Proof. Proof of the theorem is by induction. For $n = 4$ and $n = 6$, the result is true by Subsection 3.2 and illustration after Theorem 4.1 respectively. Proof of the induction step follows by Theorem 4.1. □

5. Factorization of Q_8 from a factorization of Q_4

In this section, we discuss factorizations of Q_8 obtained from those of Q_4, using properties of Cartesian product of graphs. We state a lemma, whose proof is trivial and follows from the definition of the Cartesian product of graphs.

Lemma 5.1. [11]

(a) *Let a graph G_1 be decomposed into spanning subgraphs $H_1, H_2, \ldots, H_r$ and let a graph G_2 be decomposed into spanning subgraphs $F_1, F_2, \ldots, F_r$. Then the graph $G_1 \Box G_2$ can be decomposed into spanning subgraphs $H_1 \Box F_1, H_2 \Box F_2, \ldots, H_r \Box F_r$.*

(b) *Let G_1 be a graph with components $H_1, H_2, \ldots, H_r$ and let G_2 be a graph with components $F_1, F_2, \ldots, F_s$. Then the components of $G_1 \Box\, G_2$ are $H_i \Box\, F_j$ with $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$.*

We need the first part of the lemma for $r = 2$. So we quote the following remark.

Remark 5.2. If graphs G_1 and G_2 are decomposed into two spanning subgraphs H_1, H_2

and F_1, F_2 respectively, then the graph $G_1 \square G_2$ can be decomposed into spanning subgraphs $H_1 \square F_1, H_2 \square F_2$. In fact, $G_1 \square G_2$ can be decomposed into spanning subgraphs $H_1 \square F_2$ and $H_2 \square F_1$ as well. So $G_1 \square G_2$ has two decompositions $(H_1 \square F_1) \cup (H_2 \square F_2)$ and $(H_1 \square F_2) \cup (H_2 \square F_1)$.

We recall the following lemma by Kotzig.

Lemma 5.3 ([7]). *Cartesian product of two cycles can be decomposed into two Hamiltonian cycles.*

For the decomposition of Q_8 using Q_4, we need 2-factorizations of Q_4, i.e., decomposition of Q_4 into 2-regular, spanning subgraphs. Recall the table of all possible cycle decompositions of Q_4 given in Section 3.1. In that table, we have coloured the cells having factorizations of Q_4. We call the two spanning subgraphs in each factorization of Q_4 as H_1 and H_2. Being 2-regular, H_1, H_2 consist of either a cycle or union of vertex disjoint cycles.

Now $Q_8 = Q_4 \square Q_4$. By Remark 5.2, corresponding to each pair of factorization $H_1 \sqcup H_2$ of Q_4, there exist two factorizations of Q_8, viz. $(H_1 \square H_1) \sqcup (H_2 \square H_2)$ and $(H_1 \square H_2) \sqcup (H_2 \square H_1)$. First, we give the Table 5 of factorizations of Q_4 with reference from Table 2.

Table 5.

Factorization of Q_4	H_1	H_2
$C_{16} \sqcup C_{16}$	C_{16}	C_{16}
$C_{16} \sqcup (C_{10} \uplus C_6)$	C_{16}	$C_{10} \uplus C_6$
$C_{16} \sqcup (C_8 \uplus C_4 \uplus C_4)$	C_{16}	$C_8 \uplus C_4 \uplus C_4$
$C_{16} \sqcup (C_6 \uplus C_6 \uplus C_4)$	C_{16}	$C_6 \uplus C_6 \uplus C_4$
$(C_{12} \uplus C_4) \sqcup (C_{12} \uplus C_4)$	$C_{12} \uplus C_4$	$C_{12} \uplus C_4$
$(C_{12} \uplus C_4) \sqcup (C_8 \uplus C_8)$	$C_{12} \uplus C_4$	$C_8 \uplus C_8$
$(C_8 \uplus C_8) \sqcup (C_8 \uplus C_8)$	$C_8 \uplus C_8$	$C_8 \uplus C_8$
$(C_8 \uplus C_8) \sqcup (C_4 \uplus C_4 \uplus C_4 \uplus C_4)$	$C_8 \uplus C_8$	$C_4 \uplus C_4 \uplus C_4 \uplus C_4$
$(C_4 \uplus C_4 \uplus C_4 \uplus C_4) \sqcup (C_4 \uplus C_4 \uplus C_4 \uplus C_4)$	$C_4 \uplus C_4 \uplus C_4 \uplus C_4$	$C_4 \uplus C_4 \uplus C_4 \uplus C_4$

Now we work out two factorizations of Q_8 corresponding to one of the factorizations of Q_4.

We deliberately consider a factorization, where H_1 and H_2 are non-isomorphic.

Factorization of Q_4: $C_{16} \sqcup (C_{10} \uplus C_6)$'

$$H_1 = C_{16} \text{ and } H_2 = (C_{10} \uplus C_6). \text{ So } Q_4 = H_1 \sqcup H_2.$$

Factorizations of Q_8:

$$\begin{aligned} Q_8 =& Q_4 \Box Q_4 = (H_1 \sqcup H_2) \Box (H_1 \sqcup H_2) \\ =& (H_1 \Box H_1) \sqcup (H_2 \Box H_2) \ldots (I) \\ =& (H_1 \Box H_2) \sqcup (H_2 \Box H_1) \ldots (II), \qquad \text{(by Remark 5.2.)} \end{aligned}$$

As $H_1 \neq H_2$, here (I) and (II) will give two different factorizations of Q_8 as follows.
(I).

$$\begin{aligned} Q_8 =& (H_1 \Box H_1) \sqcup (H_2 \Box H_2) \\ =& [C_{16} \Box C_{16}] \sqcup [(C_{10} \uplus C_6) \Box (C_{10} \uplus C_6)] \\ =& [C_{16} \Box C_{16}] \sqcup [(C_{10} \Box C_{10}) \uplus (C_{10} \Box C_6) \uplus (C_6 \Box C_{10}) \uplus (C_6 \Box C_6)], \qquad \text{(by Lemma 5.1(2).)} \end{aligned}$$

Now by Lemma 5.3, the Cartesian product of two cycles can be decomposed into two Hamiltonian cycles (say) C^1_{256} and C^2_{256}. So $C_{16} \Box C_{16} = C^1_{256} \sqcup C^2_{256}$, where vertex set of the cycles C^1_{256} and C^2_{256} is the same. We get cycles from the other brackets on the similar lines. Therefore

$$Q_8 = [C^1_{256} \sqcup C^2_{256}] \sqcup [(C^1_{100} \sqcup C^2_{100}) \uplus (C^1_{60} \sqcup C^2_{60}) \uplus (C^1_{60} \sqcup C^2_{60}) \uplus (C^1_{36} \sqcup C^2_{36})].$$

So we get cycle decomposition of Q_8. In fact, we can regroup the cycles as follows to get the factorization of Q_8.

$$Q_8 = C^1_{256} \sqcup C^2_{256} \sqcup [C^1_{100} \uplus C^1_{60} \uplus C^1_{60} \uplus C^1_{36}] \sqcup [C^2_{100} \uplus C^2_{60} \uplus C^2_{60} \uplus C^2_{36}].$$

(II).

$$\begin{aligned} Q_8 =& (H_1 \Box H_2) \sqcup (H_2 \Box H_1) = [C_{16} \Box (C_{10} \uplus C_6)] \sqcup [(C_{10} \uplus C_6) \Box C_{16}] \\ =& [(C_{16} \Box C_{10}) \uplus (C_{16} \Box C_6)] \sqcup [(C_{10} \Box C_{16}) \uplus (C_6 \Box C_{16})], \quad \text{(by Lemma 5.1(2).)} \end{aligned}$$

$$Q_8 = [(C^1_{160} \sqcup C^2_{160}) \uplus (C^1_{96} \sqcup C^2_{96})] \sqcup [(C^1_{160} \sqcup C^2_{160}) \uplus (C^1_{96} \sqcup C^2_{96})], \quad \text{(by Lemma 5.3)}$$

$$Q_8 = [C^1_{160} \uplus C^1_{96}] \sqcup [C^2_{160} \uplus C^2_{96}] \sqcup [C^1_{160} \uplus C^1_{96}] \sqcup [C^2_{160} \uplus C^2_{96}],$$

is the factorization of Q_8.

Observe that the two factorizations of Q_8, which we get from a single factorization of Q_4 are non-isomorphic.

Remark 5.4. Consider the decomposition of Q_4 into 8-cycles. It is actually a factorization of Q_4. From this factorization, we get a factorization of Q_8 into 64-cycles. In fact, one can choose 8 edges from each cycle in this decomposition of Q_8, such that their union forms a perfect matching.

One can repeat the above procedure to get factorizations of Q_{16} from the factorizations of Q_8 and so on to get factorizations of Q_n, where n is a power of 2.

6. Concluding remarks

In this paper, we gave an algorithm and its complexity to find all possible types of paths in Q_n. In particular, from the types of cycles, we get a complete list of all possible and impossible cycle decompositions of Q_4. Using cycle decompositions of Q_4 along with perfect matchings, various cycle decompositions of Q_6, Q_8 and in general Q_n can also be obtained. Moreover, the factorization of Q_8 using factorization of Q_4 is obtained. One can generalise the process to get factorizations of Q_n for n power of 2, from the factorizations of $Q_{n/2}$.

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