$(K_4 - e)$-supermagic labeling of triangular ladders

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ABSTRACT

A graph $G = (V, E)$ is H-supermagic if there exists a bijection f from the set $V \cup E$ to the set of integers $\{1, 2, 3, \ldots, |V|+|E|\}$, called H-supermagic labeling such that the sum of labels of all elements of every induced subgraph of G isomorphic to H is equal to the same integer and all vertex labels are in $\{1, 2, 3, \ldots, |V|\}$. We present a $(K_4 - e)$-supermagic labeling of the triangular ladder TL_{2n} for any $n \geq 2$.

Keywords: graph labeling, supermagic labeling, triangular ladder

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1. Introduction

Let G and H be graphs. Then G is said to have an H-*covering* if there exists a set $\mathcal{H} = \{H_1, H_2, \ldots, H_s\}$ of subgraphs of G, all isomorphic to H, such that every edge of G is contained in at least one graph in $\mathcal{H}$. Also G is said to have an H-*decomposition* if every edge of G is contained in exactly one graph in $\mathcal{H}$. Therefore, every H-decomposition is also an H covering, but the converse is not always true. The set $\mathcal{H}$ is called a *maximal* H-*covering* if it contains all subgraphs of G isomorphic to H.

Suppose $G(V, E)$ is a graph with an H-covering. Then a function f from $V \cup E$ to the set of integers $\{1, 2, \ldots, |V|+|E|\}$ is called an H-*magic labeling of G* if there exists a positive integer μ such that for every subgraph H_i of G with the vertex set V_i and edge set E_i the sum $w(H_i)$ (called the *weight* of H_i) satisfies

$$w(H_i) = \sum_{x \in V_i} f(x) + \sum_{e \in E_i} f(e) = \mu.$$

The graph G is then called an H-*magic graph*. If $f(V)$, the set of labels of all vertices

of V, is moreover equal to $\{1, 2, \ldots, |V|\}$, then f is called an H-*supermagic labeling of* G (or H-SML for short) and G is an H-*supermagic graph*.

Similarly, when $\mathcal{H}$ is an H-decomposition of G and the function f described above exists (that is, f is and H-SML), then $\mathcal{H}$ is called an H-*magic* or H-*supermagic decomposition* of G.

The notions of H-magic and H-supermagic labelings were introduced by Gutiérrez and Lladó [3] in 2005. While some H-magic decompositions appeared in various papers when the unique H-cover of some graph G was in fact a decomposition, the notion itself was first time formalized by Inayah, Lladó, and Moragas [4] in 2012.

Probably the most popular graph H for which the H-magic and H-supermagic property has been studied is C_m, and among those C_4 is the leader, closely followed by C_3. There are dozens of papers dealing with C_m-magic and supermagic labelings. We are not going to list them here, instead we refer the reader to section 5.4 in an excellent survey by Gallian [2].

We present an H-supermagic labeling of the triangular ladder TL_{2n} for $H = K_4 - e$. Triangular ladders were studies with respect to a C_3-supermagic labeling by Khalid, Rizvi, and Ali [5] and by Rizvi, Ali, and Hussain [6].

Disclaimer. The text of the Introduction is taken almost verbatim from author's paper [1].

2. Construction

A *triangular ladder* TL_{2n} of order $2n$ is a graph consisting of two paths P_n (called *rails*) on vertices $x_1, x_2, \ldots, x_n$ (*upper rail*) and $y_1, y_2, \ldots, y_n$ (*lower rail*) with edges $a_i = x_i x_{i+1}$ and $b_i = y_i y_{i+1}$, $i \in [1, n-1]$, respectively, and a set of edges (called *rungs*) forming two perfect matchings with edges $d_i = x_i y_i$ for $i = 1, 2, \ldots, n$ (called *vertical rungs*) and $t_i = y_i x_{i+1}$ for $i = 1, 2, \ldots, n-1$ (called *tilt rungs*).

To illustrate the labeling for readers convenience in our proofs, we place the labels in an array as follows. One can observe that the labels correspond to the placement of the labeled elements in Figure 1.
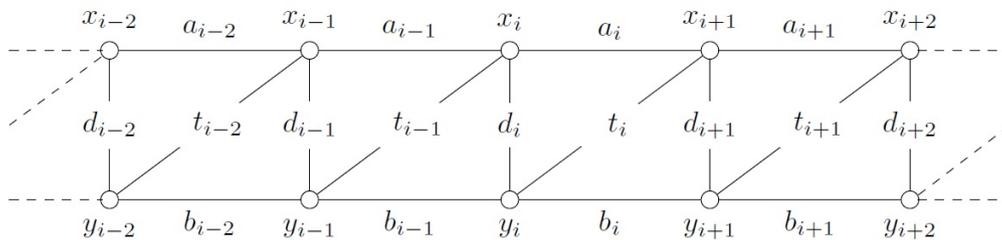

Fig. 1. Triangular ladder

$$\begin{array}{ccccccc} \ldots & f(x_{i-1}) & f(a_{i-1}) & f(x_i) & f(a_i) & f(x_{i+1}) & \ldots \\ \ldots & f(d_{i-1}) & f(t_{i-1}) & f(d_i) & f(t_i) & f(d_{i+1}) & \ldots \\ \ldots & f(y_{i-1}) & f(b_{i-1}) & f(y_i) & f(b_i) & f(y_{i+1}) & \ldots \end{array}$$

We denote by H_i^d the graph induced on vertices $x_i, x_{i+1}, y_i, y_{i+1}$ and H_i^t the graph induced on vertices $x_i, x_{i+1}, y_{i-1}, y_i$. They are shown in Figures 2 and 3, respectively.

Construction 2.1. We label the vertices as

$$f(x_i) = i, \ f(y_i) = i + n,$$

and the rails as

$$f(a_i) = 3n + i - 1, \quad \text{and} \quad f(b_i) = 2n + i.$$

Then the spokes are labeled as

$$f(d_i) = 6n - 2i - 1, \quad \text{and} \quad f(t_i) = 6n - 2i - 2.$$

Lemma 2.2. *For every $i \in [1, n]$, the weight of H_i^d is $w(H_i^d) = 25n - 5$.*

Proof. We denote by $V(H_i^d)$ and $E(H_i^d)$ the vertex and edge set of the graph H_i^d, respectively. Then

$$w(H_i^d) = w(V(H_i^d)) + w(E(H_i^d)),$$

where

$$
\begin{aligned}
w(V(H_i^d)) &= f(x_i) + f(x_{i+1}) + f(y_i) + f(y_{i+1}) \\
&= i + (i + 1) + (n + i) + (n + i + 1) \\
&= 2n + 4i + 2,
\end{aligned}
$$

and

$$
\begin{aligned}
w(H_i^d) &= d_i + d_{i+1} + a_i + t_i + b_i \\
&= (6n - 2i - 1) + (6n - 2i - 3) + (3n + i - 1) + (6n - 2i - 2) + (2n + i) \\
&= 23n - 4i - 7.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
w(H_i^d) &= w(V(H_i^d)) + w(E(H_i^d)) \\
&= (2n + 4i + 2) + (23n - 4i - 7) \\
&= 25n - 5.
\end{aligned}
$$

To visualize our calculations, in Figure 2, we highlight the induced graph H_i^d in blue. The corresponding labels are also in blue, with the sum of labels in the respective columns placed below.

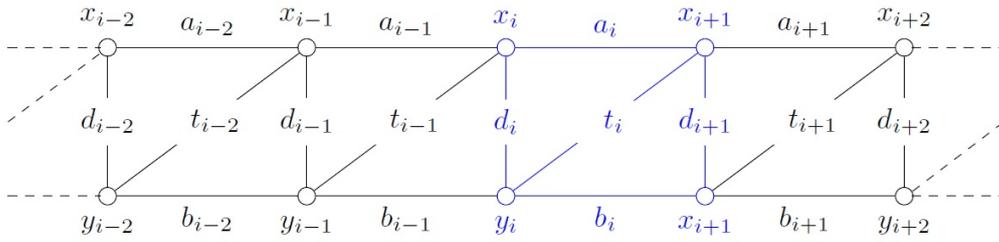| ...	$i-1$	$3n+i-2$	i	$3n+i-1$	$i+1$	...
...	$6n-2i+1$	$6n-2i$	$6n-2i-1$	$6n-2i-2$	$6n-2i-3$	...
...	$n+i-1$	$2n+i-1$	$n+i$	$2n+i$	$n+i+1$	...
...			$7n-1$	$11n-3$	$7n-1$	...

Fig. 2. Triangular ladder with induced subgraph H_i^d (in blue)

Then for $i \in [1, n-1]$ it is easy to observe that

$$w(H_i^d) = (11n - 2) + 2(7n - 1) = 25n - 5,$$

as desired.   □

Lemma 2.3. *For every $i \in [2, n-1]$, the weight of H_i^t is $w(H_i^t) = 25n + 4$.*

Proof. We have again

$$w(H_i^d) = w(V(H_i^d)) + w(E(H_i^d)),$$

where

$$\begin{aligned}
w(V(H_i^d)) &= f(x_i) + f(x_{i+1}) + f(y_{i-1}) + f(y_i) \\
&= i + (i+1) + (n+i-1) + (n+i) \\
&= 2n + 4i,
\end{aligned}$$

and

$$\begin{aligned}
9w(H_i^d) &= t_{i-1} + t_i + a_i + d_i + b_{i-1} \\
&= (6n - 2i) + (6n - 2i - 2) + (3n + i - 1) + (6n - 2i - 1) + (2n + i - 1) \\
&= 23n - 4i - 5.
\end{aligned}$$

Therefore,

$$\begin{aligned}
9w(H_i^d) &= w(V(H_i^d)) + w(E(H_i^d)) \\
&= (2n + 4i) + (23n - 4i - 5) \\
&= 25n - 5.
\end{aligned}$$

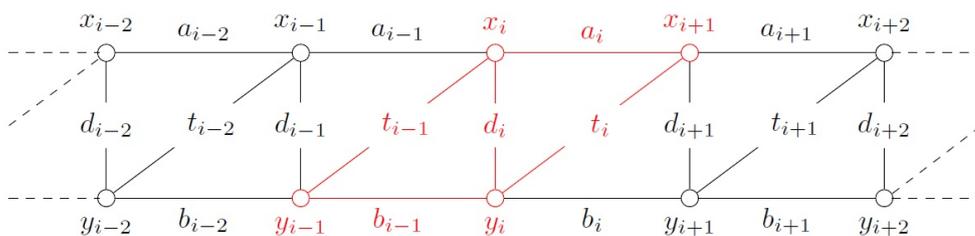In Figure 3, we highlight the induced graph H_i^t in red.

Fig. 3. Triangular ladder with induced subgraph H_i^t (in red)

We again present the array for $i \in [1, n-1]$ with the corresponding labels in red

...	$i-1$	$3n+i-2$	i	$3n+i-1$	$i+1$	...
...	$6n-2i+1$	$6n-2i$	$6n-2i-1$	$6n-2i-2$	$6n-2i-3$	...
...	$n+i-1$	$2n+i-1$	$n+i$	$2n+i$	$n+i+1$	...
...	$n+i-1$	$8n-i-1$	$7n-1$	$9n-i-3$	$i+1$	...

and observe that

$$w(H_i^t) = (n+i-1) + (8n-i-1) + (7n-1) + (9n-i-3) + (i+1) = 25n - 5,$$

which completes the proof. □

Our main result now follows immediately.

Theorem 2.4. *Let* $H = K_4 - e$. *Then the triangular ladder* TL_{2n} *is* H-*supermagic for every* $n \geq 2$ *with the magic constant* $25n + 4$.

Proof. The proof follows directly from Lemmas 2.2 and 2.3. □

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