

On k –division Fibonacci-Pell and k –division Gaussian Fibonacci-Pell polynomials and their applications

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ABSTRACT

This paper introduces two novel sequences: the k -division Fibonacci–Pell polynomials and the k -division Gaussian Fibonacci–Pell polynomials. Building on the well-known Fibonacci and Pell sequences, these new sequences are defined using a division-based approach, enhancing their combinatorial and algebraic properties. We present explicit recurrence relations, generating functions, combinatorial identities, and Binet-type formulas for these sequences. A significant contribution of the study is the factorization of the Pascal matrix via the Riordan group method using the proposed polynomials. Two distinct factorizations are derived, highlighting the algebraic structure and combinatorial interpretations of the k -division polynomials. The work not only generalizes known polynomial sequences but also provides new insights into their matrix representations and applications.

Keywords: Fibonacci sequence, Pell sequence, Pascal matrix, Riordan group

2020 Mathematics Subject Classification: 11B39, 05A15, 05A10, 11C08, 15A23, 15B36.

1. Introduction

The Fibonacci and Pell sequences can be defined as follows.

A Fibonacci sequence, denoted by $\{F(n)\}_{n=0}^{\infty}$, is defined by

$$F(n) = F(n-1) + F(n-2), \quad n \geq 0,$$

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Received 07 Mar 2025; Revised 10 Oct 2025; Accepted 23 Oct 2025; Published 04 Dec 2025.

DOI: [10.61091/jcmcc128-20](https://doi.org/10.61091/jcmcc128-20)

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with $F(0) = 0$, $F(1) = 1$.

The definition of the Pell sequence $\{P(n)\}_{n=0}^{\infty}$ is given by

$$P(n) = 2P(n-1) + P(n-2), \quad n \geq 0,$$

where $P(0) = 0$, $P(1) = 1$.

The Fibonacci and Pell sequences have characteristic polynomials $x^2 - x - 1$ and $x^2 - 2x - 1$, respectively.

There have been many studies of the Fibonacci and Pell sequences (see [11, 17, 20]), and many generalizations have been obtained. In [16], generalized Fibonacci and Lucas polynomials were introduced. In 2015, the generalized Pell sequence was introduced and its modulo m behavior was studied in [2]. In [3], the quaternion-Pell sequence was introduced and some of its properties were studied. In 2021, the generalized order k -Pell sequences were obtained in some finite groups [6]. In 2023, the k -Pell sequences of the 2-generator p -groups of nilpotency class were studied [15]. Here, using Fibonacci and Pell sequences, two new sequences are introduced.

The lower triangular Pascal matrix, $n \times n$, denoted by $P_n = [p_{ij}]$, is defined as [1]:

$$p_{ij} = \begin{cases} \binom{i-1}{j-1}, & \text{if } i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

The Riordan group was introduced in [18] as follows.

Let $R = [r_{ij}]_{i,j \geq 0}$ be an infinite matrix with complex entries and let $C_i(t) = \sum_{n \geq 0} r_{n,i} t^n$ be the generating function of the i th column of R . The matrix R is a Riordan matrix if $C_i(t) = h(t)[u(t)]^i$, where

$$h(t) = 1 + h_1 t + h_2 t^2 + h_3 t^3 + \cdots, \quad u(t) = t + u_2 t^2 + u_3 t^3 + \cdots.$$

We can express R as $R = (h(t), u(t))$ and define it as a Riordan matrix. Under matrix multiplication $*$, the set of Riordan matrices forms a group. In [10], some properties of this group were obtained, including factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. In 2011, new Pascal matrix factorizations using Fibonomial coefficients were introduced in [19]. In [4], the k -Fibonacci matrix and the Pascal matrix were studied. In 2022, a Pascal matrix factorization involving the t -extension of the p -Fibonacci matrix was given in [5]. In 2024, Kuloğlu et al. introduced the (d, k) -Fibonacci polynomials and gave a Pascal matrix factorization involving these sequences [9]. They also defined d -Tribonacci polynomials and studied some combinatorial properties in [8], and carried out similar work with Gaussian numbers in [7].

Motivated by the above results, we define new sequences. Here, we use the Fibonacci and Pell sequences and obtain new sequences, and then we study some of their properties. We also give two Pascal matrix factorizations involving the k -division and k -division Gaussian Fibonacci–Pell polynomials.

The remainder of this paper is organized as follows. In Section 2, we introduce the k -division and k -division Gaussian Fibonacci–Pell polynomials. Factorizations of the Pascal matrix involving the k -division and k -division Gaussian Fibonacci–Pell polynomials are presented in Section 3.

2. k -division Fibonacci–Pell and k -division Gaussian Fibonacci–Pell polynomials

In this section, we introduce the k -division Fibonacci–Pell and k -division Gaussian Fibonacci–Pell polynomials and obtain some results. In [12], we presented a new method for constructing sequences and named it the k -division sequence. Then, we defined the k -division Fibonacci–Pell sequences.

Definition 2.1. Let $f_1(x) = x^u + a_1x^{u-1} + \cdots + a_u$ and $g_1(x) = x^m + b_1x^{m-1} + \cdots + b_m$, where $a_i \in \mathbb{Z}$ for $1 \leq i \leq u$ and $b_j \in \mathbb{Z}$ for $1 \leq j \leq m$, be the characteristic polynomials of two arbitrary sequences of degrees u and m , respectively, with $m \geq u$. For $k \in \mathbb{N}$, the k -division sequence $\{h_n(k)\}_{n=0}^\infty$ is defined by

$$h(n, k) = x^k g_1(x) + t(x), \quad n \geq k + m, \quad (1)$$

where $t(x)$ is the remainder of the division $\frac{x^k g_1(x)}{f_1(x)}$ and

$$h(0, k) = h(1, k) = \cdots = h(m + k - 2, k) = 0, \quad h(m + k - 1, k) = 1.$$

The characteristic polynomials of the Fibonacci and Pell sequences are $x^2 - x - 1$ and $x^2 - 2x - 1$, respectively. By Definition 2.1, we define the 1-division Fibonacci–Pell sequence as follows.

Definition 2.2. If $k = 1$, the 1-division Fibonacci–Pell sequence is defined as

$$h(n, 1) = 2h(n - 1, 1) + 2h(n - 2, 1) + h(n - 3, 1), \quad n \geq 3,$$

with $h(0, 1) = h(1, 1) = 0$ and $h(2, 1) = 1$.

For $k \geq 2$, by using Definition 2.1, we have

$$\begin{aligned} k = 2, \quad & h(n, 2) = 2h(n - 1, 2) + 2h(n - 2, 2) + h(n - 3, 2), \quad n \geq 4, \\ k = 3, \quad & h(n, 3) = 2h(n - 1, 3) + h(n - 2, 3) + 3h(n - 4, 3) + 2h(n - 5, 3), \quad n \geq 5, \\ k = 4, \quad & h(n, 4) = 2h(n - 1, 4) + h(n - 2, 4) + 5h(n - 5, 4) + 3h(n - 6, 4), \quad n \geq 6, \\ & \vdots \end{aligned}$$

In general,

$$\begin{aligned} k = m, \quad & h(n, m) = 2h(n - 1, m) + h(n - 2, m) + F(m + 1)h(n - (m + 1), m) \\ & + F(m)h(n - (m + 2), m), \quad n \geq m + 2, \end{aligned}$$

with initial conditions

$$h(0, m) = h(1, m) = \cdots = h(m, m) = 0, \quad h(m + 1, m) = 1.$$

We need to show that

$$h_m(X) = X^m(X^2 - 2X - 1) + F(m+1)X + F(m), \quad (2)$$

is divisible by $f(X) = X^2 - X - 1$. Let $\gamma = \frac{1 + \sqrt{5}}{2}$ be the golden section. Then $\gamma^2 = \gamma + 1$. Also,

$$\gamma^{m+1} = F(m+1)\gamma + F(m) \quad \text{for all } m \geq 1, \quad (3)$$

a formula which can be proved by induction. Then

$$\begin{aligned} h_m(\gamma) &= \gamma^m(\gamma^2 - 2\gamma - 1) + F(m+1)\gamma + F(m) \\ &= \gamma^m((\gamma^2 - 2\gamma - 1) - \gamma) + F(m+1)\gamma + F(m) \\ &= -\gamma^{m+1} + F(m+1)\gamma + F(m) = 0, \end{aligned}$$

which shows that the polynomial in (2) has γ as a root. Since it has integer coefficients, it is divisible by the minimal polynomial of γ , which is $X^2 - X - 1$, so indeed $F(m+1)\gamma + F(m)$ is the required $t(x)$.

Here, for $k \geq 2$, we define the k -division Fibonacci–Pell polynomials as follows:

Definition 2.3. For $k \geq 2$, the k -division Fibonacci–Pell polynomials $\{h_n^k(x)\}_{n=-\infty}^{\infty}$ are defined by

$$\begin{aligned} h_{n+1}^k(x) &= 2p_1(x)h_n^k(x) + p_2(x)h_{n-1}^k(x) + F(k+1)p_3(x)h_{n-k}^k(x) \\ &\quad + F(k)p_4(x)h_{n-(k+1)}^k(x), \quad n \geq 1, \end{aligned} \quad (4)$$

where $p_i(x)$ are real-coefficient functions (or polynomials) for $1 \leq i \leq 4$, $h_n^k(x) = 0$ for $n \leq 0$, and $h_1^k(x) = 1$.

For example:

(i) If $k = 2$, we have

$$h_{n+1}^2(x) = 2p_1(x)h_n^2(x) + p_2(x)h_{n-1}^2(x) + 2p_3(x)h_{n-2}^2(x) + p_4(x)h_{n-3}^2(x), \quad n \geq 1,$$

so

$$\begin{aligned} h_0^2(x) &= 0, \\ h_1^2(x) &= 1, \\ h_2^2(x) &= 2p_1(x), \\ h_3^2(x) &= 4p_1^2(x) + p_2(x), \\ h_4^2(x) &= 8p_1^3(x) + 4p_1(x)p_2(x) + 2p_3(x), \dots \end{aligned}$$

(ii) If $k = 3$, we have

$$h_{n+1}^3(x) = 2p_1(x)h_n^3(x) + p_2(x)h_{n-1}^3(x) + 3p_3(x)h_{n-3}^3(x) + 2p_4(x)h_{n-4}^3(x), \quad n \geq 1,$$

so

$$h_0^3(x) = 0,$$

$$\begin{aligned} h_1^3(x) &= 1, \\ h_2^3(x) &= 2p_1(x), \\ h_3^3(x) &= 4p_1^2(x) + p_2(x), \\ h_4^3(x) &= 8p_1^3(x) + 4p_1(x)p_2(x), \dots \end{aligned}$$

Lemma 2.4. For $k \geq 2$, let $g_{h_n^k(x)}$ be the generating function of the k -division Fibonacci–Pell polynomials. Then

$$g_{h_n^k(x)} = \frac{r}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}}. \quad (5)$$

Proof. Let $g_{h_n^k(x)}$ be the generating function of the k -division Fibonacci–Pell polynomials. Then

$$g_{h_n^k(x)} = \sum_{n=0}^{\infty} h_n^k(x)r^n = h_0^k(x) + h_1^k(x)r + \dots + h_n^k(x)r^n + \dots \quad (6)$$

Multiply $g_{h_n^k(x)}$ successively by $2p_1(x)r$, $p_2(x)r^2$, $F(k+1)p_3(x)r^k$ and $F(k)p_4(x)r^{k+1}$, respectively. We obtain

$$\begin{aligned} g_{h_n^k(x)} &= h_0^k(x) + h_1^k(x)r + \dots + h_n^k(x)r^n + \dots, \\ 2p_1(x)r g_{h_n^k(x)} &= 2p_1(x)r h_0^k(x) + 2p_1(x)h_1^k(x)r^2 + \dots + 2p_1(x)h_n^k(x)r^{n+1} + \dots, \\ p_2(x)r^2 g_{h_n^k(x)} &= p_2(x)r^2 h_0^k(x) + p_2(x)h_1^k(x)r^3 + \dots + p_2(x)h_n^k(x)r^{n+2} + \dots, \\ F(k+1)p_3(x)r^k g_{h_n^k(x)} &= F(k+1)p_3(x)r^k h_0^k(x) + F(k+1)p_3(x)h_1^k(x)r^{k+1} + \dots \\ &\quad + F(k+1)p_3(x)h_n^k(x)r^{n+k} + \dots, \\ F(k)p_4(x)r^{k+1} g_{h_n^k(x)} &= F(k)p_4(x)r^{k+1} h_0^k(x) + F(k)p_4(x)h_1^k(x)r^{k+2} + \dots \\ &\quad + F(k)p_4(x)h_n^k(x)r^{n+k+1} + \dots. \end{aligned}$$

Using the recurrence relation for $h_n^k(x)$ and aligning coefficients of like powers of r , we obtain

$$\begin{aligned} g_{h_n^k(x)}(1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}) \\ = h_0^k(x) + h_1^k(x)r + 2p_1(x)h_0^k(x). \end{aligned}$$

Since $h_0^k(x) = 0$ and $h_1^k(x) = 1$, the right-hand side reduces to r . Therefore

$$g_{h_n^k(x)} = \frac{r}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}},$$

which proves the claim. $\square$

Lemma 2.5. For $n \geq 0$ and $k \geq 2$, we have

$$\begin{aligned} h_n^k(x) &= \sum_{\substack{n_1, n_2, n_k, n_{k+1} \geq 0 \\ n_1 + 2n_2 + kn_k + (k+1)n_{k+1} = n}} \binom{n_1 + n_2 + n_k + n_{k+1}}{n_1, n_2, n_k, n_{k+1}} \\ &\quad \times 2^{n_1} p_1^{n_1}(x) p_2^{n_2}(x) (F(k+1)p_3(x))^{n_k} (F(k)p_4(x))^{n_{k+1}}. \end{aligned}$$

Proof. Using the generating function in Lemma 2.4, we have

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n+1}^k(x) r^n &= \frac{1}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}} \\ &= \sum_{n=0}^{\infty} (2p_1(x)r + p_2(x)r^2 + F(k+1)p_3(x)r^k + F(k)p_4(x)r^{k+1})^n.\end{aligned}$$

Expanding the n th power by the multinomial theorem, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} h_{n+1}^k(x) r^n &= \sum_{n=0}^{\infty} \sum_{n_1+n_2+n_k+n_{k+1}=n} \binom{n}{n_1, n_2, n_k, n_{k+1}} (2p_1(x)r)^{n_1} (p_2(x)r^2)^{n_2} \\ &\quad \times (F(k+1)p_3(x)r^k)^{n_k} (F(k)p_4(x)r^{k+1})^{n_{k+1}} \\ &= \sum_{n=0}^{\infty} \sum_{n_1+n_2+n_k+n_{k+1}=n} \binom{n}{n_1, n_2, n_k, n_{k+1}} 2^{n_1} p_1^{n_1}(x) p_2^{n_2}(x) \\ &\quad \times (F(k+1)p_3(x))^{n_k} (F(k)p_4(x))^{n_{k+1}} r^{n_1+2n_2+kn_k+(k+1)n_{k+1}}.\end{aligned}$$

Now compare the coefficient of r^n on both sides. The exponent of r on the right-hand side is

$$n_1 + 2n_2 + kn_k + (k+1)n_{k+1} = n,$$

and for such quadruples (n_1, n_2, n_k, n_{k+1}) the corresponding coefficient is

$$\binom{n}{n_1, n_2, n_k, n_{k+1}} 2^{n_1} p_1^{n_1}(x) p_2^{n_2}(x) (F(k+1)p_3(x))^{n_k} (F(k)p_4(x))^{n_{k+1}}.$$

Renaming the upper index n of the multinomial coefficient as $n_1 + n_2 + n_k + n_{k+1}$ yields the stated formula, and the lemma follows. $\square$

Theorem 2.6. For Binet formula for k -division Fibonacci Pell polynomials $h_n^k(x)$, we have

$$h_n^k(x) = \sum_{i=1}^{k+1} \frac{B_i(x)}{1 - \delta_i(x)r} = \frac{r}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}}.$$

Proof. The following equation can be written for every value of n ,

$$\begin{aligned}h_0^k(x) &= \sum_{i=1}^{k+1} B_i(x), \\ &\vdots \\ h_n^k(x) &= \sum_{i=1}^{k+1} B_i(x) [\delta_i(x)]^n.\end{aligned}$$

Multiplying these equations by $r, r^2, r^3, \dots, r^n$ respectively, except the first equation, we obtain:

$$\begin{aligned} r^0 h_0^k(x) &= \sum_{i=1}^{k+1} B_i(x), \\ &\vdots \\ r^n h_n^k(x) &= \sum_{i=1}^{k+1} B_i(x) [\delta_i(x) r]^n. \end{aligned}$$

Adding

$$\begin{aligned} h_0^k(x) + r h_1^k(x) + r^2 h_2^k(x) + \dots + r^n h_n^k(x) \\ = \sum_{i=1}^{k+1} B_i(x) (1 + \delta_i(x) r + [\delta_i(x) r]^2 + \dots + [\delta_i(x) r]^n). \end{aligned}$$

From the left side, the generating function of the k -division Fibonacci Pell polynomials provides the equation. Thus, Here is the equation we obtain:

$$\sum_{i=1}^{k+1} \frac{B_i(x)}{1 - \delta_i(x) r} = \frac{r}{1 - 2p_1(x) r - p_2(x) r^2 - F(k+1)p_3(x) r^k - F(k)p_4(x) r^{k+1}}.$$

□

Theorem 2.7. For sum of the k -division Fibonacci Pell polynomials, $Sh_n^k(x)$, we have

$$Sh_n^k(x) = \sum_{n=0}^{\infty} h_n^k(x) = \frac{1}{1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)}.$$

Proof. This is the equation we give

$$Sh_n^k(x) = \sum_{n=0}^{\infty} h_n^k(x) = h_0^k(x) + h_1^k(x) + h_2^k(x) + \dots + h_n^k(x) + \dots.$$

By multiplying the last equation, $2p_1(x), p_2(x), \dots$, respectively then we get

$$\begin{aligned} 2p_1(x) Sh_n^k(x) &= 2p_1(x) h_0^k(x) + 2p_1(x) h_1^k(x) + 2p_1(x) h_2^k(x) + \dots \\ &\quad + 2p_1(x) h_n^k(x) + \dots, \\ p_2(x) Sh_n^k(x) &= p_2(x) h_0^k(x) + p_2(x) h_1^k(x) + p_2(x) h_2^k(x) + \dots + p_2(x) h_n^k(x) + \dots, \\ F(k+1)p_3(x) Sh_n^k(x) &= F(k+1)p_3(x) h_0^k(x) + F(k+1)p_3(x) h_1^k(x) \\ &\quad + F(k+1)p_3(x) h_2^k(x) + \dots, \\ F(k)p_4(x) Sh_n^k(x) &= F(k)p_4(x) h_0^k(x) + F(k)p_4(x) h_1^k(x) + F(k)p_4(x) h_2^k(x) + \dots \\ &\quad + F(k)p_4(x) h_n^k(x) + \dots \end{aligned}$$

If we do necessary calculation, we obtain

$$\begin{aligned} & Sh_n^k(x)[1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] \\ & = h_0^k(x)[1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] \\ & \quad + h_1^k(x)[1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] + \dots \\ & \quad + h_n^k(x)[1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] + \dots \\ & = 1. \end{aligned}$$

Therefore, we get

$$Sh_n^k(x)[1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] = 1.$$

From here, the existence of the equality in the theorem can be easily seen. $\square$

Definition 2.8. The k -division Fibonacci Pell polynomials matrix H^k for $h_n^k(x)$ is defined

$$H^k = \begin{bmatrix} 2p_1(x) & p_2(x) & F(k+1)p_3(x) & F(k)p_4(x) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Theorem 2.9. This is the representation of $h_n^k(x)$ with the condition that $n > k$.

$$(H^k)^n = \begin{bmatrix} h_{n+1}^k(x) & p_2(x)h_n^k(x) + F(k+1)p_3(x)h_{n-k+1}^k(x) + F(k)p_4(x)h_{n-k}^k(x) \\ h_n^k(x) & p_2(x)h_{n-1}^k(x) + F(k+1)p_3(x)h_{n-k}^k(x) + F(k)p_4(x)h_{n-k-1}^k(x) \\ h_{n-k}^k(x) & p_2(x)h_{n-k}^k(x) + F(k+1)p_3(x)h_{n-2k+1}^k(x) + F(k)p_4(x)h_{n-2k}^k(x) \\ h_{n-k-1}^k(x) & p_2(x)h_{n-k-1}^k(x) + F(k+1)p_3(x)h_{n-2k}^k(x) + F(k)p_4(x)h_{n-2k-1}^k(x) \\ F(k+1)p_3(x)h_n^k(x) + F(k)p_4(x)h_{n-k+1}^k(x) & F(k)p_4(x)h_n^k(x) \\ F(k+1)p_3(x)h_{n-1}^k(x) + F(k)p_4(x)h_{n-k}^k(x) & F(k)p_4(x)h_{n-1}^k(x) \\ F(k+1)p_3(x)h_{n-k}^k(x) + F(k)p_4(x)h_{n-2k+1}^k(x) & F(k)p_4(x)h_{n-k}^k(x) \\ F(k+1)p_3(x)h_{n-k-1}^k(x) + F(k)p_4(x)h_{n-2k}^k(x) & F(k)p_4(x)h_{n-k-1}^k(x) \end{bmatrix}.$$

Proof. By induction on n , it can be proved easily. $\square$

Corollary 2.10. For $n, m \geq 0$, we get

$$\begin{aligned} h_{n+m+1}^k(x) &= h_{n+1}^k(x)h_{m+1}^k(x) + p_2(x)h_m^k(x)h_n^k(x) \\ & \quad + F(k+1)p_3(x)[h_{n-k+1}^k(x)h_m^k(x) + h_n^k(x)h_{m-k}^k(x)] \\ & \quad + F(k)p_4(x)[h_{n-k}^k(x)h_m^k(x) + h_{n-k+1}^k(x)h_{m-k}^k(x) + h_n^k(x)h_{m-k-1}^k(x)]. \end{aligned}$$

Proof. We have $(H^k)^{m+n} = (H^k)^n(H^k)^m$.

By multiplying and adding the first row in matrix $(H^k)^n$ and the first column in matrix $(H^k)^m$, we obtain the 1st row and 1st column element of the matrix given above. $\square$

Here, for $k \geq 2$, we define the k -division Gaussian Fibonacci-Pell polynomials as follows:

Definition 2.11. for $k \geq 2$, the defined the k -division Gaussian Fibonacci-Pell polynomials $\{Gh_n^k(x)\}_{n=0}^\infty$ is defined as for $n \geq 1$

$$Gh_{n+1}^k(x) = 2p_1(x)Gh_n^k(x) + p_2(x)Gh_{n-1}^k(x) + F(k+1)p_3(x)Gh_{n-k}^k(x) + F(k)p_4(x)Gh_{n-(k+1)}^k(x), \quad (7)$$

with $p_i(x)$ be a real coefficient for $1 \leq i \leq 4$, $Gh_0^k(x) = 0$ for $n \leq 0$ and $Gh_1^k(x) = 2p_1(x) + i$.

As an example, if $k = 2$, then

$$Gh_{n+1}^2(x) = 2p_1(x)Gh_n^2(x) + p_2(x)Gh_{n-1}^2(x) + 2p_3(x)Gh_{n-2}^2(x) + p_4(x)Gh_{n-3}^2(x), \quad n \geq 1, \quad (8)$$

so,

$$\begin{aligned} Gh_0^2(x) &= 0, \quad Gh_1^2(x) = 2p_1(x) + i, \\ Gh_2^2(x) &= 4p_1^2(x) + 2p_2(x)i, \quad Gh_3^2(x) = 8p_1^3(x) + 4p_1^2(x)i + 2p_1(x)p_2(x) + p_2(x)i, \\ &\vdots \end{aligned}$$

Similar to Lemma 2.4, the following Lemma is proved.

Lemma 2.12. For $k \geq 2$, let $g_{h_n^k}(x)$ be the generating function of the k -division Gaussian Fibonacci-Pell polynomials. Then,

$$g_{h_n^k}(x) = \frac{r(2p_1(x) + i)}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}}. \quad (9)$$

Lemma 2.13. For $n \geq 0$ and $k \geq 2$, we get

$$\begin{aligned} Gh_n^k(x) &= (2p_1(x) + i) \sum_{\substack{n_1, n_2, n_k, n_{k+1} \\ 1+n_1+2n_2+kn_k+(k+1)n_{k+1}=n}} \binom{n_1 + n_2 + n_k + n_{k+1}}{n_1, n_2, n_k, n_{k+1}} \\ &\quad \times 2p_1^{n_1}(x)p_2^{n_2}(x)F(k+1)p_3^{n_k}(x)F(k)p_4^{n_{k+1}}(x)r^n. \end{aligned}$$

Proof. Similar to Lemma 2.5, it can proved. □

Theorem 2.14. The Binet formula for k -division Gaussian Fibonacci Pell polynomials $Gh_n^k(x)$ is:

$$Gh_n^k(x) = \sum_{i=1}^{k+1} \frac{B_i(x)}{1 - \lambda_i(x)r} = \frac{r(2p_1(x) + i)}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}}.$$

Proof. For each value of n , we can write

$$Gh_0^k(x) = \sum_{i=1}^{k+1} d_i(x),$$

$$\vdots$$

$$Gh_n^k(x) = \sum_{i=1}^{k+1} d_i(x) [\lambda_i(x)]^n.$$

Multiplying the equations, except the first equation, by $r, r^2, r^3, \dots, r^n$ respectively, we get the following equations :

$$r^0 Gh_0^k(x) = \sum_{i=1}^{k+1} d_i(x),$$

$$\vdots$$

$$r^n Gh_n^k(x) = \sum_{i=1}^{k+1} d_i(x) [\lambda_i(x) r]^n.$$

The above equation can be added side by side to give us

$$Gh_0^k(x) + r Gh_1^k(x) + r^2 Gh_2^k(x) + \dots + r^n Gh_n^k(x)$$

$$= \sum_{i=1}^{k+1} d_i(x) (1 + \lambda_i(x) r + [\lambda_i(x) r]^2 + \dots + [\lambda_i(x) r]^n).$$

The equation obtained on the left side gives the generating function of the k -division Gaussian Fibonacci Pell polynomials. Thus, we get following equation:

$$\sum_{i=1}^{k+1} \frac{d_i(x)}{1 - \lambda_i(x) r} = \frac{r(2p_1(x) + i)}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}}.$$

□

Theorem 2.15. *Let $SGh_n^k(x)$ represent the sum of the k -division Gaussian Fibonacci Pell polynomials. Then, we obtain the following result:*

$$SGh_n^k(x) = \sum_{n=0}^{\infty} Gh_n^k(x) = \frac{2p_1(x) + i}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)r^{k+1}}.$$

Proof. We get

$$SGh_n^k(x) = \sum_{n=0}^{\infty} Gh_n^k(x) = Gh_0^k(x) + Gh_1^k(x) + Gh_2^k(x) + \dots + Gh_n^k(x) + \dots$$

Multiplying the last equation by $2p_1(x), p_2(x), F(k+1)p_3(x), F(k)p_4(x)$, respectively then we obtain

$$2p_1(x) SGh_n^k(x) = 2p_1(x) Gh_0^k(x) + 2p_1(x) Gh_1^k(x) + 2p_1(x) Gh_2^k(x) + \dots$$

$$+ 2p_1(x) Gh_n^k(x) + \dots,$$

$$p_2(x) SGh_n^k(x) = p_2(x) Gh_0^k(x) + p_2(x) Gh_1^k(x) + p_2(x) Gh_2^k(x) + \dots$$

$$\begin{aligned}
 &+ p_2(x) Gh_n^k(x) + \dots, \\
 F(k+1)p_3(x) SGh_n^k(x) &= F(k+1)p_3(x) Gh_0^k(x) + F(k+1)p_3(x) Gh_1^k(x) \\
 &+ F(k+1)p_3(x) Gh_2^k(x) + \dots, \\
 F(k)p_4(x) SGh_n^k(x) &= F(k)p_4(x) Gh_0^k(x) + F(k)p_4(x) Gh_1^k(x) \\
 &+ F(k)p_4(x) Gh_2^k(x) + \dots
 \end{aligned}$$

If we do necessary calculation, we get

$$\begin{aligned}
 &SGh_n^k(x) [1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] \\
 &= Gh_0^k(x) [1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] \\
 &\quad + Gh_1^k(x) [1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] + \dots \\
 &\quad + Gh_n^k(x) [1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] + \dots \\
 &= 2p_1(x) + i.
 \end{aligned}$$

Thus, we have

$$SGh_n^k(x) [1 - 2p_1(x) - p_2(x) - F(k+1)p_3(x) - F(k)p_4(x)] = 2p_1(x) + i.$$

From here, the existence of the equality in the theorem can be easily seen. $\square$

3. Factorization of k -division Fibonacci-Pell and k -division Gaussian Fibonacci-Pell polynomials

In section, we introduce the infinite k -division Fibonacci-Pell and k -division Gaussian Fibonacci-Pell polynomials. Also, Using these sequences, we compute two Factorization of the Pascal matrix.

First, we define infinite k -division Fibonacci-Pell polynomials matrix as follows:

Definition 3.1. The infinite k -division Fibonacci-Pell polynomials matrix is denoted by $H^k(x) = [H_{p_1, p_2, p_3, p_4, i, j(x)}]$ and defined as:

$$\begin{aligned}
 H^k(x) &= \begin{bmatrix} h_1^k(x) & 0 & 0 & 0 & 0 & \dots \\ h_2^k(x) & h_1^k(x) & 0 & 0 & 0 & \dots \\ h_3^k(x) & h_2^k(x) & h_1^k(x) & 0 & 0 & \dots \\ h_4^k(x) & h_3^k(x) & h_2^k(x) & h_1^k(x) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 0 & 0 & 0 & \dots \\ 2p_1(x) & 1 & 0 & 0 & \dots \\ 4p_1^2(x) + p_2(x) & 2p_1(x) & 1 & 0 & \dots \\ 8p_1^3(x) + 4p_1(x)p_2(x) + kp_3(x) & 4p_1^2(x) + p_2(x) & 2p_1(x) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \\
 &= (g_{H^k(x)}, f_{H^k(x)}).
 \end{aligned}$$

Matrix $H^k(x)$ is a Riordan matrix.

Lemma 3.2. *In the first column, the generator function is as follows:*

$$\begin{aligned} g_{H^k(x)}(r) &= \sum_{n=0}^{\infty} H_{p_1, p_2, p_3, p_4, i, j(x)} r^n \\ &= \frac{1}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

Proof. The first column of matrix $H^k(x)$ follows

$$(1, 2p_1(x), 4p_1^2(x) + p_2(x), 8p_1^3(x) + 4p_1(x)p_2(x) + kp_3(x), \dots)^T,$$

so, this is the generating function for the first column of $H^k(x)$ matrix:

$$\begin{aligned} 1 + (2p_1(x))r + (4p_1^2(x) + p_2(x))r^2 + (8p_1^3(x) + 4p_1(x)p_2(x) + kp_3(x))r^3 + \dots \\ = h_1^k(x) + h_2^k(x)r + h_3^k(x)r^2 + \dots, \end{aligned}$$

so, using (5),

$$\begin{aligned} g_{h_n^k(x)} &= h_0^k(x) + h_1^k(x)r + h_2^k(x)r^2 + \dots + h_n^k(x)r^n + \dots \\ &= \frac{r}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

Thus, can be written

$$\begin{aligned} r(h_1^k(x) + h_2^k(x)r + \dots + h_n^k(x)r^{n-1} + \dots) \\ = \frac{r}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}, \\ h_1^k(x) + h_2^k(x)r + \dots + h_n^k(x)r^{n-1} + \dots \\ = \frac{1}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} g_{H^k(x)}(r) &= \sum_{n=0}^{\infty} H_{p_1, p_2, p_3, p_4, i, j(x)} r^n \\ &= \frac{r}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

□

Form the Riordan matrix, we get $f_{H^k(x)}(r) = r$. Thus, we have

$$\begin{aligned} H^k(x) &= (g_{H^k(x)}(r), f_{H^k(x)}(r)) \\ &= \left(\frac{1}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}, r \right). \end{aligned}$$

Here, we get the k -division Fibonacci-Pell polynomials matrix factorization of Pascal matrix. First, we define a matrix $D^k(x) = (d_{i,j}^k(x))$ as :

$$d_{i,j}^k(x) = \binom{i-1}{j-1} - 2p_1(x) \binom{i-2}{j-1} - p_2(x) \binom{i-3}{j-1} - F(k+1)p_3(x) \binom{i-k-2}{j-1} - F(k)p_4(x) \binom{i-k-3}{j-1}. \quad (10)$$

Thus,

$$D^k(x) = \begin{bmatrix} 1 & 1 & 0 & \dots \\ 1-2p_1(x) & 1 & 0 & \dots \\ 1-2p_1(x)-p_2(x) & 2-2p_1(x) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Theorem 3.3. *The infinite Pascal matrix can be factorized in the following manner:*

$$P(x) = H^k(x) * D^k(x).$$

Proof. Using the definition of infinite Pascal matrix and infinite k -division Fibonacci-Pell polynomials, we have

$$P = \left(\frac{1}{1-r}, \frac{r}{1-r} \right),$$

$$H^k(x) = \left(\frac{1}{1-2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}, r \right).$$

Here, we can get the Riordan representation of the infinite matrix

$$D^k(x) = (g_{D^k(x)}(r), f_{D^k(x)}(r)),$$

as follows:

$$D^k(x) = \begin{bmatrix} 1 & 1 & 0 & \dots \\ 1-2p_1(x) & 1 & 0 & \dots \\ 1-2p_1(x)-p_2(x) & 2-2p_1(x) & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

By utilizing the first column of the matrix $D^k(x)$, we derive:

$$g_{D^k(x)}(r) = \left(\frac{1-2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}{1-r} \right).$$

Also, using the matrix $D^k(x)$, we have

$$f_{D^k(x)}(r) = \frac{r}{1-r}.$$

Therefore, $D^k(x) = (g_{D^k(x)}(r), f_{D^k(x)}(r))$. Then, we have result. □

Now, we define a matrix $E^k(x) = (e_{i,j}^k(x))$ as follows:

$$e_{i,j}^k(x) = \binom{i-1}{j-1} - 2p_1(x) \binom{i-2}{j} - p_2(x) \binom{i-3}{j+1} - F(k+1)p_3(x) \binom{i-k-2}{j+k} - F(k)p_4(x) \binom{i-k-3}{j+k+1}.$$

$$-F(k)p_4(x) \binom{i-k-3}{j+k+1},$$

so,

$$E^k(x) = \begin{bmatrix} 1 & 1 & 0 & \dots \\ 1-2p_1(x) & 1 & 0 & \dots \\ 1-4p_1(x)-p_2(x) & 2-2p_1(x) & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

In a similar manner to Theorem 3.3, the following Lemma is proved.

Lemma 3.4. *The infinite Pascal matrix can be factorized in the following manner:*

$$P(x) = H^k(x) * E^k(x).$$

The infinite k -division Gaussian Fibonacci-Pell polynomials matrix define as follows:

Definition 3.5. The infinite k -division Gaussian Fibonacci-Pell polynomials matrix is denoted by $GH^k(x) = [GH_{p_1, p_2, p_3, p_4, i, j(x)}]$ and is defined :

$$\begin{aligned} GH^k(x) &= \begin{bmatrix} Gh_1^k(x) & 0 & 0 & 0 & 0 & \dots \\ Gh_2^k(x) & Gh_1^k(x) & 0 & 0 & 0 & \dots \\ Gh_3^k(x) & Gh_2^k(x) & Gh_1^k(x) & 0 & 0 & \dots \\ Gh_4^k(x) & Gh_3^k(x) & Gh_2^k(x) & Gh_1^k(x) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= \begin{bmatrix} 2p_1(x) + i & 0 & 0 & 0 & \dots \\ 4p_1^2(x) + 2p_2(x)i & 2p_1(x) + i & 0 & \dots \\ 8p_1^3(x) + 2p_1(x)p_2(x) + 4p_1^2(x)i + p_2(x)i & 4p_1^2(x) + 2p_2(x)i & 2p_1(x) + i & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \\ &= (g_{GH^k(x)}, f_{GH^k(x)}), \end{aligned}$$

where $GH^k(x)$ is a Riordan matrix.

Lemma 3.6. *As a generator function for the first column, we have the following:*

$$\begin{aligned} g_{GH^k(x)}(r) &= \sum_{n=0}^{\infty} GH_{p_1, p_2, p_3, p_4, i, j(x)} r^n \\ &= \frac{2p_1(x) + i}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

Proof. The first column of matrix $GH^k(x)$ is as follows:

$$(2p_1(x) + i, 4p_1^2(x) + 2p_2(x)i, 8p_1^3(x) + 2p_1(x)p_2(x) + 4p_1(x)^2 + p_2(x)i, \dots)^T.$$

Thus, in a matrix $GH^k(x)$, the first column is generated as follows:

$$2p_1(x) + i + (4p_1^2(x) + 2p_2(x)i)r + (8p_1^3(x) + 2p_1(x)p_2(x) + 4p_1(x)^2 + p_2(x)i)r^2 + \dots$$

$$= Gh_1^k(x) + Gh_2^k(x)r + Gh_3^k(x)r^2 + \dots$$

So, using (8),

$$\begin{aligned} g_{Gh_n^k(x)} &= Gh_0^k(x) + Gh_1^k(x)r + Gh_2^k(x)r^2 + \dots + Gh_n^k(x)r^n + \dots \\ &= \frac{r(2p_1(x) + i)}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

Thus, can be written

$$\begin{aligned} &r(Gh_1^k(x) + Gh_2^k(x)r + \dots + Gh_n^k(x)r^{n-1} + \dots) \\ &= \frac{r(2p_1(x) + i)}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}. \end{aligned}$$

Therefore,

$$\begin{aligned} &Gh_1^k(x) + Gh_2^k(x)r + \dots + Gh_n^k(x)r^{n-1} + \dots \\ &= \frac{2p_1(x) + i}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}, \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} g_{GH^k(x)}(r) &= \sum_{n=0}^{\infty} GH_{p_1, p_2, p_3, p_4, i, j}(x) r^n \\ &= \frac{r(2p_1(x) + i)}{1 - 2p_1(x)r - p_2(x)r^2 - F_{k+1}p_3(x)r^k - F_k p_4(x)x^{k+1}}. \end{aligned}$$

□

Form the Riordan matrix, we get $f_{GH^k(x)}(r) = r$. Therefore, we have

$$\begin{aligned} GH^k(x) &= (g_{GH^k(x)}(r), f_{GH^k(x)}(r)) \\ &= \left(\frac{2p_1(x) + i}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}, r \right). \end{aligned}$$

Here, we get the k -division Gaussian Fibonacci-Pell polynomials matrix factorization of Pascal matrix. The amatrix $O^k(x) = \frac{1}{2p_1(x) + i}(o_{i,j}^k(x))$ is defined as:

$$\begin{aligned} o_{i,j}^k(x) &= \binom{i-1}{j-1} - 2p_1(x) \binom{i-2}{j-1} - p_2(x) \binom{i-3}{j-1} - F(k+1)p_3(x) \binom{i-k-2}{j-1} \\ &\quad - F(k)p_4(x) \binom{i-k-3}{j-1}. \end{aligned} \tag{11}$$

Thus,

$$O^k(x) = \begin{bmatrix} 1 & & & & \\ \frac{2p_1(x) + i}{1 - 2p_1(x)} & 1 & 0 & \dots & \\ \frac{2p_1(x) + i}{1 - 2p_1(x) - p_2(x)} & \frac{2p_1(x) + i}{2 - 2p_1(x)} & 0 & \dots & \\ \frac{2p_1(x) + i}{2p_1(x) + i} & \frac{2p_1(x) + i}{2p_1(x) + i} & \dots & \dots & \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

Theorem 3.7. *Pascal's infinite matrix can be factored as follows:*

$$P(x) = H^k(x) * O^k(x).$$

Proof. From Theorem 3.3 and Lemma 3.6, we have

$$P = \left(\frac{1}{1-r}, \frac{r}{1-r} \right),$$

$$H^k(x) = \left(\frac{2p_1(x) + i}{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}, r \right).$$

Here, we can get the Riordan representation of the infinite matrix $O^k(x) = (g_{O^k(x)}(r), f_{O^k(x)}(r))$ as follows:

$$O^k(x) = \begin{bmatrix} 1 & 1 & 0 & \dots \\ \frac{2p_1(x) + i}{1 - 2p_1(x)} & 1 & 0 & \dots \\ \frac{2p_1(x) + i}{1 - 2p_1(x) - p_2(x)} & \frac{2p_1(x) + i}{2 - 2p_1(x)} & 0 & \dots \\ \frac{2p_1(x) + i}{2p_1(x) + i} & \frac{2p_1(x) + i}{2p_1(x) + i} & \dots & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Using the first column of the matrix $O^k(x)$, we have

$$g_{O^k(x)}(r) = \frac{1}{2p_1(x) + i} \left(\frac{1 - 2p_1(x)r - p_2(x)r^2 - F(k+1)p_3(x)r^k - F(k)p_4(x)x^{k+1}}{1 - r} \right).$$

Also, using the matrix $D^k(x)$, we get

$$f_{O^k(x)}(r) = \frac{r}{1 - r}.$$

Therefore, $O^k(x) = (g_{O^k(x)}(r), f_{O^k(x)}(r))$. Then, we have result. $\square$

Now, we define a matrix $S^k(x) = \frac{1}{2p_1(x) + i} (s_{i,j}^k(x))$ as :

$$S_{i,j}^k(x) = \binom{i-1}{j-1} - 2p_1(x) \binom{i-2}{j} - p_2(x) \binom{i-3}{j+1} - F(k+1)p_3(x) \binom{i-k-2}{j+k} - F(k)p_4(x) \binom{i-k-3}{j+k+1}, \quad (12)$$

so,

$$S^k(x) = \frac{1}{2p_1(x) + i} \begin{bmatrix} 1 & 1 & 0 & \dots \\ 1 - 2p_1(x) & 1 & 0 & \dots \\ 1 - 4p_1(x) - p_2(x) & 2 - 2p_1(x) & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Lemma 3.8. *Pascal's infinite matrix can be factored as follows:*

$$P(x) = H^k(x) * S^k(x).$$

Proof. A similar proof is given in Theorem 7 for this Lemma. $\square$

4. Conclusion

In this study, the introduction of k -division Fibonacci-Pell and k -division Gaussian Fibonacci-Pell polynomials enriches the field of combinatorial mathematics and linear algebra. By leveraging the Riordan group framework, the authors successfully factorized the Pascal matrix, revealing new structural properties and potential applications of these polynomial sequences. In [13], we defined the 1-division 3-Lehmer-Pell sequences as follows:

Definition 4.1. For $M = -1$, the 1-division 3-Lehmer-Pell sequences, denoted by $\{hL_n(k, L)\}_{n=0}^{\infty}$, are

$$hL_n(1, 3) = \begin{cases} 3hL_{n-1}(1, 3) + 3hL_{n-2}(1, 3) + hL_{n-3}(1, 3) & n \text{ odd,} \\ hL_{n-1}(1, 3) + hL_{n-2}(1, 3) + hL_{n-3}(1, 3) & n \text{ even,} \end{cases}$$

where $hL_0(1, 3) = hL_1(1, 3) = 0$ and $hL_2(1, 3) = 1$.

Thus, we have $\{hL_n(1, 3)\}_{n=0}^{\infty} = \{0, 0, 1, 3, 2, 16, 11, 83, 56, 428, 2289, \dots\}$. Also, in [14] we have

Definition 4.2. The 1-division 3-Jacobsthal-Pell sequence $\{h_n(1, 3)\}_{n=0}^{\infty}$ is

$$h_n(1, 3) = 3h_{n-1}(1, 3) + 5h_{n-2}(1, 3) + h_{n-3}(1, 3), \quad n \geq 3,$$

with initial conditions $h_0(1, 3) = h_1(1, 3) = 0$ and $h_2(1, 3) = 1$.

We can do all the definitions, theorems, and lemmas proven for these two families as future work.

Our suggestions for future research are:

- Analytical Properties: Further exploration of the zeros, orthogonality, and limit behaviors of the proposed polynomials could yield deeper mathematical insights.
- Generalizations: Extending these polynomials to encompass other classical sequences (e.g., Jacobsthal or Horadam) may uncover broader connections.
- Applications: Investigating potential uses in cryptography, coding theory, and signal processing could enhance the practical relevance of the results.
- Graphical Analysis: Visualization of the polynomial behaviors and matrix factorizations could offer intuitive understanding and attract broader interest.
- Computational Algorithms: Developing efficient algorithms for computing these polynomials and their matrix factorizations would be beneficial for large-scale applications.

This work provides a solid foundation for future explorations in both theoretical and applied mathematics.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

- [1] R. Brawer and M. Pirovino. The linear algebra of the Pascal matrix. *Linear Algebra and its Applications*, 174:13–23, 1992. [https://doi.org/10.1016/0024-3795\(92\)90038-C](https://doi.org/10.1016/0024-3795(92)90038-C).
- [2] Ö. Deveci and E. Karaduman. The Pell sequences in finite groups. *Utilitas Mathematica*, 96:263–279, 2015.
- [3] Ö. Deveci and A. G. Shannon. The quaternion-Pell sequence. *Communications in Algebra*, 46(12):5403–5409, 2018. <https://doi.org/10.1080/00927872.2018.1468906>.
- [4] S. Falcon. The k -Fibonacci matrix and the Pascal matrix. *Central European Journal of Mathematics*, 9:1403–1410, 2011. <https://doi.org/10.2478/s11533-011-0089-9>.
- [5] M. Hashemi and E. Mehraban. Factorization of the t -extension of the p -Fibonacci matrix and Pascal matrix. *Journal of Mathematical Modeling*, 2021. <https://doi.org/10.22124/jmm.2021.18678.1597>.
- [6] M. Hashemi and E. Mehraban. The generalized order k -Pell sequences in some special groups of nilpotency class 2. *Communications in Algebra*, 50(4):1768–1784, 2021. <https://doi.org/10.1080/00927872.2021.1988959>.
- [7] B. Kuloğlu and E. Özkan. On generalized (k, r) -Gauss Pell numbers. *Journal of Science and Arts*, 3:617–624, 2021. <https://doi.org/10.46939/J.Sci.Arts-21.3-a02>.
- [8] B. Kuloğlu and E. Özkan. d -tribonacci polynomials and their matrix representations. *Wseas Transactions on Mathematics*, 22:204–212, 2023. <https://doi.org/10.37394/23206.2023.22.25>.
- [9] B. Kuloğlu and E. Özkan. Matrix representation of the (d, k) -Fibonacci polynomials. *Bulletin of the Transilvania University of Brasov Series III Mathematics and Computer Science*, 4(66):185–200, 2024. <https://doi.org/10.31926/but.mif.2024.4.66.2.11>.
- [10] G. Y. Lee, J. S. Kim, and S. G. Lee. Factorizations and eigenvalues of Fibonacci and symmetric Fibonacci matrices. *Fibonacci Quarterly*, 40:203–211, 2002. https://doi.org/10.1080/00150517.2002.12428645?urlappend=%3Futm_source%3Dresearchgate.net%26medium%3Darticle.
- [11] E. Mehraban, Ö. Deveci, and E. Hincal. The t -Fibonacci sequences in the 2-generator p -groups of nilpotency class 2. *Notes on Number Theory and Discrete Mathematics*, 29(4):827–841, 2023. <https://doi.org/10.7546/nntdm.2023.29.4.827-841>.
- [12] E. Mehraban, Ö. Deveci, and E. Hincal. k -division Fibonacci-Pell sequences and their applications. *Utilitas Mathematica*, 123:87–97, 2025. <https://doi.org/10.61091/um123-06>.

- [13] E. Mehraban, T. A. Gulliver, Ö. Deveci, and E. Hincal. Affine-hill cipher key generation using self-invertible matrices from k -division L -lehmer-Pell sequences. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 124:47–58, 2025. <https://doi.org/10.61091/jcmcc124-03>.
- [14] E. Mehraban, T. A. Gulliver, and E. Hincal. A new blind signature from k -division t -jacobsthal-Pell sequences on the heisenberg group. *Applied Mathematics E-note*. preprint.
- [15] E. Mehraban and M. Hashemi. Fibonacci length and the generalized order k -Pell sequences of the 2-generator p -groups of nilpotency class 2. *Journal of Algebra and Its Applications*, 22(3), 2023. <https://doi.org/10.1142/S0219498823500615>. Article ID 2350061.
- [16] A. Nalli and P. Haukkanen. On generalized Fibonacci and Lucas polynomials. *Chaos Solitons Fractals*, 42:3179–3186, 2009. <https://doi.org/10.1016/j.chaos.2009.04.048>.
- [17] E. Özkan and M. Uysal. d -Gaussian Fibonacci, d -Gaussian Lucas polynomials and their matrix representations. *Ukrainian Mathematical Journal*, 75(4):562–585, 2023. <https://doi.org/10.1007/s11253-023-02217-3>.
- [18] L. W. Shapiro, S. Getu, W. J. Woan, and L. C. Woodson. The riordan group. *Discrete Applied Mathematics*, 34(1-3):229–239, 1991. [https://doi.org/10.1016/0166-218X\(91\)90088-E](https://doi.org/10.1016/0166-218X(91)90088-E).
- [19] D. Tasci, N. Tuglu, and M. Asci. On fibo-Pascal matrix involving k -Fibonacci and k -Pell matrices. *Arabian Journal for Science and Engineering*, 36:1031–1037, 2011. <https://doi.org/10.1007/s13369-011-0091-3>.
- [20] M. Tastan, E. Ozkan, and A. G. Shannon. The generalized k -Fibonacci polynomials and generalized k -Lucas polynomials. *Notes on Number Theory and Discrete Mathematics*, 27:148–158, 2021. <https://doi.org/10.7546/nntdm.2021.27.2.148-158>.

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