

On the numerical semigroup generated by $\left\{ b^{k+n+i} + \frac{b^{n+i}-1}{b-1} \mid i \in \mathbb{N} \right\}$

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ABSTRACT

Let k, b, n be positive integers such that $b \geq 2$. Denote by $S(k, b, n)$ the numerical semigroup generated by $\left\{ b^{k+n+i} + \frac{b^{n+i}-1}{b-1} \mid i \in \mathbb{N} \right\}$. In this paper, we give formulas for computing the embedding dimension and the Frobenius number of $S(k, b, n)$.

Keywords: numerical semigroups, embedding dimension, Frobenius number

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1. Introduction and preliminaries

Let \mathbb{N} be the set of nonnegative integers. A *numerical semigroup* is a submonoid S of \mathbb{N} under addition such that $\mathbb{N} \setminus S$ is finite. The elements of $G(S) = \mathbb{N} \setminus S$ are called the *gaps* of S . An important invariant of S is the largest integer in $G(S)$, known as the *Frobenius number* of S and denoted by $F(S)$.

Given a nonempty subset A of \mathbb{N} , we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A , that is, $\langle A \rangle = \{ \lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N} \setminus \{0\}, a_1, \dots, a_n \in A, \lambda_1, \dots, \lambda_n \in \mathbb{N} \}$. If $M = \langle A \rangle$, then we say that A is system of generators of M . In addition, if no proper subset of A generates M , then we say that A is a *minimal system of generators* of M . It is shown that every submonoid of $(\mathbb{N}, +)$ has a unique minimal system of generators and such a system is finite [8]. Moreover, $\langle A \rangle$ is a numerical semigroup if and only if $\gcd(A) = 1$. The cardinality of the minimal system of generators of a numerical semigroup S is called the *embedding dimension* of S and is denoted by $e(S)$.

The Frobenius problem (see [4, 8]) is to find formulas for computing the Frobenius

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number and the genus of a numerical semigroup. The problem was solved by Sylvester in [9] for numerical semigroups with embedding dimension two. Recently, this problem for numerical semigroups with embedding dimension three was solved by Tripathi in [10]. However, this problem remains open for numerical semigroups with embedding dimension greater than three.

A *repunit* is a number like 11, 111, or 1111 that contains only the digit 1. The set of repunits in base b is $\{\frac{b^n-1}{b-1} \mid n \in \mathbb{N}_+\}$. In particular, repunits in binary is well-known *Mersenne numbers*. A positive integer x is a *Thabit number* if $x = 3 \cdot 2^n - 1$ for some $n \in \mathbb{N}$. The concept of Mersenne [7] (repunit [6], Thabit [5]) numerical semigroups was introduced by Rosales, Branco and Torráo. Moreover, they gave the formulas for the Frobenius number, the genus and the embedding dimension of the three kinds of numerical semigroups. In [3], we generalized Thabit numerical semigroups to the class of numerical semigroups generated by $\{(2^k - 1) \cdot 2^{n+i} - 1 \mid i \in \mathbb{N}\}$, where k, n are positive integers such that $2 \leq k \leq 2^n$. In [2], we generalize Thabit numerical semigroups along another line to the class of numerical semigroup generated by $\{b^{n+1+i} + \frac{b^{n+i}-1}{b-1} \mid i \in \mathbb{N}\}$, where $b \geq 2$ is an integer. In this paper, we further generalize to the numerical semigroup generated by $\{b^{k+n+i} + \frac{b^{n+i}-1}{b-1} \mid i \in \mathbb{N}\}$ and denote it by $S(k, b, n)$, where k, b, n are positive integers such that $b \geq 2$. The main purpose of the remainder of this paper is to give the formulas for the the embedding dimension and the Frobenius number of $S(k, b, n)$.

2. The embedding dimension

Let k, b, n be positive integers such that $b \geq 2$. Then $S(k, b, n)$ is a submonoid of $(\mathbb{N}, +)$. Denote by s_i the integer $b^{k+n+i} + \frac{b^{n+i}-1}{b-1}$ for all $i \in \mathbb{N}$. It is clear that $s_{i+1} = bs_i + 1$ for all $i \in \mathbb{N}$. Hence $\gcd(s_i, s_{i+1}) = 1$ and so $S(k, b, n)$ is a numerical semigroup.

Lemma 2.1 (Lemma 2 [5]). *Let s, t be integers and let A be nonempty set of integers such that $M = \langle A \rangle$. Then the following conditions are equivalent:*

- (1) $sa + t \in M$ for all $a \in A$;
- (2) $sm + t \in M$ for all $m \in M \setminus \{0\}$.

Let $T(k, b, n)$ be a submonoid of $(\mathbb{N}, +)$ generated by $\{(b-1) \cdot b^{k+n+i} + b^{n+i} - 1 \mid i \in \mathbb{N}\}$. Denote $(b-1) \cdot b^{k+n+i} + b^{n+i} - 1$ by t_i for all $i \in \mathbb{N}$. It is easy to see that $s_i = \frac{t_i}{b-1}$. Define a map $f : T(k, b, n) \rightarrow S(k, b, n)$ by $f(t) = \frac{t}{b-1}$ for all $t \in T(k, b, n)$. Then f is a monoid isomorphism. Consequently, if A is the minimal system of generators of $T(k, b, n)$, then $\{\frac{a}{b-1} \mid a \in A\}$ is the minimal system of generators of $S(k, b, n)$.

Lemma 2.2. $T(k, b, n) = \langle \{t_i \mid i = 0, 1, \dots, n+k\} \rangle$.

Proof. Denote $M = \langle \{t_i \mid i = 0, 1, \dots, n+k\} \rangle$. Then $M \subseteq T(k, b, n)$. Next we prove another inclusion. If $i \in \{0, 1, \dots, n+k-1\}$, then $bt_i + b - 1 = t_{i+1} \in M$. If $i = n+k$, then $bt_i + b - 1 = t_{n+k+1} = (b-1) \cdot b^{2k+2n+1} + b^{k+2n+1} - 1 = (t_0 - b^k + 2)t_0 + t_k + (b^k - 1)t_n \in M$. From Lemma 2.1, we obtain that $bt + b - 1 \in M$ for all $t \in M \setminus \{0\}$. By induction, we can

easily show that $t_i \in M$ for all $i \geq n + k + 1$. Therefore, we complete our proof. \square

Theorem 2.3. *The set $\{t_i \mid i = 0, 1, \dots, n + k\}$ is the minimal system of generators of $T(k, b, n)$.*

Proof. By Lemma 2.2, we know that $\{t_i \mid i = 0, 1, \dots, n + k\}$ is a system of generators of $T(k, b, n)$. Next we need only prove that $t_j \notin \langle \{t_0, t_1, \dots, t_{j-1}\} \rangle$ for any $j \in \{1, 2, \dots, n + k\}$. Suppose that there exists $j \in \{1, 2, \dots, n + k\}$ such that $t_j \in \langle \{t_0, t_1, \dots, t_{j-1}\} \rangle$. Then $t_j = \sum_{i=0}^{j-1} a_i t_i = \sum_{i=0}^{j-1} a_i [(b-1) \cdot b^{k+n+i} + b^{n+i}] - \sum_{i=0}^{j-1} a_i$ for some $a_0, \dots, a_{j-1} \in \mathbb{N}$. Hence $\sum_{i=0}^{j-1} a_i \equiv 1 \pmod{(b-1) \cdot b^{k+n} + b^n}$ and so $\sum_{i=0}^{j-1} a_i = 1 + s[(b-1) \cdot b^{k+n} + b^n]$ for some $s \in \mathbb{N}_+$. Thus $\sum_{i=0}^{j-1} a_i \geq 1 + (b-1) \cdot b^{k+n} + b^n$. Consequently, $(b-1) \cdot b^{k+n+j} + b^{n+j} - 1 = t_j \geq \sum_{i=0}^{j-1} a_i t_0 \geq [1 + (b-1) \cdot b^{k+n} + b^n][(b-1) \cdot b^{k+n} + b^n - 1] = (b-1)^2 \cdot b^{2k+2n} + 2(b-1) \cdot b^{k+2n} + b^{2n} - 1$, which is impossible. Therefore, we obtain our conclusion. \square

As a consequence of the previous statement and theorem, we have the following result.

Corollary 2.4. *The minimal system of generators of $S(k, b, n)$ is $\{s_i \mid i = 0, 1, \dots, n + k\}$ and so $e(S(k, b, n)) = n + k + 1$.*

3. The Frobenius number

Let S be a numerical semigroup and $m \in S \setminus \{0\}$. The *Apéry set* (see [1, 8]) of m in S is $Ap(S, m) = \{s \in S \mid s - m \notin S\}$. It is well known from [8] that $Ap(S, m) = \{w(0), w(1), \dots, w(m-1)\}$, where $w(i)$ is the least element of S congruent with i modulo m for all $i \in \{0, 1, \dots, m-1\}$. Our aim purpose is to give an explicit description of the elements in the set $Ap(S(k, b, n), s_0)$.

Lemma 3.1. *Let $Ap(S(k, b, n), s_0) = \{w(0), w(1), \dots, w(s_0 - 1)\}$. Then $w(0) < w(1) < \dots < w(s_0 - 1)$.*

Proof. We show that $w(i) < w(i + 1)$ for all $i \in \{0, 1, \dots, s_0 - 2\}$. Since $w(i + 1) \in S(k, b, n)$, there exist a_0, a_1, \dots, a_{n+k} such that $w(i + 1) = \sum_{j=0}^{n+k} a_j s_j$. Moreover, $w(i + 1) \not\equiv 0 \pmod{s_0}$ and so there exists $j \in \{1, 2, \dots, n + k\}$ such that $a_j \neq 0$. Therefore,

$$\begin{aligned} w(i + 1) - 1 &= a_0 s_0 + \dots + (a_j - 1) s_j + \dots + a_{n+k} s_{n+k} + s_j - 1 \\ &= a_0 s_0 + \dots + (a_j - 1) s_j + \dots + a_{n+k} s_{n+k} + b s_{j-1} \\ &= a_0 s_0 + \dots + (a_{j-1} + b) s_{j-1} + (a_j - 1) s_j + \dots + a_{n+k} s_{n+k} \\ &\in S(k, b, n). \end{aligned}$$

As $w(i + 1) - 1 \equiv i \pmod{s_0}$, we have $w(i) \leq w(i + 1) - 1$. \square

Lemma 3.2. $\max Ap(S(k, b, n), s_0) \leq s_n + (b - 1)s_{n+k}$.

Proof. For all $i \in \mathbb{N}$, it is easy to show that $s_i = b^i s_0 + \frac{b^i - 1}{b - 1}$. Therefore, we can obtain that

$$\begin{aligned} s_n + (b - 1)s_{n+k} &= b^n s_0 + \frac{b^n - 1}{b - 1} + (b - 1) \left(b^{k+n} s_0 + \frac{b^{k+n} - 1}{b - 1} \right) \\ &= [b^n + (b - 1)b^{k+n}]s_0 + b^{k+n} + \frac{b^n - 1}{b - 1} - 1 \\ &= [b^n + (b - 1)b^{k+n}]s_0 + (s_0 - 1). \end{aligned}$$

Hence, $s_n + (b - 1)s_{n+k} \equiv s_0 - 1 \pmod{s_0}$. Consequently, $w(s_0 - 1) \leq s_n + (b - 1)s_{n+k}$ and by Lemma 3.1 we complete our proof. \square

Let r be a positive integer. We define the set $A(r) = \{(a_1, a_2, \dots, a_r) \in \{0, 1, \dots, b\}^r \mid \text{if } 1 \leq i < j \leq r \text{ and } a_j = b, \text{ then } a_i = 0.\}$ and a relation \leq_r on $A(r)$ as follows:

$$\begin{aligned} (x_1, x_2, \dots, x_r) &\leq_r (y_1, y_2, \dots, y_r) \\ \Leftrightarrow (\forall m > 0) \ x_m = y_m \text{ or } (\exists m > 0)(\forall i > m) \ x_m < y_m, x_i = y_i. \end{aligned}$$

It is well known that \leq_r is a colexicographic order and a total order on $A(r)$. Similarly, we can define the strict order $<_r$ on $A(r)$. We denote $R(n + k) = \{(x_1, x_2, \dots, x_{n+k}) \in A(n + k) \mid (x_1, x_2, \dots, x_{n-1}, x_n, x_{n+1}, \dots, x_{n+k-1}, x_{n+k}) \leq_{n+k} (0, 0, \dots, 0, 1, 0, \dots, 0, b - 1)\}$ and its cardinality by $\sharp R(n + k)$.

Lemma 3.3. $Ap(S(k, b, n), s_0) \subseteq \left\{ \sum_{i=1}^{n+k} a_i s_i \mid (a_1, a_2, \dots, a_{n+k}) \in A(n + k) \right\}$.

Proof. Let $x \in Ap(S(k, b, n), s_0)$. We need only show that $x = \sum_{i=1}^{n+k} a_i s_i$ for some $(a_1, \dots, a_{n+k}) \in A(n + k)$. By induction over x , if $x = 0$ then the result holds obviously. Suppose that $x > 0$ and $j = \min\{i \in \{1, 2, \dots, n + k\} \mid x - s_i \in S(k, b, n)\}$. Since $x \in Ap(S(k, b, n), s_0)$, we have $x - s_j \in Ap(S(k, b, n), s_0)$. By induction hypothesis, there exists $(y_1, y_2, \dots, y_{n+k}) \in A(n + k)$ such that $x - s_j = \sum_{i=1}^{n+k} y_i s_i$. Hence $x = y_1 s_1 + \dots + (y_j + 1)s_j + \dots + y_{n+k} s_{n+k}$. Next we check that $(y_1, \dots, y_j + 1, \dots, y_{n+k}) \in A(n + k)$.

(1) To prove $(y_1, \dots, y_j + 1, \dots, y_{n+k}) \in \{0, 1, \dots, b\}^{n+k}$, it suffices to show that $y_j + 1 \neq b + 1$. If $y_j + 1 = b + 1$, then $(y_j + 1)s_j = (b + 1)s_j = bs_{j-1} + s_{j+1}$. Thus $x - s_{j-1} \in S(k, b, n)$ which contradicts the minimality of j .

(2) From the minimality of j , we obtain that $y_i = 0$ for all $1 \leq i < j$. Moreover, $y_l \neq b$ for all $j < l \leq n + k$. In fact, if there exists $j < l \leq n + k$ such that $y_l = b$, then $s_j + y_l s_l = s_j + bs_l = bs_{j-1} + s_{l+1}$. Hence $x - s_{j-1} \in S(k, b, n)$ which is impossible. \square

Lemma 3.4. $\sharp R(n + k) = b^{k+n} + \frac{b^n - 1}{b - 1}$.

Proof. We define a mapping $\phi : R(n+k) \rightarrow \mathbb{N}$ by $\phi(x_1, \dots, x_{n+k}) = \sum_{i=1}^{n+k} x_i \frac{b^i-1}{b-1}$.

(1) ϕ is strictly increasing.

Let $\mathbf{x} = (x_1, \dots, x_{n+k})$ and $\mathbf{y} = (y_1, \dots, y_{n+k})$ be two distinct elements of $R(n+k)$ with $\mathbf{x} <_{n+k} \mathbf{y}$ in the colexicographic order. Then there exists an index $m \in \{1, \dots, n+k\}$ such that $x_m < y_m$ and $x_i = y_i$ for all $i > m$. Then

$$\phi(\mathbf{y}) - \phi(\mathbf{x}) = (y_m - x_m) \cdot \frac{b^m - 1}{b - 1} + \sum_{i=1}^{m-1} (y_i - x_i) \cdot \frac{b^i - 1}{b - 1}.$$

We consider the worst-case scenario for the difference

- $y_m - x_m = 1$,
- For $i < m$, we take $y_i = 0$ and x_i as large as possible (i.e., $x_i = b - 1$ for all $i < m$).

Then $\phi(\mathbf{y}) - \phi(\mathbf{x}) \geq \frac{b^m-1}{b-1} - \sum_{i=1}^{m-1} (b-1) \cdot \frac{b^i-1}{b-1} = \frac{b^m-1}{b-1} - \sum_{i=1}^{m-1} (b^i-1) = \frac{b^m-1}{b-1} - [\frac{b^m-b}{b-1} - (m-1)] = \frac{b-1}{b-1} + (m-1) = m > 0$. Hence, ϕ is strictly increasing.

(2) The image of ϕ is $[0, s_0 - 1]$.

It is obvious that the minimum element in $R(n+k)$ is $\mathbf{o} = (0, \dots, 0)$, and the maximum element in $R(n+k)$ is $\mathbf{a}^* = (0, \dots, 0, 1, 0, \dots, 0, b-1)$, where the 1 is at position n and $b-1$ is at position $n+k$. Then $\phi(\mathbf{o}) = 0$, and $\phi(\mathbf{a}^*) = \frac{b^n-1}{b-1} + (b-1) \cdot \frac{b^{n+k}-1}{b-1} = b^{n+k} + \frac{b^n-1}{b-1} - 1 = s_0 - 1$. Since ϕ is strictly increasing and $R(n+k)$ is finite, ϕ maps $R(n+k)$ bijectively onto the integer interval $[0, s_0 - 1]$.

Therefore $\#R(n+k) = s_0 = b^{k+n} + \frac{b^n-1}{b-1}$. □

Theorem 3.5. $Ap(S(k, b, n), s_0) = \{ \sum_{i=1}^{n+k} a_i s_i \mid (a_1, \dots, a_{n+k}) \in R(n+k) \}$.

Proof. From Lemma 3.2, 3.3 and 3.4, we have

$$Ap(S(k, b, n), s_0) \subseteq \left\{ \sum_{i=1}^{n+k} a_i s_i \mid (a_1, \dots, a_{n+k}) \in R(n+k) \right\}.$$

Now consider the mapping $\psi : R(n+k) \rightarrow \mathbb{N}$ by $\psi(a_1, \dots, a_{n+k}) = \sum_{i=1}^{n+k} a_i s_i$.

We show that ψ is injective on $R(n+k)$. Suppose $\sum_{i=1}^{n+k} a_i s_i = \sum_{i=1}^{n+k} a'_i s_i$. Since $s_i = b^i s_0 + \frac{b^i-1}{b-1}$, we have

$$\sum_{i=1}^{n+k} a_i \frac{b^i - 1}{b - 1} \equiv \sum_{i=1}^{n+k} a'_i \frac{b^i - 1}{b - 1} \pmod{s_0}.$$

But both sides are in $[0, s_0 - 1]$ by the proof of Lemma 3.4, so they must be equal. The injectivity of ϕ in Lemma 3.4 implies $a_i = a'_i$. Hence, ψ is injective.

By Lemma 3.4, the set on the right has cardinality s_0 , and $Ap(S(k, b, n), s_0)$ also has cardinality s_0 . Therefore, the two sets are equal. □

Let S be a numerical semigroup and $m \in S \setminus \{0\}$. It is well known from [8] that $F(S) = \max \text{Ap}(S, m) - m$. Hence, we have the following formula for the Frobenius number of $S(k, b, n)$.

Corollary 3.6. $F(S(k, b, n)) = (b^{k+1} - b^k + 2) \cdot b^{k+2n} + \left(\frac{b^n-1}{b-1} - b^k\right) \cdot b^n - 1$.

Proof. By Theorem 3.5, the maximum element of $\text{Ap}(S(k, b, n), s_0)$ is $\psi(a^*)$, where $a^* = (0, \dots, 0, 1, 0, \dots, 0, b-1)$ is the maximum element in $R(n+k)$. We compute

$$\psi(a^*) = s_n + (b-1)s_{n+k}.$$

The Frobenius number is then

$$F(S(k, b, n)) = s_n + (b-1)s_{n+k} - s_0 = (b^{k+1} - b^k + 2) \cdot b^{k+2n} + \left(\frac{b^n-1}{b-1} - b^k\right) \cdot b^n - 1.$$

This completes the proof. \square

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References

- [1] R. Apéry. Sur les branches superlinéaires des courbes algébriques. *C. R. Acad. Sci. Paris*, 222:1198–1200, 1946.
- [2] Z. Gu. On the numerical semigroup generated by $\{b^{n+1+i} + \frac{b^{n+i}-1}{b-1} \mid i \in \mathbb{N}\}$. *Discrete Mathematics and Applications*, 30(4):257–264, 2020. https://doi.org/10.1515/dma-2020-0022?urlappend=%3Futm_source%3Dresearchgate.net%26medium%3Darticle.
- [3] Z. Gu and X. L. Tang. The frobenius problem for a class of numerical semigroups. *International Journal of Number Theory*, 13(5):1335–1347, 2017. <https://doi.org/10.1142/S1793042117500749>.
- [4] J. L. Ramírez Alfonsín. *The Diophantine Frobenius Problem*. Oxford University Press, London, 2005.
- [5] J. C. Rosales, M. B. Branco, and D. Torráo. The frobenius problem for Thabit numerical semigroups. *Journal of Number Theory*, 155:85–99, 2015. <https://doi.org/10.1016/j.jnt.2015.03.006>.
- [6] J. C. Rosales, M. B. Branco, and D. Torráo. The frobenius problem for repunit numerical semigroups. *Ramanujan Journal*, 40(2):323–334, 2016. <https://doi.org/10.1007/s11139-015-9719-3>.

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- [7] J. C. Rosales, M. B. Branco, and D. Torráo. The frobenius problem for Mersenne numerical semigroups. *Mathematische Zeitschrift*, 286(1):741–749, 2017. <https://doi.org/10.1007/s00209-016-1781-z>.
- [8] J. C. Rosales and P. A. García-Sánchez. *Numerical Semigroups*. Springer, New York, 2009.
- [9] J. J. Sylvester. On subvariants, i.e. semi-invariants to binary quantics of an unlimited order. *American Journal of Mathematics*:79–136, 1882.
- [10] A. Tripathi. Formulae for the frobenius number in three variables. *Journal of Number Theory*, 170:368–389, 2017. <https://doi.org/10.1016/j.jnt.2016.05.027>.

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