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Period of Pell-Narayana sequence in groups

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ABSTRACT

This paper studies the Pell-Narayana sequence modulo m. It starts by defining the Pell-Narayana numbers and examining their combinatorial relationships with well-known sequences and functions, including Eulerian, Catalan, and Delannoy numbers. Building on this, the concept of a Pell-Narayana orbit is introduced for a 2-generator group with generating pair $(x, y) \in G$, which allows the analysis of the periods of these orbits. The results include explicit calculations of the Pell-Narayana periods for polyhedral and binary polyhedral groups, depending on the choice of generating pair (x, y), along with a discussion of their properties. Furthermore, the paper determines the periodic lengths of Pell-Narayana orbits for the groups $Q_8, Q_8 \times \mathbb{Z}_{2m}$, and $Q_8 \times_\varphi \mathbb{Z}_{2m}$ for all $m \geq 3$.

Keywords: Pell-Narayana sequence, Pell-Narayana length, Pell-Narayana orbits, polyhedral group, binary polyhedral group

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1. Introduction

Since Fibonacci numbers exist in nature and find application in many branches of science, they have managed to attract the attention of many scientists for many years. Inspired by the Fibonacci numbers, many number sequences were defined and their properties were examined. One of the most important of these number sequences is the Narayana numbers. Narayana numbers are also associated with many other number sequences and their properties are studied [5, 20, 13, 22].

Applications of Fibonacci number sequences in the cyclic group were started with Wall [27], continued to be studied with different number sequences in different groups [7, 8, 19,

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18, 17]. Later in [16, 19], it was applied to 3-step Fibonacci numbers and in [15], these applications were moved to k-step Fibonacci numbers.

The polyhedral and binary polyhedral groups were explored in earlier works such as [26] and [4]. Building on these foundations, researchers like Knox, Kumari, and Aküzüm [12, 14, 2] extended the analysis to k-nacci sequences within finite groups. Similarly, in later studies, 3-term recurrence sequences and their generalized forms were studied in the context of finite groups and fields. [6, 3, 9, 1]. In the coming years, it is likely that research will increasingly explore the use of number sequences to investigate the structure and properties of more complex algebraic systems, such as quaternions, hybrid numbers, and dual numbers, within complex or higher-dimensional spaces. These studies may focus on the underlying group structures, symmetries, and interactions of such number systems, potentially revealing new insights into both theoretical and applied mathematics [11, 29, 28].

Broadly speaking, this area of research aims to deepen the understanding of the interplay between number sequences and algebra. By expanding the scope of work on sequences, it investigates related identities, Binet formulas, and generating functions. In this study, the relationship between the Pell-Narayana sequence and algebra is explored, starting from its defined properties and its connections with groups. Specifically, the concept of Pell-Narayana sequences is extended and analyzed. In Section 2, the Pell-Narayana sequence is studied in the context of finite groups, with particular attention to its periodic behavior modulo m. Section 3 focuses on the orbits of Pell-Narayana sequences within finite groups. Finally, the lengths of these sequences for generating pairs in groups are examined, complemented by applications designed to provide a clearer understanding of the findings.

2. The Pell-Narayana sequences

A Pell-Narayana sequence $\{PN_r\}$ is defined [21] recursively by the equation

$$PN_r = 2PN_{r-1} + PN_{r-3}, \tag{1}$$

where $PN_0 = 0,\ PN_1 = 1,\ PN_2 = 1$. it is $\{0,1,1,2,5,11,24, \dots\}$ (A078012) [24].

Kalman [3] pointed out that all sequences are essentially a linear combination of the preceding k terms of the equation defined below:

$$a_{n+k} = c_0 a_n + \cdots + c_{k-1} a_{n+k-1},$$

such that $\{c_0, c_1, \dots, c_{k-1}\} \in \mathbb{R}$. These coefficients with companion matrix can be represented in closed form as follows [10]:

$$A_k=[a_{ij}]_{k\times k}=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ c_0 & c_1 & c_2 & \dots & c_{k-2} & c_{k-1} \end{bmatrix}_{k\times k}. \tag{2}$$

Then by an inductively,

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ \vdots \\ a_{n+k-1} \end{bmatrix}. \tag{3}$$

If the first k terms of a sequence are repeated in the sequence from the beginning, the sequence is called simply the periodic sequence, and the period of this sequence is also k. For example, the sequence, $x_1, x_2, ..., x_5, x_1, x_2, ..., x_5, \dots$ is a simply periodic with period 5. On the other hand, if consecutive k terms repeat in the sequence between terms of a sequence, this sequence is called a periodic sequence, and the period of this sequence is k. For example, the sequence, $x_1, x_2, x_3, x_4, x_3, x_4, \dots$, is a periodic, period 2.

Definition 2.1. Consider a finite generated group $G = \langle A\rangle$ for the finite set $A = \{a_1, a_2, \dots., a_n\}$.

For $0 \le i \le n-1$, if $x_i = a_{i+1}$ and $x_{i+n} = \prod_{j=1}^{n} x_{i+j-1}$ then the sequence $\{x_0, x_1, \dots\}$ is called a Fibonacci orbit of G and showed by $F_A(G)$. If $F_A(G)$ is a simply periodic sequence, then the period of the sequence is called a Fibonacci length of G, shown $LEN_A(G)$ [4].

2.1. Pell–Narayana sequences modulo m

From statements (1) and (3), for the Pell-Narayana sequence, we can write the following relation

$$\begin{bmatrix} PN_{n-2} \\ PN_{n-1} \\ PN_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} PN_{n-3} \\ PN_{n-2} \\ PN_{n-1} \end{bmatrix}. \tag{4}$$

Now, let us take

$$K=[k_{ij}]_{3\times 3}=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{bmatrix},$$

which is said to be Pell-Narayana matrix. Then by an mathematical induction, for $n \geq 0$,

$$K^n \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} PN_n \\ PN_{n+1} \\ PN_{n+2} \end{bmatrix}, \tag{5}$$

is obtained.

Now, let's reduce each term of the Pell-Narayana sequence to mod m and denote the resulting new sequence with $\left\{PN_n^{(m)}\right\}$. That is,

$$\left\{PN_n^{(m)}\right\} = \left\{PN_0^{(m)}, PN_1^{(m)}, \dots\right\}.$$

These recurrences follow the same pattern as described in (1). They are part of a broader family of third-order lacunary sequences, which include the Padovan, Perrin, Pell-Narayana, Narayana, and Plastic sequences. This family has been thoroughly investigated in studies by Anderson, Horadam, Petroudi, Soykan, and Shannon [21, 23, 25].

Theorem 2.2. *The sequence* $\left\{PN_n^{(m)}\right\}$ *exhibits simple periodicity.*

Proof. Given that there are only a finite number of possible term triplets, specifically m^3, the sequence eventually repeats. Once the triplet begins to repeat, all subsequent terms are generated by iterating this repeating pattern.

From (1), we have

$$PN_{n+3} = 2PN_{n+2} + PN_n,$$

so, if

$$PN_{i+2}^{(m)} = PN_{j+2}^{(m)},$$
$$PN_{i+1}^{(m)} = PN_{j+1}^{(m)},$$
$$PN_i^{(m)} = PN_j^{(m)},$$

then $PN_{i-j+2}^{(m)} = PN_2^{(m)}$, $PN_{i-j+1}^{(m)} = PN_1^{(m)}$ and $PN_{i-j}^{(m)} = PN_0^{(m)}$.

This demonstrates that the sequence $\left\{PN_n^{(m)}\right\}$ is a simply periodic.

Let $\mathrm{Per}(PN^{(m)})$ the smallest period of the Pell Narayana sequence $\left\{PN_n^{(m)}\right\}$ and let p_i represent distinct primes. If $m = \prod_{i=1}^{t} {p_i}^{e_i} (t \geq 1\ )$ therefore we write $\mathrm{Per}(PN^{(m)}) = \mathrm{lcm}[\mathrm{Per}(PN^{({p_i}^{e_i})})]$, the least common multiple of the $\mathrm{Per}(PN^{({p_i}^{e_i})})$. □

Theorem 2.3. *Let* $m = \prod_{i=1}^{t} {p_i}^{e_i} (t \geq 1\ )$ *and* p_i *represent distinct primes, then* $\left\{PN_n^{(m)}\right\}$ $= \mathrm{lcm}[\mathrm{Per}(PN_n^{({p_i}^{e_i})})]$, *the least common multiple of the* $\mathrm{Per}(PN_n^{({p_i}^{e_i})})$.

Proof. In addition to defining the length of the period of $\left\{PN_{\mathrm{n}}^{({p_i}^{e_i})}\right\}$ with $\mathrm{Per}(PN^{({p_i}^{e_i})})$, We can also express this by stating that the sequence $\left\{PN_n^{({p_i}^{e_i})}\right\}$ repeats only after blocks

of length $u \cdot \mathrm{Per}(PN_n^{(p_i^{e_i})})$, where $u \in \mathbb{N}$. This follows from the previously established result regarding the period length of $\left\{PN_n^{(m)}\right\}$, given as $\mathrm{Per}(PN_n^{(m)})$.This means that for all values of i, $\left\{PN_{\mathrm{n}}^{(p_i{}^{e_i})}\right\}$ repeats after $\mathrm{Per}(PN_n^{(m)})$.

Therefore, $\mathrm{Per}(PN_n^{(m)})$ takes the form $u \cdot \mathrm{Per}(PN_n^{(p_i^{e_i})})$ for all values of i, where any such u contributes to a period of the sequence $\left\{PN_n^{(m)}\right\}$. Consequently, we conclude that $\mathrm{Per}(PN_n^{(m)}) = \mathrm{lcm}[\mathrm{Per}(PN_n^{(p_i^{e_i})})]$, as required. □

Theorem 2.4 (Matrix interpretation and CRT decomposition)**.** *Let*

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix},$$

be the Pell–Narayana companion matrix satisfying

$$K^n \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} PN_n \\ PN_{n+1} \\ PN_{n+2} \end{pmatrix}. \tag{6}$$

Then for any modulus $m > 1$, the period of the Pell–Narayana sequence modulo m satisfies

$$(\mathrm{Per}(PN_n^{(m)}) = \mathrm{ord}_m(K; s_0),$$

where $\mathrm{ord}_m(K; s_0)$ denotes the smallest positive integer t for which

$$K^t s_0 \equiv s_0 \pmod{m}, \qquad s_0 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

If $\gcd(\det K, m) = 1$, then K is invertible modulo m and the above equality reduces to

$$(\mathrm{Per}(PN_n^{(m)}) = \mathrm{ord}_m(K),$$

the multiplicative order of K in the group $\mathrm{GL}_3(\mathbb{Z}_m)$.

Proof. From relation (6), each state vector of the sequence modulo m can be written as

$$s_n = K_m^n s_0, \qquad K_m = K \bmod m.$$

The sequence is periodic if and only if there exists a smallest $T > 0$ such that $s_T \equiv s_0 \pmod{m}$, that is,

$$K_m^T s_0 \equiv s_0 \pmod{m}.$$

Hence the period equals the order of K_m acting on the initial vector s_0, which proves the first statement.

For the decomposition, let the prime factorization of m be $m = \prod_{i=1}^{t} p_i^{e_i}$. By the Chinese Remainder Theorem, there is a ring isomorphism

$$\mathbb{Z}_m \cong \prod_{i=1}^{t} \mathbb{Z}_{p_i^{e_i}},$$

which induces

$$K_m \mapsto (K_{p_1^{e_1}}, \ldots, K_{p_t^{e_t}}), \qquad s_0 \mapsto (s_0^{(1)}, \ldots, s_0^{(t)}).$$

Then

$$K_m^T s_0 \equiv s_0 \pmod{m} \iff K_{p_i^{e_i}}^T s_0^{(i)} \equiv s_0^{(i)} \pmod{p_i^{e_i}} \text{ for all } i.$$

Let

$$T_i = \mathrm{ord}_{p_i^{e_i}}(K; s_0^{(i)}) = \mathrm{Per}(PN_n^{(p_i^{e_i})}).$$

Each T_i divides any common period T, and conversely,

$$T = \mathrm{lcm}(T_1, \ldots, T_t),$$

satisfies the congruences above simultaneously by the CRT. Therefore,

$$\boxed{\mathrm{Per}(PN_n^{(m)}) = \mathrm{ord}_m(K; s_0) = \mathrm{lcm}[\mathrm{Per}(PN_n^{(p_i^{e_i})})].}$$

This gives an explicit matrix-based justification of the period decomposition formula stated in Theorem 2.3. □

Example 2.5. Consider the Pell–Narayana matrix

$$K = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 2 \end{pmatrix}.$$

For the modulus $m = 6$, we have the prime factorization $6 = 2 \cdot 3$. From Example 2.5, it is known that

$$\left\{PN_n^{(2)}\right\} = 3, \qquad \left\{PN_n^{(3)}\right\} = 13.$$

By Theorem 2.4,

$$\left\{PN_n^{(6)}\right\} = \mathrm{lcm}[\left\{PN_n^{(2)}\right\}, \left\{PN_n^{(3)}\right\}] = \mathrm{lcm}(3, 13) = 39.$$

Indeed, computing powers of K modulo 6 confirms that

$$K^{39} \equiv I_3 \pmod{6},$$

so the Pell–Narayana sequence modulo 6 has period 39. This illustrates the identification $\mathrm{Per}(PN_n^{(m)}) = \mathrm{ord}_m(K)$ and verifies the lcm formula obtained via the Chinese Remainder Theorem.

Corollary 2.6. *For p_i, (p_i is a prime) if it consists of even numbers such as 2, 4 or 6, the simple periodicity is provided, otherwise periodicity is provided. We can illustrate this with examples below.*

Example 2.7. $\left\{PN_n^{(2)}\right\} = \{0,1,1,0,1,1,\dots\} \Longrightarrow \mathrm{Per}(PN^{(2)}) = 3$. For $m = 4$, $e_1 = 2$, $\left\{PN_n^{(4)}\right\} = \{0,1,1,2,1,3,0,1,1,2,1,3,0,1,1,\dots\} \Longrightarrow \mathrm{Per}(PN^{(4)}) = 6$.

For $m = 3$,

$$\begin{aligned} PN_n^{(3)} =& \{0,1,1,2,2,2,0,2,0,0,2,1,2,0,1,1,1,0,1,0,0,1,2,1 \\ & ,0,2,2,1,1,1,0,1,0,0,1,2,1,\dots\} \Longrightarrow \mathrm{Per}(PN^{(3)}) = 13. \end{aligned}$$

For $m = 6$,

$$\begin{aligned} PN_n^{(6)} =& \{0,1,1,2,5,5,0,5,3,0,5,1,2,3,1,4,5,5,2,3,5,0,3,5,4,5,3,4,1,5 \\ & ,2,5,3,2,3,3,2,1,5,0,1,1,2,5,5,0,\dots\} \Longrightarrow \mathrm{Per}(PN^{(6)}) = 39. \end{aligned}$$

$$\mathrm{Per}(PN^{(6)}) = lcm\left(\mathrm{Per}\left(PN^{(2)}\right), \mathrm{Per}\left(PN^{(3)}\right)\right) = lcm\,(3,13) = 39,$$

as required.

For $m = 5$,

$$\begin{aligned} PN_n^{(5)} =& \{0,1,1,2,0,1,4,3,2,3,4,0,3,0,0,3,2,4,1,4,2,0,40,0,4,3,1,1, \\ & 0,1,3,1,3,4,4,1,1,1,3,2,0,3,3,1,0,3,2,4,1,\dots\} \Longrightarrow \mathrm{Per}(PN^{(5)}) = 31. \end{aligned}$$

Let

$$A = [a_{ij}]_{(k+1)\times(k+1)},$$

where a_{ij} are integers, and let

$$A \bmod m = (a_{ij} \bmod m).$$

Let

$$\langle PN\rangle_{p^\alpha} = \{K^i \bmod p^\alpha \mid i \geq 0\},$$

be a cyclic group, and let

$$\left|\langle PN\rangle_{p^\alpha}\right|,$$

denote its order. From (5), we have

$$\mathrm{Per}(PN^{(p^\alpha)}) = \left|\langle PN\rangle_{p^\alpha}\right|.$$

Theorem 2.8. *For $t \in Z^+$, $\mathrm{Per}(PN^{(p)}) = \mathrm{Per}(PN^{(p^t)})$. In this case, we have the relation:*

$$\mathrm{Per}(PN^{(p^\alpha)}) = p^{\alpha-t} \cdot \mathrm{Per}(PN^{(p)}), \quad \alpha \geq t.$$

Specifically, if $\mathrm{Per}(PN^{(p)}) \neq \mathrm{Per}(PN^{(p^2)})$, then we obtain the formula:

$$\mathrm{Per}(PN^{(p^\alpha)}) = p^{\alpha-1} \cdot \mathrm{Per}(PN^{(p)}), \quad \alpha > 1.$$

Proof. For $q \in Z^+$, Since $K^{\mathrm{Per}(N^{(p^{q+1})})} \equiv I \pmod{p^{q+1}}$ and $K^{\mathrm{Per}(N^{(p^{q+1})})} \equiv I \pmod{p^q}$, it follows that $\mathrm{Per}(PN^{(p^q)})$ divides $\mathrm{Per}(PN^{(p^{q+1})})$, where I is the identity matrix. Moreover, we know that:

$$K^{\mathrm{Per}(PN^{(p^q)})} = I + (a_{ij}^{(q)} p^q),$$

so we have:

$$K^{\mathrm{Per}(PN^{(p^q)})p} = (I + a_{ij}^{(q)} p^q)^p = \sum_{i=0}^{p} \binom{p}{i} (a_{ij}^{(q)} p^q)^i \equiv I \pmod{p^{q+1}}.$$

This shows that $\mathrm{Per}(PN^{(p^{q+1})})$ divides $\mathrm{Per}(PN^{(p^q)})p$. Thus, we conclude:

$$\mathrm{Per}(PN^{(p^{q+1})}) = \mathrm{Per}(PN^{(p^q)}) \quad \text{or} \quad \mathrm{Per}(PN^{(p^{q+1})}) = \mathrm{Per}(PN^{(p^q)})p.$$

Additionally, $\mathrm{Per}(PN^{(p^{q+1})}) = \mathrm{Per}(PN^{(p^q)})p$ if and only if there exists an $a_{ij}^{(q)}$ that is not divisible by p. Since $\mathrm{Per}(PN^{(p^t)}) \neq \mathrm{Per}(PN^{(p^{t+1})})$, it implies there exists an $a_{ij}^{(t+1)}$ not divisible by p. Therefore, we conclude that:

$$\mathrm{Per}(PN^{(p^{t+1})}) \neq \mathrm{Per}(PN^{(p^{t+2})}).$$

By induction on t, the proof is completed. □

Let K be the companion matrix of a sequence $PN(n)$. Denote by

$$T_{p^q} := \mathrm{Per}(PN(p^q)),$$

the period modulo p^q. Then we have the following lemma and theorem describing the prime-power lifting behavior.

Lemma 2.9 (Prime-power dichotomy)**.** *Suppose*

$$K^{T_{p^q}} \equiv I + p^q A, \quad A \in M_n(\mathbb{Z}).$$

(a) *If* $A \equiv 0 \pmod{p}$, *then*

$$K^{T_{p^q}} \equiv I \pmod{p^{q+1}} \quad \textit{and} \quad T_{p^{q+1}} = T_{p^q}.$$

That is, the period stabilizes.

(b) *If there exists at least one entry of* A *with* $A_{ij} \not\equiv 0 \pmod{p}$, *then*

$$K^{T_{p^q}} \not\equiv I \pmod{p^{q+1}},$$

but

$$(K^{T_{p^q}})^p \equiv I \pmod{p^{q+1}} \quad \Rightarrow \quad T_{p^{q+1}} = p\, T_{p^q}.$$

That is, the period multiplies exactly by p.

Example 2.10. For $p = 2$ and $q = 1$,

$$K^{\mathrm{Per}(PN^{(4)})} \equiv K^6 \equiv I \pmod 4,$$

so that,

$$K^6 = \begin{pmatrix} 9 & 4 & 20 \\ 20 & 9 & 44 \\ 44 & 20 & 97 \end{pmatrix}_{mod4} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I.$$

Also

$$K^{\mathrm{Per}\left(N^{(2)}\right)2} = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 2 \begin{pmatrix} 4 & 2 & 10 \\ 10 & 4 & 22 \\ 22 & 10 & 48 \end{pmatrix} \right)^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} (mod 2^2).$$

For $t = 1$ and $\alpha = 2$, we obtain

$$\mathrm{Per}\left(PN^{(p^\alpha)}\right) = p^{\alpha - t}\mathrm{Per}\left(PN^{(p)}\right) = \mathrm{Per}\left(PN^{\left(2^2\right)}\right) = 2^1\mathrm{Per}\left(PN^{(2)}\right),$$

where $\mathrm{Per}\left(PN^{(2)}\right) = 3$ *and* $\mathrm{Per}\left(PN^{\left(2^2\right)}\right) = 6$ from Example 2.7. Also, in this example for $\mathrm{Per}\left(PN^{(p)}\right) \neq \mathrm{Per}\left(PN^{\left(p^2\right)}\right)$, $\mathrm{Per}(PN^{(p^\alpha)}) = p^{\alpha-1}\mathrm{Per}(PN^{(p)})$ is provided.

Lemma 2.11 (Equivalence of Scalar and State-Vector Periods)**.** *Let $PN(n)$ be a sequence generated by a linear recurrence with companion matrix K. Denote by*

$$T_{scalar} := \mathrm{Per}(PN(p^q)),$$

the scalar period modulo p^q, i.e., the smallest positive integer such that

$$PN(n + T_{scalar}) \equiv PN(n) \pmod{p^q} \quad \textit{for all } n.$$

Denote by

$$T_{vector} := \textit{the smallest } T \textit{ such that } K^T \equiv I \pmod{p^q},$$

the state-vector (companion matrix) period modulo p^q. Then we have

$$T_{scalar} = T_{vector}.$$

Proof. Let v_n denote the state vector corresponding to the sequence $PN(n)$, so that $v_{n+1} = Kv_n$. Then

$$v_{n+T_{\mathrm{vector}}} = K^{T_{\mathrm{vector}}} v_n \equiv I\, v_n = v_n \pmod{p^q}.$$

In particular, the first component of v_n is $PN(n)$, so

$$PN(n + T_{\mathrm{vector}}) \equiv PN(n) \pmod{p^q}.$$

By minimality of T_{scalar} and T_{vector}, we conclude

$$T_{\mathrm{scalar}} = T_{\mathrm{vector}}.$$

□

3. Pell-Narayana orbits of finite groups

Definition 3.1. For j-generating ($2 \leq j \leq 3$), the Pell-Narayana orbits of finite groups are defined as follows:

i. Let $(x_0, x_1) \in X$ be the generating pair for a 2-generator group G, in that case, the sequence $\{s_i\}$ of elements of G defines the Pell-Narayana orbit $PN(G)_{(x_0,x_1)}$:

$$s_0 = x_0,\ s_1 = x_1,\ s_2 = (s_1)^2, \ldots,\ s_n = (s_{n-3})(s_{n-1})^2 \text{ for } n \geq 3.$$

ii. Let $(x_0, x_1, x_2) \in X$ be the generating triplet for a 3-generator group G, in that case, the sequence $\{s_i\}$ of elements of G defines the Pell-Narayana orbit $PN(G)_{(x_0,x_1,x_2)}$:

$$s_0 = x_0,\ s_1 = x_1, \ldots,\ s_n = (s_{n-3})(s_{n-1})^2 \text{ for } n \geq 3.$$

Theorem 3.2. *A Pell-Narayana orbit of a finite group with j-generating ($2 \leq j \leq 3$) is a periodic sequence.*

Proof. Let's consider group G with the generating pair (x_0, x_1). If the order of the group G is n, we can say that there are n^3 different 3-tuples elements in G.

This means that every 3-tuples element in the Pell-Narayana orbit of G will appear at least 2 times.

This means very clearly that the Pell-Narayana orbit is periodic sequence.

Given the periodicity here, $i, j \in Z^+$ and $i > j$ such that

$$s_{i-1} = s_{j-1},\ s_{i-2} = s_{j-2},\ s_{i-3} = s_{j-3}.$$

From the definition of the orbit in the Pell-Narayana sequence, we know that

$$s_i = (s_{i-3})(s_{i-1})^2 \text{ and } s_j = (s_{j-3})(s_{j-1})^2.$$

Thus, $s_i = s_j$ and it follows that

$$s_{i-j} = s_{j-j} = s_0 = x_0,$$

$$s_{i-j+1} = s_{j-j+1} = s_1 = x_1.$$

So, the Pell-Narayana orbit is simply periodic sequence. The proof of 3-generator groups is similarly done.

We show the periods of the orbits $PN(G)_{(x_0,\ldots,x_n)}$ with $1 \leq n \leq 2$ by $LPN(G)_{(x_0,\ldots,x_n)}$.

Then it is clear from the operations that the Pell-Narayana orbits of the finite groups depend on the selected generating set and the order for the assignments of $x_0, \ldots, x_n$; $1 \leq n \leq 2$. □

Definition 3.3. For a finite group G, group G is a Pell-Narayana sequenceable if G has a Pell-Narayana orbit such that every element of the G group appears in the sequence.

Based on the definition, it is said that all the examples given below are a Pell-Narayana sequenceable as a requirement of the expression given in Definition 3.3.

We examine Pell-Narayana orbits of finite groups, $Q_8, Q_8 \times \mathbb{Z}_{2m}$ and $Q_8 \times_\varphi \mathbb{Z}_{2m}$, $m \geq 3$.

Theorem 3.4. $LPN\left(Q_8\right)_{(x,y)} = 6$.

Proof. $LPN\left(Q_8\right)_{(x,y)}$ is

$$x, y, y^2, x, y^3, e, x, y, y^2, x, y^3, \ldots,$$

which has period $LPN\left(Q_8\right)_{(x,y)} = 6$. □

Theorem 3.5. *For each generating triplet, the period of the Pell-Narayana orbit of the group* $Q_8 \times \mathbb{Z}_{2m}$, $(m \geq 3)$ *is* $lcm\left[6,\ \mathrm{Per}\left(PN^{(2m)}\right)\right]$.

Proof. Consider the Pell-Narayana orbit $PN\left(Q_8 \times \mathbb{Z}_{2m}\right)_{(x,y,z)}$:

$$x, y, z^{a_1}, xz^{a_2}, y^3z^{a_3}, y^2z^{a_4}, xz^{a_5}, yz^{a_6}, z^{a_7}, xz^{a_8}, y^3z^{a_9}, y^2z^{a_{10}}, xz^{a_{11}}, yz^{a_{12}}, z^{a_{13}}, xz^{a_{14}}, \ldots.$$

Using the above information, we have

$$\begin{aligned}
s_0 =& x, \quad s_1 = y, \quad s_2 = z, \quad s_3 = xz^2, \ s_4 = y^3z^3, \quad s_5 = y^2z^4, \\
s_6 =& xz^5, \quad s_7 = yz^6, \quad s_8 = z^7, \quad s_9 = xz^8, \quad s_{10} = y^3z^9, \\
s_{11} =& y^2z^{10}, \quad s_{12} = xz^{11}, \quad s_{13} = yz^{12}, \quad s_{14} = z^{13}, \quad s_{6i-3} = xz^{6i-4}, \\
s_{6i-2} =& y^3z^{6i-3}, \quad s_{6i-1} = y^2z^{6i-2}, \quad s_{6i} = xz^{6i-1}, \quad s_{6i+1} = yz^{6i}, \quad s_{6i+2} = z^{6i+1}, \ldots.
\end{aligned}$$

The sequence has the form layers of length 6. So, we need an i such that $s_{6i} = x, s_{6i+1} = y$, $s_{6i+2} = z$. It is seen that the Pell-Narayana orbit $PN\left(Q_8 \times \mathbb{Z}_{2m}\right)_{(x,y,z)}$ has period $lcm\left[6,\ \mathrm{Per}\left(PN^{(2m)}\right)\right]$. □

Theorem 3.6. *For each generating triplet, the period of the Pell-Narayana orbit of the group* $Q_8 \times_\varphi \mathbb{Z}_{2m}$, $(m \geq 3)$ *is* $lcm\left[6,\ \mathrm{Per}\left(PN^{(2m)}\right)\right]$.

Proof. Consider the Pell-Narayana orbit $PN\left(Q_8 \times_\varphi \mathbb{Z}_{2m}\right)_{(x,y,z)}$:

$$x, y, z^{a_1}, xz^{a_2}, y^3z^{a_3}, y^2z^{a_4}, xz^{a_5}, yz^{a_6}, z^{a_7}, xz^{a_8}, y^3z^{a_9}, y^2z^{a_{10}}, xz^{a_{11}}, yz^{a_{12}}, z^{a_{13}}, xz^{a_{14}}, \ldots.$$

From here, we get the following sequence.

$$\begin{aligned}
s_0 =& x, \quad s_1 = y, \quad s_2 = z, \quad s_3 = xz^2, \quad s_4 = y^3z^3, \\
s_5 =& y^2z^4, \quad s_6 = xz^5, \quad s_7 = yz^6, \quad s_8 = z^7, \quad s_9 = xz^8, \quad s_{10} = y^3z^9, \\
s_{11} =& y^2z^{10}, \quad s_{12} = xz^{11}, \quad s_{13} = yz^{12}, \quad s_{14} = z^{13}, \quad s_{6i-3} = xz^{6i-4}, \\
s_{6i-2} =& y^3z^{6i-3}, \quad s_{6i-1} = y^2z^{6i-2}, \quad s_{6i} = xz^{6i-1}, \quad s_{6i+1} = yz^{6i}, \quad s_{6i+2} = z^{6i+1}, \ldots.
\end{aligned}$$

The sequence has the form layers of length 6. So, we need an i such that $s_{6i} = x, s_{6i+1} = y$, $s_{6i+2} = z$. So, we can see that the Pell-Narayana orbit $PN\left(Q_8 \times_\varphi \mathbb{Z}_{2m}\right)_{(x,y,z)}$ has period $lcm\left[6, \mathrm{Per}\left(PN^{(2m)}\right)\right]$.

The other generating triplets are proved in a similar way. □

Remark 3.7. If $\mathrm{Per}\left(PN^{(2m)}\right) \leq 2m$ and $6|\mathrm{Per}\left(PN^{(2m)}\right)$, the groups $Q_8 \times \mathbb{Z}_{2m}$ and $Q_8 \times_\varphi \mathbb{Z}_{2m}$ such that $m \geq 3$ are not Pell-Narayana sequence.

4. The Pell-Narayana Length of Generating Pairs in Groups

Theorem 4.1. *For $\mathbb{Z}_n = \langle x \rangle$ and $\mathbb{Z}_m = \langle y \rangle$, the Pell-Narayana length of $\mathbb{Z}_n \times \mathbb{Z}_m$ is*

$$\mathrm{lcm}[\mathrm{Per}(PN^{(n)}),\ \mathrm{Per}(PN^{(m)})].$$

Proof.

$$\mathbb{Z}_n \times \mathbb{Z}_m = \langle x, y :\ x^n = y^m = e, xy = yx \rangle .$$

The Pell-Narayana orbit is:
$s_0 = x,\ s_1 = y,\ s_2 = y^2,\ s_3 = xy^4,\ s_4 = x^2y^9,\ s_5 = x^4y^{20},\ s_6 = x^9y^{44}, \ldots$.
If $x_i = x,\ x_{i+1} = y,\ x_{i+2} = y^2$ exist, the proof is completed.
Let's analyze this statement more thoroughly,

$$x^{PN_{i-2}}y^{PN_i} = x = s_0, x^{PN_{i-1}}y^{PN_{i+1}} = y = s_1, x^{PN_i}y^{PN_{i+2}} = y^2 = s_2,$$

where, i is least non-trivial integer satisfying the above conditions, so

$$i = lcm[\mathrm{Per}(PN^{(n)}), \mathrm{Per}(PN^{(m)})].$$

□

4.1. Applications

In this section, let $\ell, m, n > 1$ be integers.

Definition 4.2. The polyhedral group $(\ell, m, n\ )$ is defined by

$$\left\langle x, y, z :\ x^\ell = y^m = z^n = xyz = e \right\rangle \text{ or } \left\langle x, y :\ x^\ell = y^m = (xy)^n = e \right\rangle .$$

The polyhedral group $(\ell, m, n\ )$ is finite if and only if $k = \ell mn\left(\frac{1}{\ell} + \frac{1}{m} + \frac{1}{n} - 1\right)$ is a positive. Its order is $\frac{2\ell mn}{k}$.

Definition 4.3. The binary polyhedral group is defined by the relations

$$\langle x, y, z : x^\ell = y^m = z^n = xyz \rangle,$$

or equivalently,

$$\langle x, y : x^\ell = y^m = (xy)^n \rangle.$$

This group is finite if and only if the number

$$k = \ell mn\left(\frac{1}{\ell}+\frac{1}{m}+\frac{1}{n}-1\right) = mn+n\ell+\ell m-\ell mn,$$

is positive. The order of the binary polyhedral group is given by

$$\frac{4\ell mn}{k}.$$

Theorem 4.4. *The Pell-Narayana length of this group* $(2,2,2)$ *is 3.*

Proof. From Theorem 4.1, $LPN_{x,y,y}((2,2,2)) = 3$. from $(2,2,2) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, we have

$$s_0 = x,\ s_1 = y,\ s_2 = e, s_3 = x, s_4 = y, s_5 = e, \ldots.$$

and $LPN_{x,y,y}((2,2,2)) = 3$. □

Theorem 4.5. *The Pell-Narayana length of the polyhedral group* $(2,n,2)$ *is*

$$LPN_{x,y,y}((2,n,2)) = \begin{cases} \frac{3n}{2}, & n \equiv 0(mod\ 2) \\ 3n, & n \equiv 1(mod\ 2) \end{cases}, n > 2.$$

Proof. If $\left\langle x, y :\ x^2 = y^n = (xy)^2 = e \right\rangle$, $|x| = 2$, then $|y| = n$ and $|xy| = 2$.

The Pell-Narayana orbit is:

$$x, y, y^2, xy^4, y, y^4, xy^{12}, y, y^6, xy^{24}, y, y^8, xy^{40}, \ldots.$$

$$s_0 = x,\quad s_1 = y,\quad s_2 = y^2,\quad s_3 = xy^4,\quad s_4 = y,\quad s_5 = y^4,\quad s_6 = xy^{12},\quad s_7 = y,$$
$$s_8 = y^6,\quad s_9 = xy^{24},\quad s_{10} = y,\quad s_{11} = y^8,\quad \ldots,\quad s_{3i} = xy^{2\left(i^2+i\right)},\quad s_{3i+1} = y,$$
$$s_{3i+2} = y^{2(i+1)},$$

so, we get $2i = un$ for $u,\ i \in \mathbb{N}$.

If $n \equiv 0(mod 2)$ then $n = 2k$ and $2i = u2k,\ k = \frac{i}{u},\ n = \frac{2i}{u},\ nu = 2i,\ i = \frac{nu}{2}$ for $u = 1,\ i = \frac{n}{2}$.

So, we get

$$LPN_{x,y,y}((2,n,2)) = lcm\left(\left(3, \frac{n}{2}, 3\right)\right) = 3\frac{n}{2}.$$

If $n \equiv 1(mod 2)$ then $n = 2k+1$ and $2i = u(2k+1),\ k = \frac{i-1}{2},\ n = \frac{2(i-1)}{2}+1,\ n = i$.

So, we get

$$LPN_{x,y,y}((2,n,2)) = lcm\,(3,n,3) = 3n.$$

□

Theorem 4.6. *For* $(n,2,2)$ and $(2,2,n)$. *Then*

$$LPN_{x,y,y}(G)=\begin{cases}\frac{3n}{2}, & n\equiv 0(mod\ 2)\\ 3n, & n\equiv 1(mod\ 2)\end{cases}, n>2.$$

Proof. For this group $(n,2,2)$. the following equality holds:
$\left\langle x,y:\ x^n=y^2=(xy)^2=e\right\rangle,\ |x|=n,\ \ |y|=2$ and $|xy|=2$. The Pell-Narayana orbit is

$$\begin{aligned}s_0=&x,\ s_1=y,\ s_2=e,s_3=x,\ s_4=yx^2,\ s_5=e,\ s_6=x,\ s_7=yx^4,\\ s_8=&e,\ \ldots,s_{3i}=x,\ s_{3i+1}=yx^{2i},\ s_{3i+2}=e.\end{aligned}$$

So, we want to find $i\in\mathbb{N}$ such that $2i=un$ for $u\in\mathbb{N}$.
If $n\equiv 0(mod2)$ then $n=2k$ and $2i=u2k,\ k=\frac{i}{u}$
$n=\frac{2i}{u},\ nu=2i,\ i=\frac{nu}{2}$ for $u=1,\ i=\frac{n}{2}$.
So, we get

$$LPN_{x,y,y}((n,2,2))=lcm\left(\left(\frac{n}{2},3,3\right)\right)=3\frac{n}{2}.$$

If $n\equiv 1(mod2)$ then $n=2k+1$ and $2i=u(2k+1),\ k=\frac{i-1}{2}$

$$n=\frac{2(i-1)}{2}+1,\ n=i.$$

So, we get

$$LPN_{x,y,y}((n,2,2))=lcm\,(n,3,3)=3n.$$

□

Example 4.7. For $(4,2,2)$ the Pell-Narayana length is 6. So that

$$x,\ y,\ e,\ x,\ yx^2,\ e,\ \ldots.$$

Theorem 4.8. *The Pell-Narayana length of* $\langle 2,2,2\rangle$ *is 6. This value represents the period of the Pell-Narayana sequence when considered within the structure of this specific binary polyhedral group.*

Proof. Since the group has the presentation $\left\langle x,y:\ x^2=y^2=(xy)^2\right\rangle$ and $|x|=4$, the Pell-Narayana orbit is $,y,y^2,x,\ y^3,e,x,y,y^2,x,\ y^3,e,\ldots$. So, we get $LN_{x,y,y}((2,2,2))=6$. □

Theorem 4.9. *For* $n\geq 2$, *the Pell–Narayana length of* $\langle 2,n,2\rangle$ *is denoted by*

$$LPN_{x,y,y}(\langle 2,n,2\rangle)=\begin{cases}3n, & n\equiv 0 \pmod 2,\\ 6n, & n\equiv 1 \pmod 2.\end{cases}$$

This length corresponds to the period of the Pell–Narayana sequence within the structure of $\langle 2,n,2\rangle$ *for different values of* n.

Proof. Since $\left\langle x,y:\ x^2=y^n=(xy)^2\right\rangle,\ |x|=4,\ |y|=2n,\ |xy|=4$, the Pell-Narayana orbit is

$$\begin{aligned} s_0 =& x,\ s_1=y,\ s_2=y^2,\ s_3=xy^4,\ s_4=yx^2,\ s_5=y^4,\ s_6=xy^{12},\ s_7=y,\\ s_8 =& y^6,\ s_9=xy^{24},\ s_{10}=yx^2,\ s_{11}=y^8,\ \ldots. \end{aligned}$$

For $n\equiv 0(mod\ 2)$,

$$s_{3i-1}=y^{2i},\ s_{3i}=xy^{2\left(i^2+i\right)},\ s_{3i+1}=y,$$

and for $n\equiv 1(mod\ 2)$,

$$s_{3i-1}=y^{2i},\ s_{3i}=xy^{2\left(i^2+i\right)},\ s_{3i+1}=yx^2.$$

Now we want to find $i\in\mathbb{N}$ such that $2i=un\ for\ u\in\mathbb{N}$.
For first case:

$$n=2k,\ i=uk.$$

For $u=2,\ i=n$, we have

$$LPN_{x,y,y}(\ \langle 2,n,2\rangle)=lcm\,(3,n,3)=3n.$$

For second case:

$$n=2k+1,\ 2i=u(2k+1).$$

For $u=4,\ i=2n$, we get

$$LPN_{x,y,y}(\ \langle 2,n,2\rangle)=lcm\,(3,2n,3)=6n.$$

□

Theorem 4.10. *If H is $\langle n,2,2\rangle$ or $\langle 2,2,n\rangle$ then*

$$LPN_{x,y,y}(H)=\begin{cases} 3n, & n\equiv 0(mod\ 2)\\ 6n, & n\equiv 1(mod\ 2)\end{cases},n>2.$$

Proof. If H is the group $\langle n,2,2\rangle$. From the definition polyhedral group we get the Pell-Narayana orbit is:

$$\begin{aligned} s_0=x,\ s_1=y,\ s_2=y^2,\ s_3=x,\ s_4=yx^2,\ s_5=e,\ s_6=x,\ s_7=yx^4,\ s_8=y^2,\\ s_9=x,\ s_{10}=yx^6,\ s_{11}=e,\ \ldots \end{aligned}$$

For $n\equiv 0(mod\ 2)$, we get

$$s_{3i}=x,\ s_{3i+1}=xy^{2i},\ s_{3i+2}=e.$$

So, we need $i\in\mathbb{N}$ such that $2i=un$ for $u\in\mathbb{N}$.
For first case:

$$n=2k,\ i=uk\ .$$

For $u = 2,\ i = n$, we get

$$LPN_{x,y,y}(\ \langle n, 2, 2\rangle) = lcm\,(n, 3, 3) = 3n.$$

For second case:

$$n = 2k + 1,\ 2i = u(2k + 1).$$

For $u = 4,\ i = 2n$, we get

$$LPN_{x,y,y}(\ \langle n, 2, 2\rangle) = lcm\,(2n, 3, 3) = 6n.$$

Now, let H be $\langle 2, 2, n\rangle$. Let's consider the group defined by

$$\left\langle x, y :\ x^2 = y^2 = (xy)^n \right\rangle,\ |x| = 4,\ |y| = 4,\ |xy| = 2n.$$

The proofs are like that of firstly case. □

Example 4.11. In the case $n = 4$, the Pell-Narayana length of $\langle 4, 2, 2\rangle$ is 12. The group is defined by

$$\left\langle x, y :\ x^4 = y^2 = (xy)^2 \right\rangle,\ |x| = 8,\ |y| = 4,\ |xy| = 4.$$

So, the Pell-Narayana sequence in the $\langle 4, 2, 2\rangle$ is

$$x, y, y^2, x, yx^2, e, x, yx^4, y^2, x, yx^6, e, x, y, y^2, x, yx^2, \ldots.$$

5. Conclusions

We have summarized the key characteristics and main properties of the Pell-Narayana sequence in order to facilitate the exploration of Narayana sequences modulo m. In this context, we introduced the concept of Pell-Narayana orbits within 2-generator and 3-generator finite groups. Furthermore, we derived the period of the Pell-Narayana orbit for specific groups such as Q_8, $Q_8 \times \mathbb{Z}_{2m}$, and $Q_8 \times_\varphi \mathbb{Z}_{2m}$ for $m \geq 3$. Additionally, we examined the Pell-Narayana lengths of the polyhedral and binary polyhedral groups, providing concrete examples to illustrate these findings. This work not only deepens our understanding of the Pell-Narayana sequence in various algebraic structures but also contributes to the broader study of periodic behaviors in finite groups and their associated sequences.

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