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A variation on 3-GDDs with blocks of size 4 and 3 groups

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ABSTRACT

We define 3-PBIBDs to simplify the notation used in one of Hanani's celebrated papers, where he developed important tools for the constructions of 3-designs. A special case of the 3-PBIBD, a 3-GDD$(n, 2, 4; \lambda_1, \lambda_2)$, was recently studied in [5] and [6]. In this note, we develop necessary conditions for the existence of another special case, denoted $3'$-GDD$(n, 3, 4;\ \mu_1, \mu_2)$, and provide several constructions for infinite families of these designs. We show that the necessary conditions are sufficient for the existence of $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$ when $\mu_2 = 0$ and μ_1 is arbitrary. In particular, when μ_1 is even and $\mu_2 = 0$, such designs exist for all n; and when μ_1 is odd and $\mu_2 = 0$, they exist for even n. We also provide instances of nonexistence.

Keywords: partially balanced incomplete block designs (PBIBDs), 3-design constructions, group divisible designs (GDDs), $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$, necessary and sufficient conditions, infinite families, existence and nonexistence

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1. Introduction

This study is situated within the broader context of combinatorial design theory, including classical results on t-designs and PBIBDs (see [1], [7], [2]).

A t-(v, k, λ) design, or t-design, consists of a set X of v points and a collection B of k-element subsets of X (called blocks), such that every t-element subset of X appears in exactly λ blocks. The number λ is referred to as the index of the design.

A generalization of t-designs is the concept of a 3-partially balanced incomplete block design (3-PBIBD), denoted 3-PBIBD$(n, m, k; \lambda_1, \lambda_2, \lambda_3)$. It consists of a triple (X, B, G),

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where:

- X is a set of mn points,
- G is a partition of X into m groups of size n each, and
- B is a collection of k-element subsets (blocks) of X, satisfying:

(i) every triple of points all from the same group occurs in exactly λ_1 blocks,

(ii) every triple with two points from one group and one from another occurs in exactly λ_2 blocks, and

(iii) every triple with points from three different groups occurs in exactly λ_3 blocks.

Although the group structure is used to define the association scheme, we refer to the design as 'partially balanced' to emphasize that three distinct balance parameters are involved, as opposed to only two in group divisible designs (GDDs).

When all three λ values are equal, say $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, the 3-PBIBD becomes an ordinary 3-(mn, k, λ) design. If $\lambda_2 = \lambda_3$, the design is called a 3-GDD$(n, m, k; \lambda_1, \lambda_2)$. In particular, a 3-GDD$(n, 3, 4; \mu_1, \mu_2)$ is the same as a 3-PBIBD$(n, 3, 4; \mu_1, \mu_2, \mu_2)$. These have been studied recently for $m = 3$ and $k = 4$ in [5, 6].

In this paper, we focus on a variant where $\lambda_1 = \lambda_2 \neq \lambda_3$, which we denote as $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$. A formal definition follows:

Definition 1.1. A $3'$-GDD$(n, m, k; \mu_1, \mu_2)$ is a design on a set X of mn points, partitioned into m groups of size n, with a collection of k-element blocks $\mathscr{B}$ such that:

(i) every triple contained in at most two groups occurs exactly μ_1 times, and

(ii) every triple spread across three distinct groups occurs exactly μ_2 times.

We also explore constructions and nonexistence results for various families of 3-PBIBDs. Since both 3-GDD and $3'$-GDD involve two balance parameters, we use the prime ($'$) notation to distinguish our designs from standard 3-GDDs. Recall: if $\lambda_2 = \lambda_3$, the design is a 3-GDD; if $\lambda_1 = \lambda_2$, we refer to it as a $3'$-GDD.

A special case arises when $\mu_1 = \mu_2$ in a $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$; this is simply a 3-$(3n, 4, \mu_1)$ design. Hence, the existence of such a $3'$-GDD is equivalent to that of the corresponding 3-design.

These designs are important because 3-PBIBDs appear in disguised form in Hanani's classical 1968 paper [4], which proved that the known necessary conditions for the existence of a 3-$(v, 4, \lambda)$ design are also sufficient. For example, in Lemma 3 of his paper, Hanani uses structures that are, in modern terms, $3'$-GDD$(3, 4, 4; 0, 1)$ and $3'$-GDD$(3, 6, 4; 0, 1)$. Therefore, it is natural to explore the existence of $3'$-GDDs more generally, including when $\lambda_1 \neq 0$, just as one does in the theory of 2-GDDs [3].

2. Preliminary constructions and known results

Hanani's 1968 paper [4] introduced several constructions of 3-designs using what we now describe as 3-PBIBDs. His notation, however, is different. Below is a translation of some of Hanani's terminology into the framework used in this paper:

$P[k, \lambda, v]$ denotes a 3-(v, k, λ) design,

$P(k,\lambda)$ is the set of all v such that a 3-(v,k,λ) exists,
$P''_n[k,\lambda,v]$ denotes a 3-PBIBD$(n,m,k;0,0,\lambda)$,
$P''_n(k,\lambda)$ is the set of all v for which a 3-PBIBD$(n,m,k;0,0,\lambda)$ exists,
$Q''_n[k,\lambda,nm]$ denotes a 3-PBIBD$(n,m,k;0,\lambda,\lambda)$.
Hanani proves (in our notation) the following result:

Proposition 2.1. *If λ' divides λ and n' divides n, then the existence of a 3-PBIBD$(n',m,k;0,0,\lambda')$ implies the existence of a 3-PBIBD$(n,m,k;0,0,\lambda)$.*

This means that for any positive integers t and s, if a 3-PBIBD$(n,m,k;0,0,\lambda')$ exists, then so does a 3-PBIBD$(nt,m,k;0,0,s\lambda')$.

Using this principle, and results from [4], the following 3-PBIBDs exist for any positive integer t and suitable λ:
- $8 \in P''_2(4,1) \Rightarrow$ 3-PBIBD$(2t,4,4;0,0,\lambda)$ exists,
- $12 \in P''_3(4,1) \Rightarrow$ 3-PBIBD$(3t,4,4;0,0,\lambda)$ exists,
- $18 \in P''_3(4,1) \Rightarrow$ 3-PBIBD$(3t,6,4;0,0,\lambda)$ exists,
- $16 \in P''_4(4,6) \Rightarrow$ 3-PBIBD$(4t,4,4;0,0,6\lambda)$ exists,
- $24 \in P''_4(4,6) \Rightarrow$ 3-PBIBD$(4t,6,4;0,0,6\lambda)$ exists,
- $24 \in P''_6(4,2) \Rightarrow$ 3-PBIBD$(6t,4,4;0,0,2\lambda)$ exists,
- $24 \in P''_6(4,1) \Rightarrow$ 3-PBIBD$(6t,4,4;0,0,\lambda)$ exists,
- $36 \in P''_6(4,2) \Rightarrow$ 3-PBIBD$(6t,6,4;0,0,\lambda)$ exists.

Note that a 3-PBIBD$(n,m,k;0,0,\lambda)$ cannot exist if $m < k$, since at least k distinct groups are needed to form a block of size k with no repeated group membership.

Additional known constructions from [4] include:
- $30 \in Q''_6(4,2) \Rightarrow$ 3-PBIBD$(6,5,4;0,2,2)$ exists,
- $18 \in Q''_6(4,1) \Rightarrow$ 3-PBIBD$(6,3,4;0,1,1)$ exists,
- $9,15 \in Q''_3(4,4) \Rightarrow$ 3-PBIBD$(3,3,4;0,4,4)$ and $(3,5,4;0,4,4)$ exist,
- $12,20 \in Q''_4(4,6) \Rightarrow$ 3-PBIBD$(4,3,4;0,6,6)$ and $(4,5,4;0,6,6)$ exist.

These constructions play an important role in this paper, especially when applying the following key results.

These designs play an important role in this note. For example, these designs can be used to apply our theorem given in the next section.

3. A construction theorem

The following result from [5] provides a useful building block:

Theorem 3.1. [5] *A 3-GDD$(n,2,4;0,1)$ exists for all even n, and a 3-GDD$(n,2,4;0,2)$ exists for all positive integers n.*

We now present a general construction for $3'$-GDDs using a 3-design and a suitable 3-PBIBD.

Theorem 3.2. *Suppose a 3-PBIBD$(n,3,4;0,\lambda,\lambda)$ and a 3-$(n,4,\Lambda)$ exist for some $\Lambda \geq \lambda$.*

Then a $3'$-GDD$(n, 3, 4; \Lambda, \lambda)$ exists if either $\Lambda - \lambda$ is even, or n is even.

Proof. Let V be the point set of the 3-PBIBD$(n, 3, 4; 0, \lambda, \lambda)$ with block set $\mathcal{B}$, and let G_1, G_2, G_3 be the three groups of size n.

Let $\mathcal{B}_i$ be the blocks of a 3-$(n, 4, \Lambda)$ design on group G_i, for $i = 1, 2, 3$. Let $\mathcal{B}_{ij}$ be the blocks of a 3-GDD$(n, 2, 4; 0, \Lambda - \lambda)$ on the pair of groups G_i and G_j (which exists by Theorem 3.1). Let:

$$\mathcal{B}' = \mathcal{B} \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_{12} \cup \mathcal{B}_{13} \cup \mathcal{B}_{23}.$$

We claim that $(V, \mathcal{B}')$ is a $3'$-GDD$(n, 3, 4; \Lambda, \lambda)$.

To verify this, consider any triple $T = \{a, b, c\}$ of points:

• If the three points lie in different groups, then T appears only in $\mathcal{B}$ and hence occurs exactly λ times.

• If two points are from one group and one from another, say G_i and G_j, then T occurs λ times in $\mathcal{B}$ and $(\Lambda - \lambda)$ times in $\mathcal{B}_{ij}$, for a total of Λ occurrences.

• If all three points are from the same group G_i, then T appears only in $\mathcal{B}_i$ and occurs exactly Λ times.

Hence the required balance conditions are satisfied. □

Example 3.3. Since $18 \in Q''_6(4, 1)$ and a 3-$(6, 4, 3)$ design exists, by Theorem 3.2 a $3'$-GDD$(6, 3, 4; 3, 1)$ exists.

In the next section, we derive necessary conditions for the existence of a $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$. Assuming such a design exists, we determine expressions for:

• the replication number r (i.e., the number of blocks containing a given point),

• the total number of blocks b,

• the number of blocks λ_1 containing a given pair of points from the same group (first associates), and

• the number of blocks λ_2 containing a given pair of points from different groups (second associates).

We then use these expressions to derive arithmetic conditions on the parameters μ_1 and μ_2.

4. Necessary conditions

Theorem 4.1. *If a $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$ exists, then*

(i) $r = \frac{(n-1)(7n-2)\mu_1 + 2n^2\mu_2}{6}$,
(ii) $b = \frac{n}{8}((n-1)(7n-2)\mu_1 + 2n^2\mu_2)$,
(iii) $\lambda_1 = \frac{3n-2}{2}\mu_1$,
(iv) $\lambda_2 = \frac{n\mu_2 + 2(n-1)\mu_1}{2}$.

Proof. (i) Given a point x, it appears in $\binom{n-1}{2}$ triples of Type $(3, 0)$ (triples where all three points are from the same group).

To count Type $(2,1)$ triples — those with two points from the same group and the third from another group — involving a given point x, we proceed in two steps. First, there are $2n(n-1)$ triples of Type $(2,1)$ where x and a point from the same group as x appear with a point from another group. Second, there are $n(n-1)$ Type $(2,1)$ triples in which x appears together with two points from a different group. Altogether, x appears $3n(n-1)$ times in triples of Type $(2,1)$.

So, the point x appears in $\mu_1\left(\binom{n-1}{2}+3n(n-1)\right)$ triples of Type $(3,0)$ or $(2,1)$ in the $3'$-GDD$(n,3,4;\mu_1,\mu_2)$.

Next, there are n^2 triples of Type $(1,1,1)$ containing x, which are repeated μ_2 times in the design.

On the other hand, since each block contains three triples involving x, the total number of blocks containing x (i.e., the replication number r) is obtained by dividing the total number of such triples by 3:

$$r=\frac{(n-1)\left(\frac{n-2}{2}+3n\right)\mu_1+n^2\mu_2}{3}=\frac{(n-1)(7n-2)\mu_1+2n^2\mu_2}{6}.$$

(ii) Since there are $3n$ points, each appearing in r blocks, and each block contains 4 points,

$$b=\frac{3nr}{4}=\frac{n}{8}((n-1)(7n-2)\mu_1+2n^2\mu_2).$$

(iii) Let (x_1,x_2) be any first associate pair and suppose it occurs in λ_1 blocks. The triples containing (x_1,x_2) are of Type $(3,0)$ or of Type $(2,1)$ only. There are $(n-2)$ triples of Type $(3,0)$ and $2n$ triples of Type $(2,1)$ containing (x_1,x_2). Since each of these triples occurs μ_1 times, and each block that contains the first associate pair (x_1,x_2) includes two such triples, we have:

$$\lambda_1=\frac{(3n-2)}{2}\mu_1.$$

(iv) Second associate pairs occur in triples of Type $(2,1)$ and Type $(1,1,1)$. Let (x,y) be any second associate pair. There are $2(n-1)$ triples of Type $(2,1)$ and n triples of Type $(1,1,1)$ containing (x,y). Since a Type $(2,1)$ triple appears μ_1 times, and a Type $(1,1,1)$ triple appears μ_2 times, and each block containing the pair (x,y) includes two such triples, we have:

$$\lambda_2=\frac{n\mu_2+2(n-1)\mu_1}{2}.$$

□

Lemma 4.2. *For n even, b is an integer regardless of the values of μ_1 and μ_2. For n odd, λ_1 is an integer only when μ_1 is even, and λ_2 is an integer only when μ_2 is even.*

Proof. When n is even, n^2 is divisible by 4, and the term $2n^2\mu_2$ is divisible by 8. Moreover, since $n(7n-2)\equiv 0 \pmod 8$ for even n, it follows that $b=\frac{n}{8}((n-1)(7n-2)\mu_1+2n^2\mu_2)$ is always an integer for any values of μ_1 and μ_2.

When n is odd, consider the expressions for $\lambda_1 = \frac{(3n-2)}{2}\mu_1$ and $\lambda_2 = \frac{n\mu_2+2(n-1)\mu_1}{2}$. For λ_1 to be an integer, $(3n-2)\mu_1$ must be even, which requires μ_1 to be even. Similarly, for λ_2 to be an integer, $n\mu_2$ must be even, which requires μ_2 to be even. Thus, both μ_1 and μ_2 must be even when n is odd. □

Since the values of b, r, λ_1, and λ_2 must be integers, we derive necessary conditions for the existence of a $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$ by analyzing the parameters accordingly. While one could construct a complicated table of cases, several useful observations simplify the process. These ultimately lead to the final conditions stated in Theorem 4.3.

For even n, verifying the conditions is relatively straightforward. We consider the cases $n \equiv 0, 2, 4 \pmod 6$. For odd n, we begin by noting that the requirement for λ_1 and λ_2 to be integers implies that both μ_1 and μ_2 must be even. Using this fact, we derive further conditions from the requirement that b and r must also be integers. For b, we examine the cases $n \equiv 1, 3, 5, 7 \pmod 8$, and for r, we consider $n \equiv 1, 3, 5 \pmod 6$. These yield allowable values for μ_1 and μ_2, which we then combine to obtain the final necessary conditions based on $n \pmod{24}$.

Theorem 4.3. *The necessary conditions for the existence of a $3'$-GDD$(n, 3, 4;\ \mu_1, \mu_2)$ are:*

- *for $n \equiv 0 \pmod 6$, $\mu_1 \equiv 0 \pmod 3$ and μ_2 is free,*
- *for $n \equiv 2, 4 \pmod 6$, μ_1 is free and $\mu_2 \equiv 0 \pmod 3$,*
- *for $n \equiv 1, 5, 13, 17 \pmod{24}$, $\mu_1 \equiv 0 \pmod 2$ and $\mu_2 \equiv 0 \pmod{12}$,*
- *for $n \equiv 3, 15 \pmod{24}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_1 \equiv \mu_2 \pmod 4$,*
- *for $n \equiv 7, 11, 19, 23 \pmod{24}$, $\mu_1 \equiv \mu_2 \pmod 4$ and $\mu_2 \equiv 0 \pmod 6$,*
- *for $n \equiv 9, 21 \pmod{24}$, $\mu_1 \equiv 0 \pmod 6$ and $\mu_2 \equiv 0 \pmod 4$.*

There are other necessary conditions for the existence of a $3'$-GDD. For example, we have the following two results:

Theorem 4.4. *For $n \equiv 3 \pmod{24}$, if $\mu_1 = 6t$ and $\mu_2 = 2s$, then t and s must have the same parity in order for b to be an integer.*

Theorem 4.5. *If a $3'$-GDD$(n, 3, 4, \mu_1, \mu_2)$ exists, then*

(i) $\mu_1 \geq \dfrac{n\mu_2}{3(n-1)}$,

(ii) $b \geq \dfrac{3n^2(n-1)\mu_1 + n^3\mu_2}{4}$,

(iii) $n^3\mu_2 \equiv 0 \pmod 2$.

Proof. Observe that the triples of the Type $(1, 1, 1)$ occur in the blocks of Type $(2, 1, 1)$ only. These blocks have two $(2, 1)$ triples as well as two $(1, 1, 1)$ triples. From this, we must ensure that there are enough $(2, 1)$ triples in the design. Hence,

$$3n^2(n-1)\mu_1 \geq n^3\mu_2,$$

which gives

$$\mu_1 \geq \frac{n\mu_2}{3(n-1)}.$$

The inequality for b is derived from considering the number of blocks required to cover all triples without any of the form $(3,0)$, and

The final condition follows because the total number of $(1,1,1)$ triples is n^3, each repeated μ_2 times, and these must be distributed evenly across blocks — each block containing two such triples — implying

$$n^3\mu_2 \equiv 0 \pmod 2.$$

□

5. Further existence results for 3′-GDDs

5.1. $n = 3$

Example 5.1. A 3-PBIBD$(3,3,4;0,1,2)$ is given below where the groups are $\{1,2,3\}$, $\{a,b,c\}$, and $\{x,y,z\}$:

1	1	1	2	2	2	3	3	3
a	b	c	a	b	c	a	b	c
x	x	y	x	y	x	y	x	x
y	z	z	z	z	y	z	y	z
a	a	a	b	b	b	c	c	c
x	y	z	x	y	z	x	y	z
2	1	1	1	1	2	1	2	1
3	2	3	2	3	3	3	3	2
x	x	x	y	y	y	z	z	z
1	2	3	1	2	3	1	2	3
a	b	a	b	a	a	a	a	b
c	c	b	c	b	c	b	c	c

Theorem 5.2. *For positive integer t and nonnegative integer s of the same parity, a 3′-GDD$(3,3,4;6t,2s)$ exists. Hence, necessary conditions are sufficient for existence when $n = 3$.*

Proof. Let G_i, $i = 1,2,3$, be the groups.

The blocks of a 3′-GDD$(3,3,4;6t,2s)$ for given t and s are:

- the blocks of t copies of 3-PBIBD$(3,3,4;6,1,0)$,
- the blocks of s copies of 3-PBIBD$(3,3,4;0,1,2)$, and
- $\frac{6t-(t+s)}{2}$ copies of 3-GDD$(3,2,4;0,2)$ on G_i and G_j, $1 \leq i,j \leq 3$, which exist by Theorem 3.1. □

5.2. $n = 6$

A $3'$-GDD$(6, 3, 4; 3, 1)$ on groups G_i, $i = 1, 2, 3$, can be obtained by taking a copy of 3-$(6, 4, 3)$ on each of the groups, along with two copies of 3-GDD$(6, 2, 4; 0, 1)$ on G_i, G_j for $1 \leq i < j \leq 3$, which exist by Theorem 3.1, and a copy of $Q''_6(4, 1) =$ 3-PBIBD$(6, 3, 4; 0, 1, 1)$.

Theorem 5.3. *A $3'$-GDD$(6, 3, 4; 3t, 3s + 1)$ exists whenever $3s < 3(t - 1)$.*

Proof. Take a $3'$-GDD$(6, 3, 4; 3, 1)$ along with the following:
- $3s$ copies of $Q''_6(4, 1) =$ 3-PBIBD$(6, 3, 4; 0, 1, 1)$,
- $3(t - 1) - 3s$ copies of 3-GDD$(6, 2, 4; 0, 1)$ on G_i and G_j for $1 \leq i < j \leq 3$,
- $(t - 1)$ copies of 3-$(6, 4, 3)$ on each G_i $(i = 1, 2, 3)$. □

In general, using x copies of 3-PBIBD$(6, 3, 4; 0, 1, 1)$ and using $(3s - x)$ copies of a 3-GDD$(6, 2, 4; 0, 1)$ on G_i and G_j for $1 \leq i < j \leq 3$, a 3-PBIBD$(6, 3, 4; 0, 3s, x)$ exists. Hence, using s copies of a 3-$(6, 4, 3)$ on each group, we can construct a $3'$-GDD$(6, 3, 4; 3s, x)$ for any x.

Theorem 5.4. *For any positive integers s and x, a $3'$-GDD$(6, 3, 4; 3s, x)$ exists. Hence, the necessary conditions are sufficient for existence when $n = 6$.*

5.3. $\mu_2 = 0$

Note that when the block size is 4 but only 3 groups, then a $3'$-GDD cannot exist with $\mu_1 = 0$, but $\mu_2 = 0$ is possible.

Theorem 5.5. *A $3'$-GDD$(n, 3, 4, \mu_1, 0)$ exists if and only if a 3-$(n, 4, \mu_1)$ exists. If n is odd, then μ_1 must be even.*

Proof. Use a 3-$(n, 4, \mu_1)$ on each of G_1, G_2, and G_3, along with either:
- μ_1 copies of 3-GDD$(n, 2, 4; 0, 1)$ on groups G_i, G_j, for $1 \leq i < j \leq 3$, or
- $\frac{\mu_1}{2}$ copies of 3-GDD$(n, 2, 4; 0, 2)$ on the same pairs of groups.

If n is odd, a 3-$(n, 4, \mu_1)$ does not exist when μ_1 is odd. □

5.4. *Some miscellaneous constructions*

Theorem 5.6. *A $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$ exists for $\mu_1 \equiv 0 \pmod 3$, $\mu_2 \equiv 0 \pmod 3$, and $\mu_1 \geq \mu_2$ when n is even.*

Proof. Let $\mu_1 = 3s + 3t$ and $\mu_2 = 3s$. The blocks of a $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$ consist of the blocks of:
- $3t$ copies of 3-GDD$(n, 2, 4; 0, 1)$ on G_i, G_j for $1 \leq i < j \leq 3$,
- s copies of 3-$(3n, 4, 3)$,
- t copies of 3-$(n, 4, 3)$ on each G_i, $1 \leq i \leq 3$. □

Corollary 5.7. *A $3'$-GDD$(n,3,4;12t,12s)$ exists for $t \geq s$.*

Proof. When $t = s$, a 3-$(3n,4,12t)$ exists for any n which gives the required $3'$-GDD. For $t > s$, the blocks of a 3-$(3n,4,12s)$ together with the blocks of a 3-$(n,4,12(t-s))$ which exists for any n, form the collection of blocks of the required design. □

Theorem 5.8. *If n is even, then a $3'$-GDD$(n,3,4;n,3(n-1))$ exists.*

Proof. Since n is even, a 3-$(n,4,n)$ design exists. Place one such design on each group G_1, G_2, and G_3. These blocks cover all triples of type $(3,0)$ and $(2,1)$ exactly n times, satisfying $\mu_1 = n$.

Next, we add blocks formed as follows: for every unordered pair $\{a,b\} \subseteq G_i$, take every unordered pair $\{x,y\}$ with $x \in G_j$, $y \in G_k$, where $i,j,k \in \{1,2,3\}$ and $i \neq j \neq k \neq i$, and form the block $\{a,b,x,y\}$.

Each such block contains two $(2,1)$ triples and two $(1,1,1)$ triples. Each $(2,1)$ triple appears in n blocks, and each $(1,1,1)$ triple appears in $n-1$ blocks for each G_i, totaling $\mu_2 = 3(n-1)$.

Thus, this construction yields a valid $3'$-GDD$(n,3,4;n,3(n-1))$. □

On the other hand, as a 3-$(3n,4,3)$ exists for n even and a 3-$(n,4,1)$ exists for $n \equiv 2,4 \pmod 6$, we have:

Theorem 5.9. *For $n \equiv 2,4 \pmod 6$, a $3'$-GDD$(n,3,4;\mu_1,3t)$ exists for any positive integer t, and for all $\mu_1 > 3t$.*

Theorem 5.10. *If a 3-$(3n,4,\mu_1)$ and a 3-$(2n,4,\mu_2)$ exist, then a $3'$-GDD$(n,3,4;\mu_1+\mu_2,\mu_1)$ exists.*

Proof. Let G_1, G_2, and G_3 be the three groups of size n. Take a 3-$(3n,4,\mu_1)$ on $G_1 \cup G_2 \cup G_3$. This contributes μ_1 to every triple of type $(3,0)$, $(2,1)$, and $(1,1,1)$. Then, add one copy of a 3-$(2n,4,\mu_2)$ design on each pair $G_i \cup G_j$, where $1 \leq i < j \leq 3$. These additional blocks contribute only to triples of type $(3,0)$ and $(2,1)$, and each such triple appears in exactly one of the group pairs. Hence, the index for type $(3,0)$ and $(2,1)$ triples increases by μ_2, while the index for type $(1,1,1)$ triples remains μ_1. Thus, we obtain a $3'$-GDD$(n,3,4;\mu_1+\mu_2,\mu_1)$. □

In the next section, we construct certain 3-PBIBD$(n,3,4;\lambda_1,\lambda_2,\lambda_3)$ designs. These designs are not studied in isolation, but are directly relevant to our objectives. As demonstrated in the previous section and in Theorem 6.10, such 3-PBIBDs can be used in the construction of $3'$-GDD$(n,3,4;\mu_1,\mu_2)$ designs.

6. 3-PBIBD$(n, 3, 4; 0, \lambda_2, \lambda_3)$ designs

The necessary conditions for the existence of a 3-PBIBD$(n, 3, 4; 0, \lambda_2, \lambda_3)$ can be formulated as follows. To distinguish them from earlier conditions, we use the superscript * notation when applicable.

Theorem 6.1. *If a 3-PBIBD$(n, 3, 4; 0, \lambda_2, \lambda_3)$ exists, then:*

- *The replication number r^* is*

$$r^* = \frac{3\lambda_2 n(n-1) + n^2\lambda_3}{3}.$$

- *The total number of blocks b^* is*

$$b^* = \frac{3n^2(n-1)\lambda_2 + n^3\lambda_3}{4}.$$

- *The number of blocks containing a first associate pair, denoted λ_1^*, is*

$$\lambda_1^* = n\lambda_2.$$

- *The number of blocks containing a second associate pair, denoted λ_2^*, is*

$$\lambda_2^* = \frac{2(n-1)\lambda_2 + n\lambda_3}{2}.$$

- *Finally, the values of λ_2 and λ_3 must satisfy the inequality*

$$\lambda_3 \leq \frac{3(n-1)}{n}\lambda_2 \quad \textit{which is strictly less than } 3\lambda_2.$$

Theorem 6.2. *The following necessary conditions are obtained by using the values of r^*, b^*, λ_1^*, and λ_2^*, as these values must be integers:*

- $n \equiv 0 \pmod 6$ *then any λ_2, λ_3,*
- $n \equiv 1, 5 \pmod{12}$ *then* $\lambda_3 \equiv 0 \pmod{12}$,
- $n \equiv 2, 4, 8, 10 \pmod{12}$ *then* $\lambda_3 \equiv 0 \pmod 3$,
- $n \equiv 9 \pmod{12}$ *then* $\lambda_3 \equiv 0 \pmod 4$,
- $n \equiv 3 \pmod{12}$ *then either λ_2 odd and* $\lambda_3 \equiv 2 \pmod 4$ ***or*** *λ_2 even and* $\lambda_3 \equiv 0 \pmod 4$,
- $n \equiv 7, 11 \pmod{12}$ *then either λ_2 odd and* $\lambda_3 \equiv 6 \pmod{12}$ ***or*** *λ_2 even and* $\lambda_3 \equiv 0 \pmod{12}$.

We give the following results as applications of the above conditions.

Example 6.3. A 3-PBIBD$(n, 3, 4; 0, \lambda_2, \lambda_3)$ does not exist for $n \equiv 1, 2 \pmod 3$ and $\lambda_3 \equiv 1, 2 \pmod 3$ by the conditions forced by r^*.

Example 6.4. A 3-PBIBD$(n, 3, 4; 0, 1, 3)$ does not exist by the conditions forced by the inequality between λ_2 and λ_3.

Theorem 6.5. *A* 3-*PBIBD*$(n, 3, 4; 0, n, 3(n-1))$ *exists.*

Proof. Group size n cannot be 1 since the block size is 4 and there are only 3 groups. The blocks of this design are constructed by taking the union of each pair of points from a group (i.e., each edge of a complete graph with vertices labeled by a group) with all the edges of a $K_{n,n}$, whose parts are labeled by the other two groups.

In other words, the blocks are:

$$\begin{aligned} &\{\{a_r, a_s, b_i, c_j\} \mid 1 \leq r < s \leq n,\ 1 \leq i, j \leq n\} \\ \cup\ &\{\{b_r, b_s, a_i, c_j\} \mid 1 \leq r < s \leq n,\ 1 \leq i, j \leq n\} \\ \cup\ &\{\{c_r, c_s, b_i, a_j\} \mid 1 \leq r < s \leq n,\ 1 \leq i, j \leq n\}, \end{aligned}$$

where the groups are $\{a_1, \ldots, a_n\}$, $\{b_1, \ldots, b_n\}$, and $\{c_1, \ldots, c_n\}$. □

For group size 2, the necessary conditions force the smallest values of λ_2 and λ_3 as 2 and 3 respectively, and the above theorem gives this design:

Corollary 6.6. *A* 3-*PBIBD*$(2, 3, 4; 0, 2, 3)$ *is the smallest possible design of this form and it exists.*

Theorem 6.7. *A* 3-*PBIBD*$(6, 3, 4; 0, 2, 3)$ *exists.*

Proof. A 3-PBIBD$(6, 3, 4; 0, 2, 3)$ is constructed as follows. Partition each group G_i into two subgroups G_{i1} and G_{i2} of 3 points each, for $i = 1, 2, 3$.

Take a copy of $P_3''[4, 1, 12]$, i.e., a 3-PBIBD$(3, 4, 4; 0, 0, 1)$ (constructed in Hanani [4]), on each 4-group combination of the form $\{G_{i1}, G_{i2}, G_{ab}, G_{cd}\}$, where $a \neq c \neq i$, $1 \leq i \leq 3$, $1 \leq a, c \leq 3$, $a \neq c$, and $1 \leq b, d \leq 2$.

These account for all $(1, 1, 1)$ triples, covering each three times, and most $(2, 1)$ triples—specifically those where the pair of elements from the same group are from different subgroups of the group.

To cover the remaining $(2, 1)$ triples—those where both elements from the same group are in the same subgroup—we use a 3-GDD$(3, 2, 4; 0, 2)$ on each pair (G_{i1}, G_{i2}) for $i = 1, 2, 3$.

The combined blocks from all these constructions form the required design. □

Corollary 6.8. *When* $n \equiv 0 \pmod 3$*, a* 3-*PBIBD*$(2n, 3, 4; 0, 2, 3)$ *exists.*

The proof is a generalization of the construction of 3-PBIBD$(6, 3, 4; 0, 2, 3)$, as the existence of a 3-PBIBD$(3, 4, 4; 0, 0, 1)$ implies the existence of a 3-PBIBD$(3t, 4, 4; 0, 0, 1)$. In other words, one can extend the above construction to other 3-PBIBDs by using a 3-PBIBD$(n, 4, 4; 0, 0, 1)$ from Hanani's proposition together with 3-GDDs from Theorem 3.1. For example, this approach can be used to construct a 3-PBIBD$(2t, 3, 4; 0, 2, 3)$.

Theorem 6.9. *A* 3-*PBIBD*$(4t, 3, 4; 0, 2, 3)$ *exists for any positive integer* t*.*

Proof. We follow the same construction as in the case when $n \equiv 0 \pmod 3$, but instead of a 3-PBIBD$(3t, 4, 4; 0, 0, 1)$, we use a 3-PBIBD$(2t, 4, 4; 0, 0, 1)$. Recall that a 3-PBIBD$(2, 4, 4; 0, 0, 1)$ exists, and by Hanani's proposition, this implies the existence of a 3-PBIBD$(2t, 4, 4; 0, 0, 1)$ for any positive integer t. □

From the above results, we conclude that a 3-PBIBD$(n, 3, 4; 0, 2, 3)$ exists for all $n \equiv 0, 2, 3, 4 \pmod 6$.

Theorem 6.10. *A* $3'$-*GDD*$(n, 3, 4; 2, 3)$ *exists for all* $n \equiv 2, 4 \pmod 6$.

Proof. Since a 3-$(n, 4, 2)$ exists for all $n \equiv 2, 4 \pmod 6$, and a 3-PBIBD$(n, 3, 4; 0, 2, 3)$ has been constructed above, we can combine their blocks to obtain the required $3'$-GDD. □

In contrast, we have another non-existence result:

Theorem 6.11. *A* 3-*PBIBD*$(n, 3, 4; 0, 1, 2)$ *does not exist for* $n \equiv 2, 4 \pmod 6$.

Proof. Suppose a 3-PBIBD$(n, 3, 4; 0, 1, 2)$ exists for $n \equiv 2, 4 \pmod 6$. Then, using two copies of a 3-$(n, 4, 1)$ on each group and a 3-GDD$(n, 2, 4; 0, 1)$ on each pair of groups, we can construct a 3-$(3n, 4, 2)$. However, a 3-$(3n, 4, 2)$ does not exist. □

7. Summary

This paper introduces and studies a new variation of group divisible designs, denoted as $3'$-GDDs, in which the number of blocks containing a triple depends on how the points of the triple are distributed among the groups. Specifically, a $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$ is a block design on $3n$ points partitioned into 3 groups of size n each, with block size 4, such that every triple consisting of points from at most two groups occurs in exactly μ_1 blocks, and every triple formed from three distinct groups occurs in exactly μ_2 blocks. This structure generalizes the concept of 3-PBIBDs and differs from standard 3-GDDs in how the balance conditions are applied.

The reformulation of part of Hanani's classical 1963 results using more accessible notation and terminology is expected to be useful for researchers in the area. Hanani's structures, such as P''_n and Q''_n, are reinterpreted in the language of $3'$-GDDs and PBIBDs. This reinterpretation enables a clearer understanding of these foundational structures.

We derive necessary conditions for the existence of $3'$-GDD$(n, 3, 4; \mu_1, \mu_2)$, including exact formulas for the number of blocks, replication number, and block intersection numbers, and expresses these conditions in terms of divisibility and congruence relations on μ_1, μ_2, and n. These are further simplified into modular conditions based on n mod 24.

A variety of existence results are established, including complete existence for $3'$-GDDs with $\mu_2 = 0$ when n is even and μ_1 is even, and for all n when μ_1 is even. Infinite families of constructions are provided, especially for small values such as $n = 3$ and $n = 6$, using known 3-designs, group divisible designs, and PBIBDs. The paper includes examples where $\mu_1 < \mu_2$ and discusses cases where the necessary conditions are also sufficient.

The paper also constructs several 3-PBIBD$(n, 3, 4; 0, \lambda_2, \lambda_3)$ designs, which are used as building blocks in constructing new $3'$-GDDs. It gives both existence results and nonexistence proofs based on parameter constraints, including parity and modular arguments, and highlights some of the smallest possible designs of these types.

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