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Decomposition of complete graphs into disconnected unicyclic bipartite graphs with seven edges and more than eight vertices

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ABSTRACT

In this paper, we continue investigation of decompositions of complete graphs into graphs with seven edges. The spectrum has been completely determined for such graphs with at most six vertices. The spectrum for bipartite graphs is completely known for graphs with seven or eight vertices. In this paper we completely solve the case of disconnected unicyclic bipartite graphs with seven edges by studying the remaining graphs with nine or ten vertices.

Keywords: graph decomposition, G-design, ρ-labeling

2020 Mathematics Subject Classification: 05C51, 05C78.

1. Introduction

Graph decompositions have been extensively studied for decades and became one of the classical themes in graph theory. Decomposition of complete graphs into mutually isomorphic subgraphs is probably the most popular topic within this area. We say that a graph G *decomposes* K_n if there exist subgraphs $G_1, G_2, \ldots, G_s$ of K_n, all isomorphic to G, such that every edge of K_n appears in exactly one copy G_i of G. One selects a class of graphs $\mathcal{G}$, finite or infinite, and finds complete graphs that admit a decomposition into all graphs in $\mathcal{G}$. Such a collection of complete graphs is then called the *spectrum* of $\mathcal{G}$.

In this paper we continue the effort to find the spectrum for all graphs with a given (small) number of edges by determining which complete graphs they decompose. Almost all graphs with up to six edges have been fully classified, as well as almost all graphs

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with eight edges. For a detailed overview, we refer the reader to Adams, Bryant, and Buchanan [1]. For graphs with seven edges, much less is known. An overview of known results is presented in Section 2.

We continue in this direction by determining the spectrum for all disconnected unicyclic bipartite graphs with seven edges and nine or ten vertices decomposing complete graphs. A *unicyclic graph* is a simple finite graph without loops containing exactly one cycle. There are three such graphs, either with three or four components. We call them G_1, G_2, G_3 and denote $\mathcal{G} = \{G_i \mid i = 1, 2, 3\}$. They are presented in Figure 1.

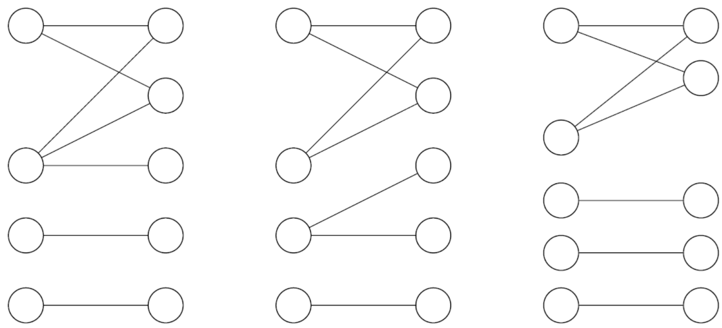

Fig. 1. Graphs G_1–G_3 (left to right)

The obvious necessary condition for K_n to be decomposable into a graph with seven edges is $n \equiv 0, 1 \pmod 7$.

For $n \equiv 0, 1 \pmod{14}$ the decompositions are based on Rosa-type labelings, introduced by Rosa in 1967 [13]. This is done in Section 4. In Section 5 for $n \equiv 7 \pmod{14}$ we decompose K_n first into graphs $K_{7,7}$ and $K_{14} - K_7$ and then each of them into the desired graph G_i using again mostly labeling techniques. Finally, in Section 6 for $n \equiv 8 \pmod{14}$, we use labeling techniques with two different starters.

Disclaimer. Since this paper is a direct continuation of paper [10] by the third author and Kubesa, some parts of Sections 1, 2, and 3 are similar or identical to those in [10].

2. Known results

For an excellent survey on G-designs, we refer the reader to [1]. There is a thorough section on graphs with at most six vertices, which covers a good number of graphs with seven edges.

For graphs with at most six edges and those with eight edges the spectrum is almost completely known except for a small number of cases; the class of graphs with seven edges is still wide open.

Graphs with seven edges and five vertices are always connected and the spectrum was determined by Bermond, Huang, Rosa, and Sotteau [4] with 11 possible exceptions. Five of them for $K_5 - K_{1,3}$ were constructed by Chang [7]. For $K_5 - (P_3 \cup P_2)$, five were constructed by Li and Chang [12] and the remaining one by Adams, Bryant, and Buchanan [1].

Theorem 2.1 (Bermond et al. [4], Chang [7], Li and Chang [12], Adams, Bryant, and Buchanan [1]). *There exists a G-decomposition of K_n for a graph G on seven edges and five vertices if and only if*

1. $G = K_5 - K_{1,3}, n \equiv 0, 1 \pmod 7, n \geq 14$, *or*
2. $G = K_5 - K_3, n \equiv 1, 7 \pmod{14}$, *or*
3. $G = K_5 - (P_3 \cup P_2), n \equiv 0, 1 \pmod 7, n \neq 8, 14$, *or*
4. $G = K_5 - (P_4), n \equiv 0, 1 \pmod 7, n \neq 8$.

Blinco [5] and Tian, Du, and Kang [14] studied connected graphs with seven edges and six vertices. The only disconnected graph with seven edges and six vertices is $K_4 \cup K_2$ and the spectrum for this graph was also found in [14].

Theorem 2.2 (Blinco [5], Tian et al. [14]). *There exists a G-decomposition of K_n for a graph G on seven edges and six vertices if and only if $n \equiv 0, 1 \pmod 7$ except for eight exceptions when $n = 7$ or $n = 8$.*

All graphs with seven edges and seven vertices are either connected and unicyclic or disconnected. A complete solution for connected bipartite (and necessarily unicyclic) graphs was obtained by Froncek and Kubesa in [9].

Theorem 2.3 (Froncek, Kubesa [9]). *There exists a G-decomposition of K_n for a connected bipartite unicyclic graph G on seven edges and seven vertices if and only if $n \equiv 0, 1 \pmod 7$ except for three exceptions when $n = 7$ and two exceptions when $n = 8$.*

The only remaining connected class for seven edges and seven vertices is then unicyclic tripartite graphs. To our knowledge, this class has not received any attention yet.

Since the solution is not covering the whole spectrum, we pose our first open problem.

Problem 2.4. *Complete the G-decomposition spectrum for connected tripartite graphs on seven edges and seven vertices (which are necessarily unicyclic).*

We do not know any result on the spectrum of disconnected graphs with seven edges and seven vertices. Our second open problem is then the following.

Problem 2.5. *Determine the G-decomposition spectrum for disconnected graphs on seven edges and seven vertices.*

Connected graphs with seven edges and eight vertices are trees, which were investigated by Huang and Rosa [11].

Theorem 2.6 (Huang, Rosa [11]). *There exists a G-decomposition of K_n for a connected graph G on seven edges and eight vertices (that is, a tree) if and only if $n \equiv 0, 1 \pmod 7, n \geq 8$ except for nine exceptions when $n = 8$.*

For disconnected bipartite graphs with seven edges and eight vertices, a complete solution was obtained by Froncek and Kubesa in [10].

Theorem 2.7 (Froncek, Kubesa [10]). *There exists a G-decomposition of K_n for a disconnected bipartite unicyclic graph G on seven edges and eight vertices if and only if $n \equiv 0, 1 \pmod 7, n \geq 8$ with three exceptions when $n = 8$.*

In this paper we completely determine the spectrum for disconnected unicyclic bipartite graphs on seven edges by solving the cases with nine or ten vertices.

The remaining graphs with seven edges and more than seven vertices are necessarily disconnected. The spectrum for forests was completely determined by Banegas and Freyberg [3].

Theorem 2.8 (Banegas, Freyberg [3]). *There exists a G-decomposition of K_n for a forest G on seven edges if and only if $n \equiv 0, 1 \pmod 7, n \geq 14$.*

A partial result for unicyclic ones was obtained recently by Banegas et al. [2].

Theorem 2.9 (Benagas et al. [2]). *There exists a G-decomposition of K_n for a disconnected tripartite unicyclic graph G on seven edges whenever $n \equiv 0, 1 \pmod{14}, n \geq 14$.*

This leaves us with the third open problem.

Problem 2.10. *Complete the G-decomposition spectrum for disconnected graphs on seven edges and more than seven vertices.*

3. Definitions and tools

The following definition has been used in different variations for years, and we present it just for the sake of completeness.

Definition 3.1. Let H be a graph. A *decomposition* of the graph H is a collection of pairwise edge-disjoint subgraphs $\mathcal{D} = \{G_1, G_2, \ldots, G_s\}$ such that every edge of H appears in exactly one subgraph $G_i \in \mathcal{D}$.

We say that the collection forms a *G-decomposition* of H (also known as an *(H, G)-design*) if each subgraph G_i is isomorphic to a given graph G. If H is the complete graph K_n, then we can use just the term *G-design*.

Because we focus solely on decompositions of complete graphs, we only use the term G-decomposition or G-design. The following definition was used by Rosa [13] already in 1967. We are unsure whether this was the first appearance of that definition though.

Definition 3.2. A G-decomposition of the complete graph K_n is *cyclic* if there exists an ordering $(x_0, x_1, \ldots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n

defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \dots, n-1$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo n.

As above, we are not sure when the following definition was stated first time using the exact terminology. A construction using this type of decomposition was used already by Huang and Rosa [11] in 1978.

Definition 3.3. A G-decomposition of the complete graph K_n is 1-*rotational* if there exists an ordering $(x_0, x_1, \dots, x_{n-1})$ of the vertices of K_n and a permutation φ of the vertices of K_n defined by $\varphi(x_j) = x_{j+1}$ for $j = 0, 1, \dots, n-2$ and $\varphi(x_{n-1}) = x_{n-1}$ inducing an automorphism on $\mathcal{D}$, where the addition is performed modulo $n-1$.

We will use the interval notation $[k, n]$ for the set of consecutive integers $\{k, k+1, k+2, \dots, n\}$. When $k = 1$, the interval is denoted simply by $[n]$.

One of the basic and most useful tools for finding G-designs is the following labeling.

Definition 3.4 (Rosa [13])**.** Let G be a graph with n edges. A *ρ-labeling* (sometimes also called *rosy labeling*) of G is an injective function $f\colon V(G) \to [0, 2n]$ that induces the *length function* $\ell\colon E(G) \to [n]$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\},$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [n].$$

A graph G possessing a ρ-labeling decomposes the complete graph, as proved by Rosa in 1967.

Theorem 3.5 (Rosa [13])**.** *Let G be a graph with n edges. A cyclic G-decomposition of the complete graph K_{2n+1} exists if and only if G admits a ρ-labeling.*

When a graph G with n edges has a vertex w of degree one and $G - w$ admits a ρ-labeling, a modification of ρ-labeling can be used to find a G-decomposition of K_{2n}. Such labeling is known as 1-*rotational ρ-labeling* and was first used by Huang and Rosa in [11], although a formal definition was not stated there.

Definition 3.6 (Huang, Rosa [11])**.** Let G be a graph with n edges and edge ww' where $\deg(w) = 1$. A 1-*rotational ρ-labeling* of G consists of an injective function $f\colon V(G) \to [0, 2n-2] \cup \{\infty\}$ with $f(w) = \infty$ that induces a *length function* $\ell\colon E(G) \to [n-1] \cup \{\infty\}$ which is defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n - 1 - |f(u) - f(v)|\},$$

for $u, v \neq w$ and

$$\ell(ww') = \infty$$

with the property that

$$\{\ell(uv) : uv \in E(G)\} = [n-1] \cup \{\infty\}.$$

This technique was used in [11] and proved only for particular graphs studied in that paper. The following theorem is considered folklore.

Theorem 3.7. *Let G be a graph with n edges. If G admits a 1-rotational ρ-labeling, then there exists a 1-rotational G-decomposition of the complete graph K_{2n}.*

As we observed before, a necessary condition for K_n to admit a G-design for a graph G with 7 edges is that the number of edges in K_n must be divisible by 7, implying $n \equiv 0, 1 \pmod 7$. For the graphs we are interested in, the above theorems only allow decompositions of K_{14} and K_{15}. Therefore, we will need additional tools, which are some more restrictive modifications of ρ-labeling yet also produce G-decompositions of larger complete graphs; that is, K_{2nk+1} for any $k \geq 1$ when G has n edges.

Definition 3.8 (El-Zanati, Vanden Eynden, Punnim [16])**.** Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. A *ρ^+-labeling* of G is a ρ-labeling f with the additional property that for every $u \in U$ and $v \in V$ if $uv \in E(G)$, then $f(u) < f(v)$. A *σ^+-labeling* is a *ρ^+-labeling* without wrap-around edges, that is, the length function is defined as

$$\ell(uv) = f(v) - f(u).$$

These conditions guarantee decompositions of K_{2nk+1}, as proved by El-Zanati, Vanden Eynden and Punnim [16].

Theorem 3.9 (El-Zanati, Vanden Eynden, Punnim [16])**.** *Let G be a bipartite graph with n edges. If G admits a ρ^+-*labeling *or a σ^+-labeling, then there exists a cyclic G-decomposition of the complete graph K_{2nk+1} for every $k \geq 1$.*

Notice that mentioning the *σ^+-labeling* in Theorem 3.9 is in fact redundant, because every *σ^+-labeling* is a *ρ^+-labeling*. We mention it there explicitly for clarity because in what follows we will be only using *σ^+-labeling* in our constructions.

To decompose complete graphs with $2nk$ vertices into graphs with n edges, we will use the 1-rotational σ^+-labeling.

Definition 3.10. Let G be a bipartite graph with n edges, vertex w of degree one and an edge ww'. A *1-rotational σ^+-labeling* of G is a 1-rotational ρ-labeling with the additional property that for every $u \in U$ and $v \in V$ if $u, v \neq w$ and $uv \in E(G)$, then $f(u) < f(v)$ and the length function is defined as

$$\ell(uv) = f(v) - f(u),$$

for $u, v \neq w$,

$$f(w) = \infty, \text{ and } \ell(ww') = \infty.$$

The following analogues of the above theorems were proved recently. Bunge [6] proved the following.

Theorem 3.11 (Bunge [6])**.** *Let G be a bipartite graph with n edges and with $w \in V(G)$ such that $\deg(w) = 1$. If $G - w$ admits a ρ^+-labeling, then there exists a 1-rotational G decomposition of K_{2nk} for any positive integer k.*

It is easy to see that when we have a σ^+-labeling where the longest edge is ww', vertex w is of degree one and all other vertices have labels at most $2n-2$, the labeling can be transformed to a 1-rotational σ^+-labeling.

The following theorem is more restrictive in the labeling properties but allows us to use the same labeling for G-decompositions of both K_{2nk+1} and K_{2nk}.

Theorem 3.12 (Fahnenstiel, Froncek [8])**.** *Let G be a bipartite graph with n edges and a vertex of degree one. If G admits a σ^+-labeling with the additional property that the edge of length n is pendant and no vertex label is greater than $2n-2$, then there exists a cyclic G-decomposition of the complete graph K_{2nk+1} and a 1-rotational G-decomposition of the complete graph K_{2nk} for every $k \geq 1$.*

In our constructions, we will also need to decompose complete bipartite graphs. The tools are similar, based on labelings as well. An equivalent of ρ-labeling for bipartite graphs is called bilabeling and has been used for years by numerous authors. The following definition is adapted from [15].

Definition 3.13. Let G be a bipartite graph with n edges and a vertex bipartition $U \cup V$. A

ρ-bilabeling of G is a function $f : V(G) \to [0, n-1]$ that is injective when restricted to sets U and V and the induced length function defined as

$$\ell(uv) = (f(v) - f(u)) \pmod{n},$$

has the property that

$$\{\ell(uv) : uv \in E(G)\} = [0, n-1].$$

The following theorem was proved in a simpler form independently by many authors; e.g., in [15].

Theorem 3.14. *Let G be a bipartite graph with n edges. If G admits a ρ-bilabeling, then there exists a G-decomposition of the complete bipartite graph $K_{nk,nm}$ for every $k, m \geq 1$.*

4. Decompositions of K_n for $n \equiv 0, 1 \pmod{14}$

Our decompositions of K_n for $n \equiv 1 \pmod{14}$ are based on σ^+-labelings of the respective graphs.

Theorem 4.1. *There exists a G_i-decomposition of the complete graph K_{14k} and K_{14k+1} into each graph $G_i \in \mathcal{G}$ for every $k \geq 1$.*

Proof. Each graph $G_i \in \mathcal{G}$ has a σ^+ labeling shown in Figure 2, with no vertex label bigger than $2n - 2 = 12$. The longest edge of length 7 is pendant with one endvertex labeled 12. Therefore, a decomposition exists by Theorem 3.12. □

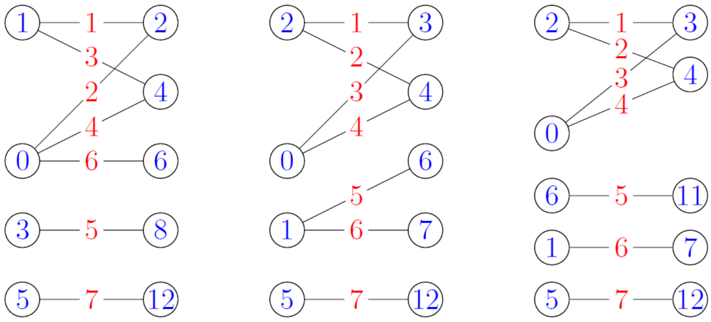

Fig. 2. σ^+-labelings of G_1–G_3 (left to right)

5. Decompositions of K_n for $n \equiv 7 \pmod{14}$

In this case, we let $n = 14k + 7$ and first decompose K_{14k+7} into $2k + 1$ graphs $K_{14} - K_7$ and $(k - 1)(2k + 1)$ copies of $K_{7,7}$. Then we use it to find a G-decomposition of K_{14k+7}.

We first partition the vertex set of K_{14k+7} into $2k+1$ sets $X_i, i = 1, 2, \ldots, 2k+1$, where $X_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,6}\}$. Then we take the complete bipartite graph $K_{7,7}$ with partite sets X_i and X_{i+1} (and with X_{2k+1} and X_1 when $i = 2k + 1$) and add the edges of the complete graph K_7 induced on X_i to obtain our graphs $K_{14} - K_7$. We denote these graphs H_i where the index i corresponds the index of X_i which is filled by K_7. For the remaining pairs of sets X_i, X_j, $i, j \in \{1, 2, \ldots, 2k + 1\}, j \neq i, j \neq i \pm 1$, again computed modulo $2k + 1$, we obtain the graphs $K_{7,7}$ with bipartition X_i, X_j.

To show that $K_{7,7}$ is G-decomposable, it is enough to find a ρ-bilabeling of G. They are shown in Figure 3.

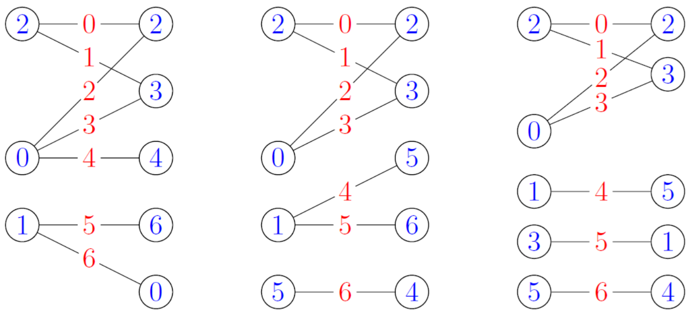

Fig. 3. ρ-bilabelings of G_1–G_3 (left to right)

A lemma follows immediately.

Lemma 5.1. *The complete bipartite graph $K_{7,7}$ is G-decomposable for every $G \in \mathcal{G}$.*

Construction 5.2. In Figure 4 we show a G_1-decomposition of the graph $H_i \cong K_{14} - K_7$. A vertex $x_{i,j}$ is depicted as j_i, where j is also the label.

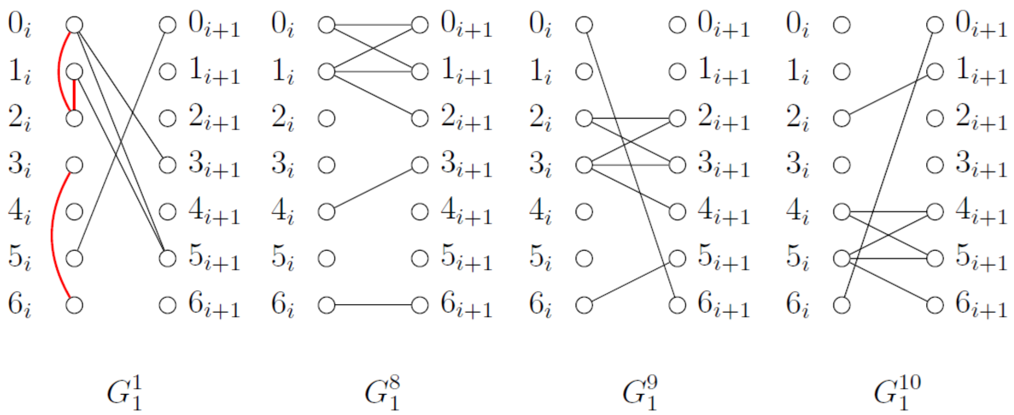

Fig. 4. G_1 in $K_{14} - K_7$

The edge lengths are calculated in two different ways depending on whether the edge belongs to K_7 or $K_{7,7}$. The edges in K_7 are red and the length is calculated as in ρ-labeling, that is

$$\ell(x_{i,a}x_{i,b}) = \ell(a_i b_i) = \min\{|a-b|, 7-|a-b|\}.$$

One can observe that the lengths in G_1^1 with respect of the complete graph K_7 induced on X_i are in $[3]$ and thus satisfy the conditions on ρ-labeling of a graph with three edges.

In $K_{7,7}$ the edges are black and we use a ρ-bilabeling with edge lengths

$$\ell(x_{i,a}x_{i+1,b}) = \ell(a_i b_{i+1}) = (b-a) \mod 7.$$

Again in G_1^1 the bilabeling lengths are $[2,5]$. Rotating the graphs G_1^1 with step one seven times, we use all edges of these lengths. Here by a *rotation with step t* we mean a permutation $\varphi(x_{i,j}) = x_{i,j+t}$ where the addition in the second subscript is performed modulo 7. The other 21 edges with bilabeling lengths 0, 1, and 6, are used for the remaining copies G_1^8, G_1^9, G_1^{10}.

Construction 5.3. For a G_2-decomposition of $H_i \cong K_{14} - K_7$, we rotate the graph G_2^1 shown in Figure 5 only six times. This is so because we need to "borrow" one edge from the seventh rotation and place it to what will be the graph G_2^8. Namely, the edge $3_i 5_{i+1}$ will be placed in G_2^8 and replaced by $2_i 1_{i+1}$. The edges are colored green in Figure 6. We need this adjustment because it is impossible to find three copies of G_2 made up form edges of lengths $0, 1$ and 6 between the two partite sets of the partition.

Construction 5.4. Similarly as in Construction 5.3, we rotate the graph G_2^1 shown in Figure 5 only six times. This time we "borrow" the edge $0_i 1_i$ from the seventh rotation and place it to what will be the graph G_2^9. Then the edge $0_i 6_{i+1}$ will be placed in G_2^7 for the same reasons as in the previous construction. It is in Figure 8 colored green again.

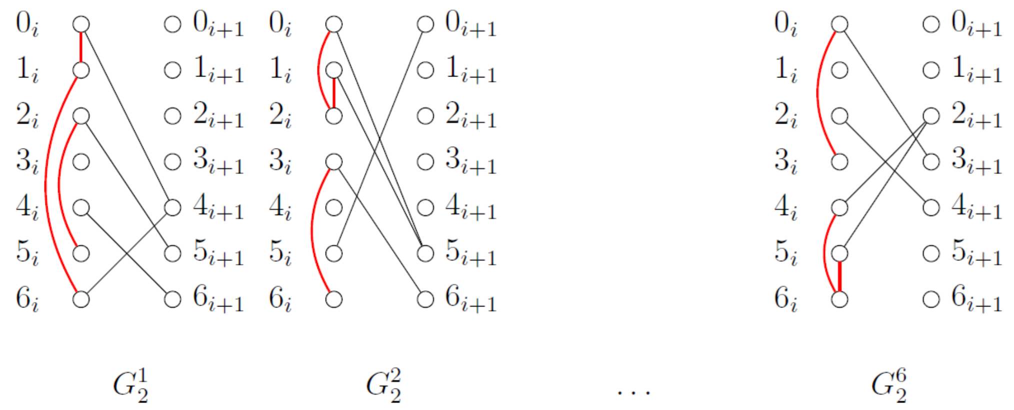

Fig. 5. $G_2^1, G_2^2, \ldots, G_2^6$ in $K_{14} - K_7$

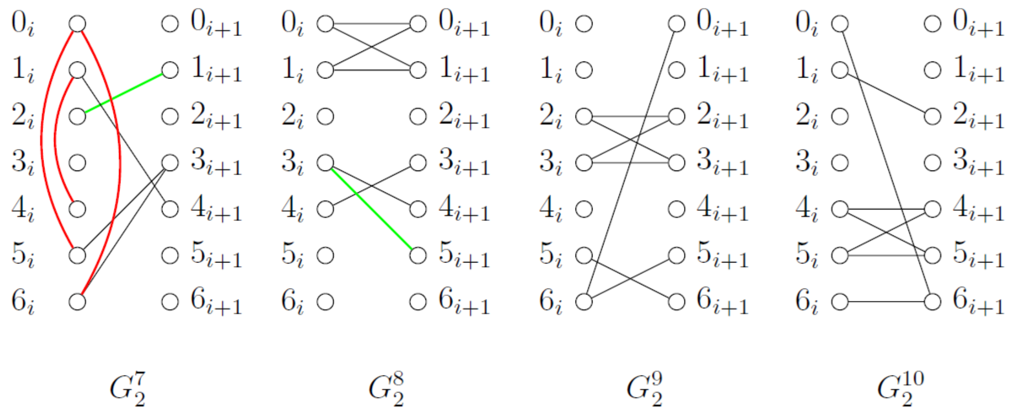

Fig. 6. G_2 with swap in $K_{14} - K_7$

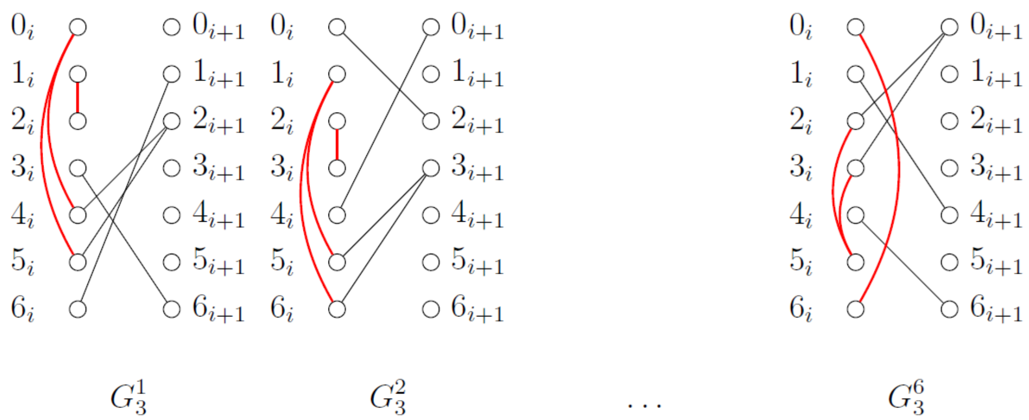

Fig. 7. $G_3^1, G_3^2, \ldots, G_3^6$ in $K_{14} - K_7$

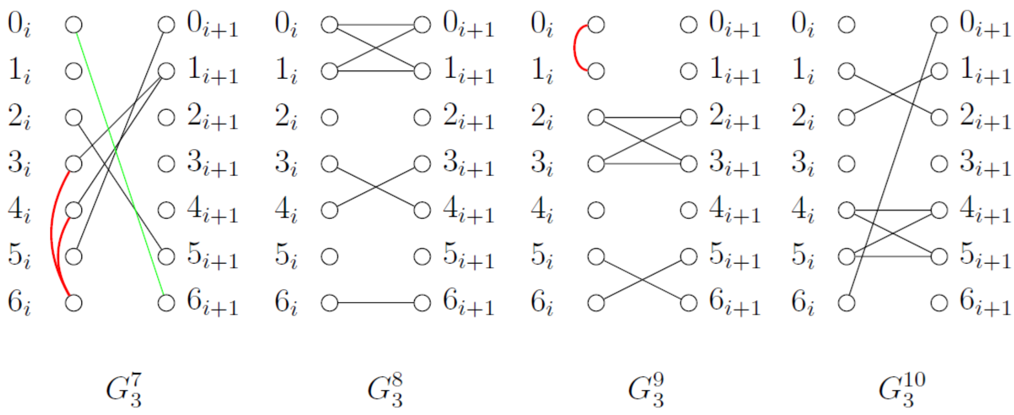

Fig. 8. $G_3^7, G_3^8, G_3^9, G_3^{10}$ in $K_{14} - K_7$

The construction immediately yield our next lemma.

Lemma 5.5. *The graph $K_{14} - K_7$ is G-decomposable for every $G \in \mathcal{G}$.*

Proof. The decompositions are shown in Figures 4–8 and explained in Constructions 5.2–5.4. □

Theorem 5.6. *The complete graph K_{14k+7} is G-decomposable for every graph $G \in \mathcal{G}$ if and only if $k \geq 1$.*

Proof. First we recall that K_{14k+7} is decomposable into $2k+1$ copies of $K_{14} - K_7$ and $(k-1)(2k+1)$ copies of $K_{7,7}$.

Each $K_{7,7}$ is G-decomposable by Lemma 5.1 while $K_{14} - K_7$ is G-decomposable by Lemma 5.5. The condition $k \geq 1$ is obvious. □

6. Decompositions of K_n for $n \equiv 8 \pmod{14}$

In this case, we will use another method based on σ^+-type labeling. Because the number of different edge lengths in K_{14k+8} is not a multiple of seven, we will need two different starters. One will use edge lengths 1, 2, and $7k+3$, each of them twice, and length $7k+4$ once. This is so because there are $7k+8$ edges of each length $1, 2, \ldots, 14k+3$ but only $7k+4$ edges of length $7k+4$. We call this starter G_i^1; the other starter consists of graphs $G_i^2, G_i^3, \ldots, G_i^{k+1}$. Labelings of G_i^1 and G_i^2 are shown in Figures 9–13.

By *rotation with step t* we mean the mapping $i \mapsto i + t$ for every $i \in [0, 14k+7]$ computed modulo $14k+8$.

The starter G_1^1 will be rotated with step one $7k+4$ times, covering all $(7k+4)(3 \cdot 2+1) = 49k+28$ edges of lengths $1, 2, 7k+3$ and $7k+4$. Then k starters $G_1^2, G_1^3, \ldots, G_1^{k+1}$ contain edges of seven different lengths each and will be rotated with step one $14k+8$ times, covering $7k(14k+8) = 98k^2 + 56k$ edges of lengths $3, 4, \ldots, 14k+3$. Therefore, we cover

$$(49k+28) + (98k^2 + 56k) = (49k+28)(2k+1)$$

$$\begin{aligned} &= 7(7k+4)(2k+1) \\ &= (7k+4)(14k+7) \\ &= (14k+8)(14k+7)/2, \end{aligned}$$

edges, which is exactly the number of edges in K_{14k+8}.

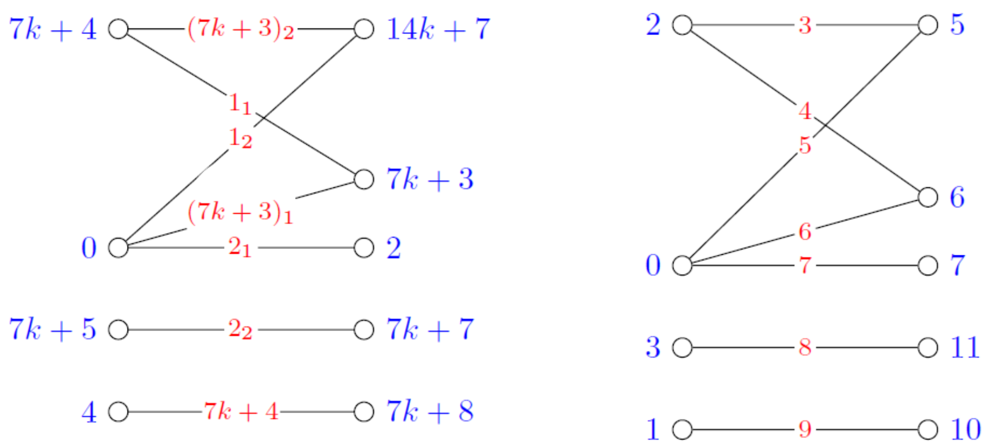

Fig. 9. G_1^1 (left) and G_1^2 (right)

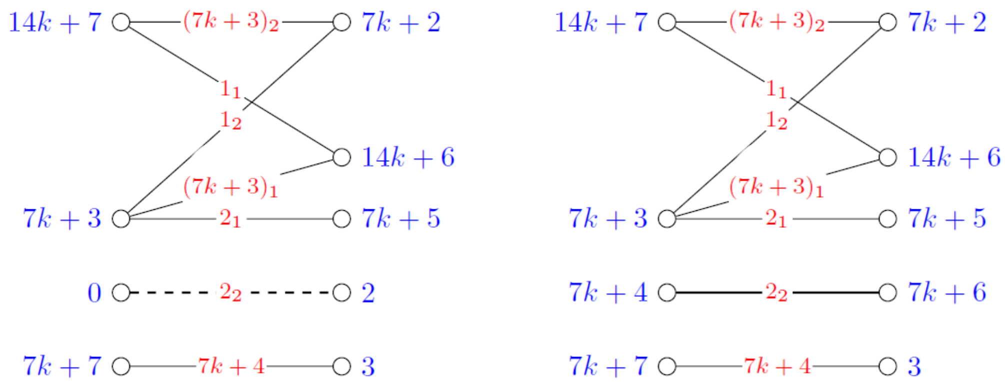

Fig. 10. The $(7k+4)$-th copy of G_1^1 before (L) and after adjustment (R)

Construction 6.1. In the graph G_1^1, let uv be an edge of length a and $f(v) = f(u) + a$ where the addition is performed modulo $14k+8$. Then we call the vertex u the *leading vertex*. For edges with the same length, the one with the smaller leading vertex label has the subscript 1 and the other one subscript 2. The subscript will not change even when in some of the following copies the labels will be ordered conversely.

We label G_1^1 as in Figure 9 and rotate it with step one, obtaining $7k+4$ copies. The edges of length 1 have the leading vertices $7k+3$ and $14k+7$, therefore are exactly $7k+4$ rotations with step one apart. Hence, after $7k+4$ rotations with step one, the $7k+4$ copies of G_1^1 do not repeat an edge of length 1. Similarly, the edges of length $7k+3$ have leading vertices 0 and $7k+4$ and the same argument applies.

However, the leading vertices of edges of length 2 are 0 and $7k+5$ and therefore differ by $7k+3$ rotations with step one only. This means that without an adjustment, the edge $(0,2)$ has images $(0,2),(1,3),\ldots,(7k+3,7k+5)$ while $(7k+5,7k+7)$ has images $(7k+5,7k+7),(7k+6,7k+8),\ldots,(0,2)$. In particular, the edge $(0,2)$ is used twice while $(7k+4,7k+6)$ not at all. Hence, in the $(7k+4)$-th copy of G_1^1 we use the edge $(7k+4,7k+6)$ instead of $(0,2)$, as shown in Figure 10.

Finally, the only edge of length $7k+4$ is $(4,7k+8)$ and we have used every edge of length $1,2,7k+3$ and $7k+4$ exactly once.

For the other starter, a labeling of G_1^2 is shown in Figure 9. For the remaining graphs $G_1^j, j=2,\ldots,k+1$ we define the labeling recursively. Let the vertices in the partite set X (with the smaller labels) be x_s and in the partite set Y (with the higher labels) be y_t. Denote the labeling function f piecewise, where f_j is the labeling for the graph G_1^j. Function f_2 is shown in Figure 9. For $j=3,4,\ldots,k+1$ we define

$$f_j(x_s)=f_{j-1}(x_s),$$

and

$$f_j(y_t)=f_{j-1}(y_t)+7.$$

This way every graph G_1^j in the starter contains seven consecutive lengths, $7(j-2)+3,7(j-2)+4,\ldots,7(j-2)+9$.

Each G_1^j will now be rotated with step one $14k+8$ times. Hence, every edge of K_{14k+8} appears in exactly one rotated copy of some G_1^j and thus we have a valid G_1-decomposition.

Construction 6.2. The construction of G_2^1 is essentially the same as in Construction 6.1.

We label G_2^1 as in Figure 11 and rotate it again with step one. The edges of length 1 and $7k+3$ behave exactly the same as in Construction 6.1.

The leading vertices of edges of length 2, namely 1 and $7k+6$, differ again by $7k+3$ rotations with step one only. The edge $(1,3)$ has images $(1,3),(2,4),\ldots,(7k+2,7k+4)$ while $(7k+6,7k+8)$ has images $(7k+6,7k+8),(7k+7,7k+9),\ldots,(1,3)$. The edge $(1,3)$ is used twice while $(7k+5,7k+7)$ not at all. Hence, in the $(7k+4)$-th copy of G_2^1 we use the edge $(7k+5,7k+7)$ instead of $(1,3)$.

Finally, the only edge of length $7k+4$ is $(1,7k+5)$ and we have used every edge of the four used lengths exactly once.

The rest of the construction is identical to Construction 6.1.

Construction 6.3. The construction of G_3^1 is even simpler than the previous two. It is so because for all three pairs of edges of the same length, $1,2,$ and $7k+3$ we are able to place the same-length edges at distance $7k+4$ apart and no edge of the complete graph is used twice (or not at all). A labeling is shown in Figure 13. The rest of the construction is identical to the previous two.

We now again have all ingredients needed for the complete result on this subclass for $n\equiv 8 \pmod{14}$.

Theorem 6.4. *The complete graph K_{14k+8} is G-decomposable for every graph $G \in \mathcal{G}$ if and only if $k \geq 1$.*

Proof. Follows directly from Constructions 6.1–6.3 and the fact that the graphs have more than eight vertices. □

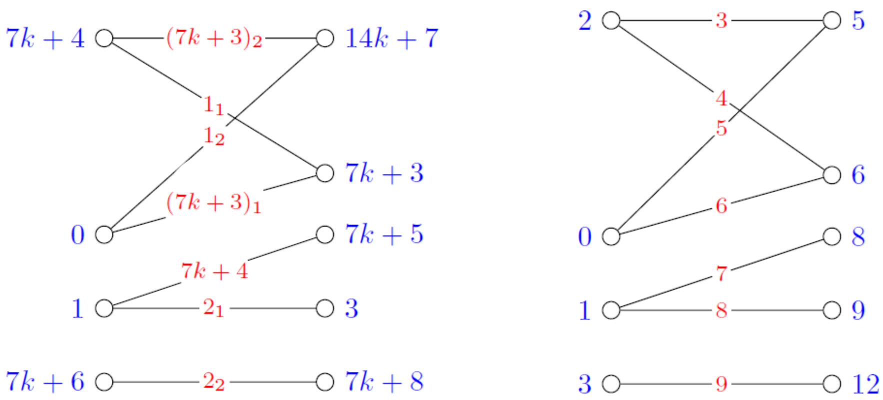

Fig. 11. G_2^1 (left) and G_2^2 (right)

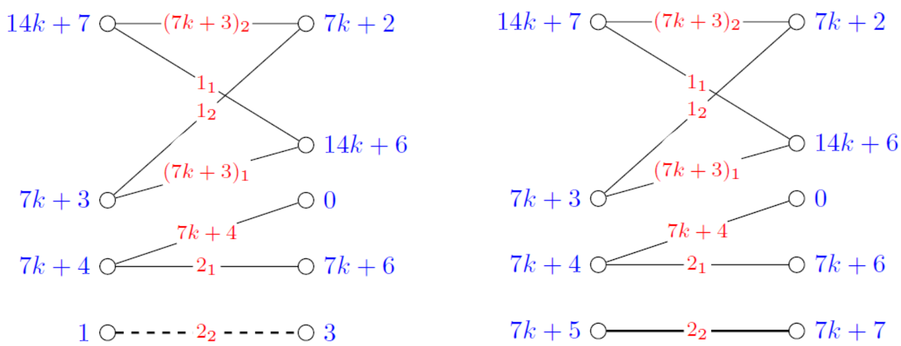

Fig. 12. The $(7k+4)$-th copy of G_2^1 before (L) and after adjustment (R)

7. Conclusion

Our main result now follows.

Theorem 7.1. *The complete graph K_n has a G-decomposition for any $G \in \mathcal{G}$ if and only if $n \equiv 0, 1 \pmod 7, n > 8$.*

Proof. The case of $n \equiv 0, 1 \pmod{14}$ is covered by Theorem 4.1. The case of $n \equiv 7 \pmod{14}$ is proved in Theorem 5.6 and for $n \equiv 8 \pmod{14}$ in Theorem 6.4. □

We now combine our result with previous results by Froncek and Kubesa stated in Theorems 2.3 and 2.7 to obtain the complete spectrum for bipartite unicyclic graphs on seven edges.

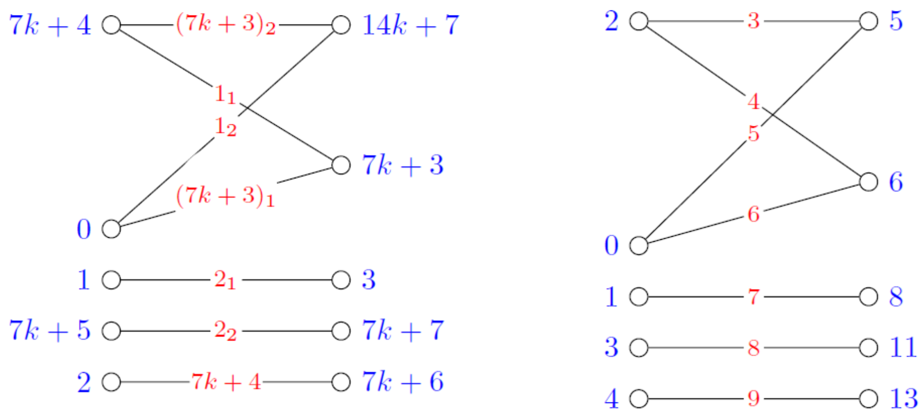

Fig. 13. G_3^1 (left) and G_3^2 (right)

Theorem 7.2. *The complete graph K_n has a G-decomposition for any bipartite unicyclic graph on 7 edges if and only if $n \equiv 0, 1 \pmod 7$ and $n \geq |V(G)|$ except for three exceptions when $n = 7$ and five exceptions when $n = 8$ (two connected and three disconnected).*

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