

Decomposition of complete graphs on $n \equiv 0, 1 \pmod{14}$ vertices into disconnected unicyclic tripartite graphs on seven edges

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ABSTRACT

Let G be a disconnected tripartite unicyclic graph on seven edges with two or more connected components. We prove that G decomposes the complete graph K_n whenever $n \equiv 0, 1 \pmod{14}$ using labeling techniques.

Keywords: graph decomposition, G -design, tripartite graph, ρ -labeling

2020 Mathematics Subject Classification: 05C78, 05C51.

1. Introduction

A *decomposition* of a graph K is a collection of pairwise edge-disjoint subgraphs $\Gamma = \{G_1, G_2, \dots, G_k\}$ of K such that $E(K) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_k)$. We call such a collection Γ a *G -decomposition* if all of its members are isomorphic to some graph G . We refer to a G -decomposition of K_n as a *G -design of order n* . If we take the vertex set of K to be \mathbb{Z}_n where $|V(K)| = n$, and $\pi : v \mapsto v + 1$ is an automorphism of Γ , we call Γ *cyclic*. If instead we take the vertex set of K to be $\mathbb{Z}_{n-1} \cup \{\infty\}$ and π is an automorphism of Γ (with $\infty + 1 = \infty$ by definition), we call Γ *1-rotational*. In 1967, Rosa [12] developed a family of graph labelings now referred to as ‘Rosa-type labelings’ which ensure various G -decompositions of infinite families of complete graphs exist if G admits such a labeling.

A graph is *unicyclic* if it contains exactly one cycle. If a unicyclic graph on 7 edges admits a ρ -tripartite and a 1-rotational ρ -tripartite labeling (defined in Section 4), then it decomposes K_{14t+1} and K_{14t} for $t \geq 1$. In this paper we show that the statement

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above holds for disconnected tripartite unicyclic graphs on 7 edges. A G -decomposition of K_n into graphs on 7 edges can exist only if 7 divides $|E(K_n)|$. Therefore, a G -design of order n for a graph G on 7 edges can only be found if $n \equiv 0, 1 \pmod{7}$. The labeling methods discussed in this paper cannot be used to find a G -design of K_n if $n \equiv 7, 8 \pmod{14}$ because the number of distinct edge lengths in such a K_n is not a multiple of 7. Therefore, the scope of this paper is limited to the case where $n \equiv 0, 1 \pmod{14}$.

2. Known results

The decomposition spectrum for graphs with up to six edges has been completely determined, although the latest results on some small classes of graphs with five or six edges have not been published yet. For an overview, see, [8].

Graphs with seven edges and five vertices were investigated in [3], [4], and [11]. Cui [7], Blinco [4], and Tian, Du, and Kang [13] studied connected graphs with seven edges and six vertices.

The only disconnected graph with seven edges and six vertices is $K_4 \cup K_2$. The spectrum for this graph was determined by Tian, Du and Kang [13].

All graphs with seven edges and seven vertices are either unicyclic or disconnected. We are not aware of any attempt to classify disconnected graphs with seven edges and seven vertices.

Froncek and Kubesa classified connected unicyclic bipartite graphs with seven edges [8] and disconnected bipartite graphs with seven edges and eight vertices (which are necessarily unicyclic) [9]. Decompositions for disconnected unicyclic bipartite graphs with seven edges and more than eight vertices were completely characterized by Banegas, Carlson, and Froncek [1].

Connected graphs with seven edges and eight vertices are trees, which were investigated by Huang and Rosa [10]. Forests with seven edges were studied by Banegas and Freyberg [2].

In this paper, we focus on disconnected tripartite unicyclic graphs.

3. Catalog of graphs

Below we present a catalog of all disconnected unicyclic tripartite graphs on seven edges. There are 27 such graphs; 25 of them contain a triangle (shown in Figures 1–25), and 2 contain a pentagon (Figures 26–27).

Figures 1–15 show triangular graphs with two components, Figures 16–22 triangular graphs with three components, Figures 23–24 with triangular graphs with four components, and Figure 25 the triangular graph with five components. Figures 26 and 27 show the pentagonal graphs with two and three components, respectively.

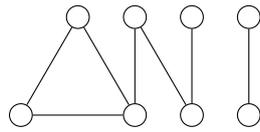


Fig. 1. G_1

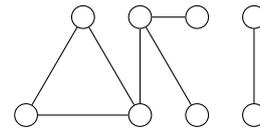


Fig. 2. G_2

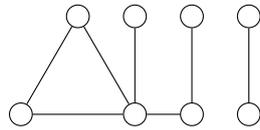


Fig. 3. G_3

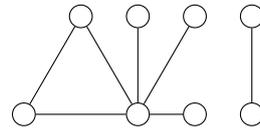


Fig. 4. G_4

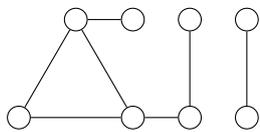


Fig. 5. G_5

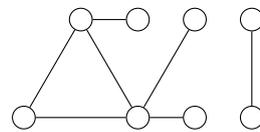


Fig. 6. G_6

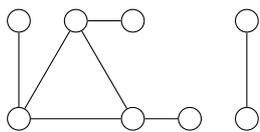


Fig. 7. G_7

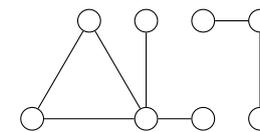


Fig. 8. G_8

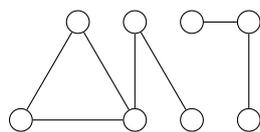


Fig. 9. G_9

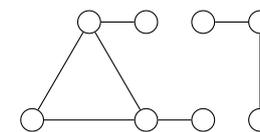


Fig. 10. G_{10}

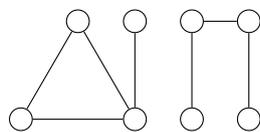


Fig. 11. G_{11}

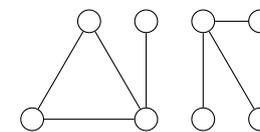


Fig. 12. G_{12}

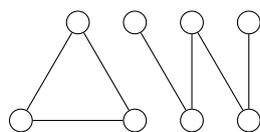


Fig. 13. G_{13}

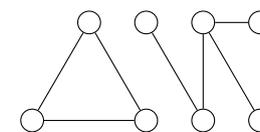
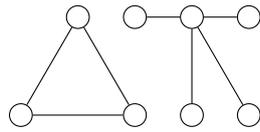
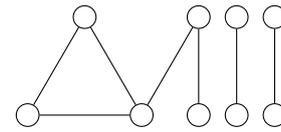
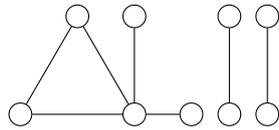
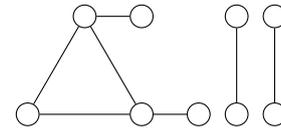
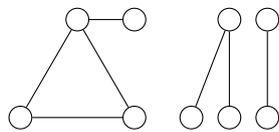
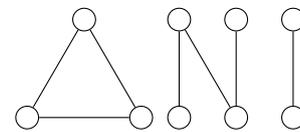
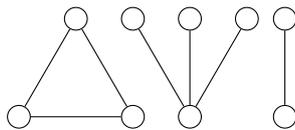
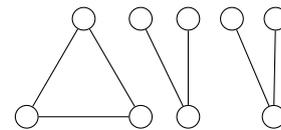
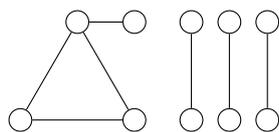
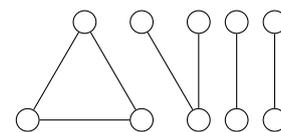
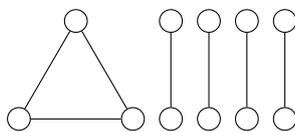
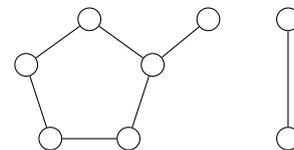
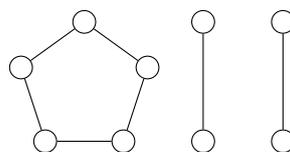


Fig. 14. G_{14}

Fig. 15. G_{15} Fig. 16. G_{16} Fig. 17. G_{17} Fig. 18. G_{18} Fig. 19. G_{19} Fig. 20. G_{20} Fig. 21. G_{21} Fig. 22. G_{22} Fig. 23. G_{23} Fig. 24. G_{24} Fig. 25. G_{25} Fig. 26. G_{26} Fig. 27. G_{27}

4. Tools: definitions and theorems

Some definitions and tools taken from previous results in graph decompositions are shown here. These tools help identify if a graph G decomposes a certain K_m by labeling the vertices and edges of G such that certain conditions are met. However, before getting into more definitions and theorems, consider how a decomposition of the graph K_n into a subgraph G must behave. Note that K_m has $\frac{m(m-1)}{2}$ edges. Since we want K_m to decompose into isomorphic copies of G , the number of edges of G must divide $\frac{m(m-1)}{2}$. As stated in the introduction, for the graphs that we are working with (all on 7 edges), they can only decompose complete graphs of form K_m where $m \equiv 0, 1 \pmod{7}$, as 7 divides $\frac{m(m-1)}{2}$ only when $m \equiv 0, 1 \pmod{7}$.

In 1967, Rosa [12] introduced a definition of what he then called a graph ρ -valuation. This laid foundations for later generalizations or modifications that we will list below.

Definition 4.1. [12] Let G be a graph with n edges. A ρ -labeling of G is a one-to-one function $f : V(G) \rightarrow \{0, 1, \dots, 2n\}$ inducing a function $\ell : E(G) \rightarrow \{1, 2, \dots, n\}$ defined as

$$\ell(uv) = \min\{|f(u) - f(v)|, 2n + 1 - |f(u) - f(v)|\},$$

with the property that $\{\ell(uv) : uv \in E(G)\} = \{1, 2, \dots, n\}$.

This definition is the foundation upon which everything else we will talk about is built. Finding ρ -labelings is instrumental in identifying decompositions of complete graphs, and many other methods striving to achieve the same goal are variations of ρ -labelings.

The following definition is a slight variation on ρ -labelings that is only defined for bipartite graphs.

Definition 4.2. [14] An *ordered ρ -labeling* is a ρ -labeling f for a bipartite graph G with bipartition $\{A, B\}$ such that $f(a) < f(b)$ for $a \in A$ and $b \in B$ when a and b are neighbors.

When we can ‘rotate’ one copy of G through the whole K_m to obtain a G -design, we speak about a *cyclic decomposition*.

Definition 4.3. Let $V(K_m) = \mathbb{Z}_m$, and let G be a subgraph of K_m . The *length* of an edge $ij \in E(G)$ is defined as $\min\{|i - j|, m - |i - j|\}$. The act of *clicking* G implies applying the permutation $i \rightarrow i + 1$ to $V(G)$. A G -decomposition Γ of K_m is *cyclic* if clicking is an automorphism of Γ .

Clicking can be intuitively thought of as visualizing a subgraph of some K_m on top of that K_m with vertices arranged around a regular polygon with m sides, and then "rotating" all vertices by one vertex clockwise to create a new subgraph isomorphic to the original. If one imagines continuing this act of clicking m times, we get an intuitive idea leading to the following theorem:

Theorem 4.4. [12]. *A cyclic decomposition of the complete graph K_{2n+1} into subgraphs*

isomorphic to a given graph G with n edges exists if and only if there exists a ρ -labeling of the graph G .

We see from Theorem 4.4 that we can only form a cyclic decomposition into subgraphs with 7 edges for the complete graph K_{15} . However, our goal is to find G -designs for all K_{2nk+1} and Theorem 4.4 only covers one of the cases of our k , where $k = 1$. Luckily, the Theorem was extended by El-Zanati, Vanden Eynden and Punnim [14] into a more general result about complete graphs of the form K_{2nk+1} .

Theorem 4.5. [14] *Let G be a bipartite graph with n edges. If G admits an ordered ρ -labeling, then there exists a cyclic G -decomposition of K_{2nk+1} for any positive integer k .*

Definitions 4.1, 4.2 and Theorem 4.5 together create a relatively simple technique to find decompositions. Most other labelings of graphs in decomposition problems stem from a variation of ρ -labelings.

A variation extends the utility of ordered ρ -labelings (which only apply to bipartite graphs) to tripartite graphs. The changes needed such that there is a similar theorem to 4.5 for tripartite graphs necessarily requires a bit more of a complex labeling. Such generalization was obtained by Bunge, Chantasartrassmee, El-Zanati, and Vanden Eynden [6].

Definition 4.6. [6] Let G be a tripartite graph with n edges and vertex tripartition $\{A, B, C\}$. A ρ -tripartite labeling of G is a ρ -labeling f that satisfies the following:

- $f(a) < f(v)$ for all $av \in E(G)$ with $a \in A$.
- For every edge $bc \in E(G)$ with $b \in B$ and $c \in C$, there exists an edge $b'c' \in E(G)$ with $b' \in B$ and $c' \in C$ where $|f(c) - f(b)| + |f(c') - f(b')| = 2n$.
- for any $b \in B$ and $c \in C$, $|f(b) - f(c)| \neq 2n$.

The definition provides us with an important result that is one of the two main building blocks in our constructions.

Theorem 4.7. [6] *Let G be a tripartite graph with n edges. If G admits a ρ -tripartite labeling, then there exists a cyclic G -decomposition of K_{2nk+1} for any positive integer k .*

Theorem 4.7 further expands the set of graphs we can decompose complete graphs into, specifically allowing for decompositions into tripartite graphs. The following definitions by Bunge [5] allow finding G -designs for complete graphs K_{2nk} .

Definition 4.8. Let $V(K_m) = \mathbb{Z}_{m-1} \cup \{\infty\}$, and let G be a subgraph of K_m . The *length* of an edge $ij \in E(G)$ where $i, j \neq \infty$ is defined again as $\min\{|i - j|, m - 1 - |i - j|\}$. The act of *clicking* G implies applying the same permutation $i \rightarrow i + 1$ to $V(G)$ for integer vertices, with the convention $\infty + 1 \rightarrow \infty$. A G -decomposition Γ of K_m is *1-rotational* if clicking is an automorphism of Γ .

Observe that this definition is very similar to that of a cyclic decomposition. Visually,

when looking at K_m , it looks like the standard way to draw K_m save for a vertex that lies in the center. Clicking a subgraph is rotating that subgraph about the center vertex.

Definition 4.9. [5] Let G be a graph with n edges, no isolated vertices, and a vertex w of degree 1. A 1-rotational ρ -labeling of G is a one-to-one function $f : V(G) \rightarrow [0, 2n - 2] \cup \{\infty\}$ where $f(w) = \infty$ and such that f is a ρ -labeling of $G - \{w\}$.

Definition 4.10. [5] Let G be a tripartite graph with n edges, vertex tripartition $\{A, B, C\}$, and edge uw where $\deg w = 1$. A 1-rotational ρ -tripartite labeling of G is a 1-rotational ρ -labeling f that satisfies the following:

- $f(w) = \infty$
- $f(a) < f(v)$ for all $av \in E(G)$ with $a \in A, v \neq w$.
- For every edge $bc \in E(G)$ with $b \in B$ and $c \in C$, there exists an edge $b'c' \in E(G)$ with $b' \in B$ and $c' \in C$ where $|f(c) - f(b)| + |f(c') - f(b')| = 2n$.

Definition 4.10 is to Definition 4.6 as the definition of a 1-rotational decomposition is to a cyclic decomposition. These tripartite labelings allow us to find decompositions after invoking the following theorem.

Theorem 4.11. [5] *Let G be a tripartite graph with n edges and a vertex of degree 1. If G admits a 1-rotational ρ -tripartite labeling, then there exists a 1-rotational G -decomposition of K_{2nk} for any positive integer k .*

5. Labelings and main result

Below we present ρ -tripartite labelings and 1-rotational ρ -tripartite labelings for all 27 disconnected unicyclic tripartite graphs on 7 edges. In each figure, the ρ -tripartite labeling is on the left, and the 1-rotational ρ -tripartite labeling is on the right. The partite sets A, B, C are in one row each, and the row is labeled by the corresponding letter in the middle between the two labelings.

Now we are ready to prove our main result.

Theorem 5.1. *Every disconnected tripartite unicyclic graph on seven edges decomposes complete graphs K_{14k} and K_{14k+1} for every $k \geq 1$.*

Proof. The assertion follows directly from Theorems 4.7 and 4.11 and our labelings in Figures 28–54. □

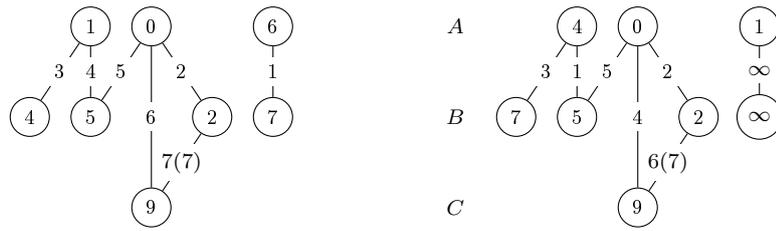


Fig. 28. G_1

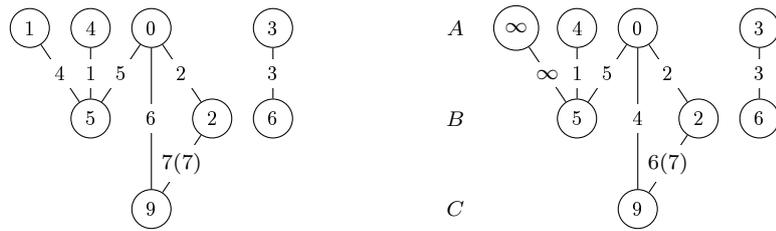


Fig. 29. G_2

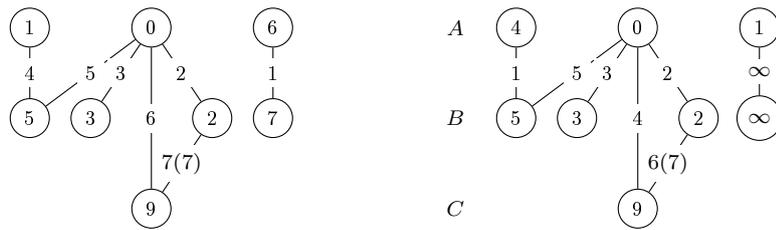


Fig. 30. G_3

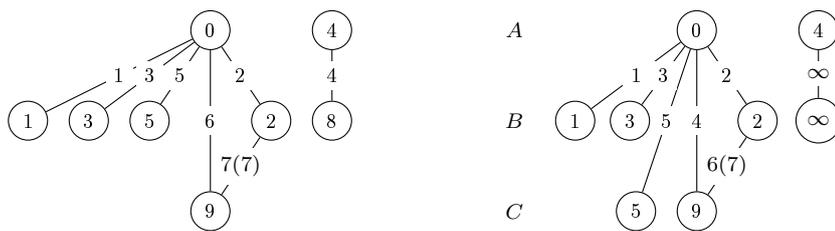


Fig. 31. G_4

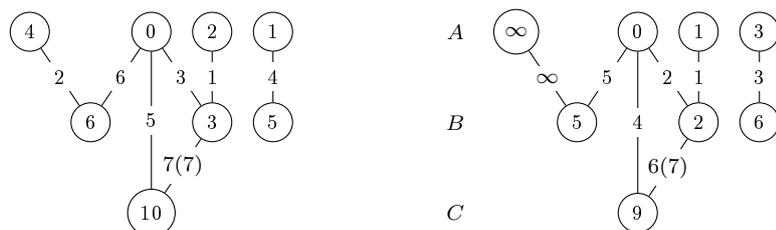


Fig. 32. G_5

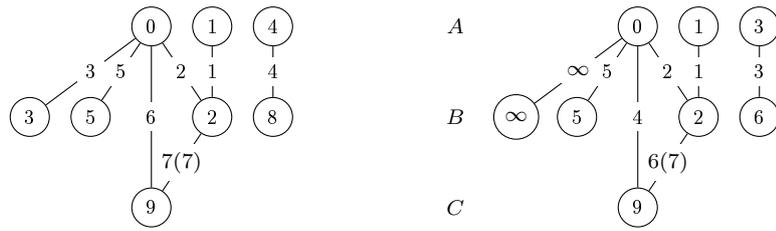


Fig. 33. G_6

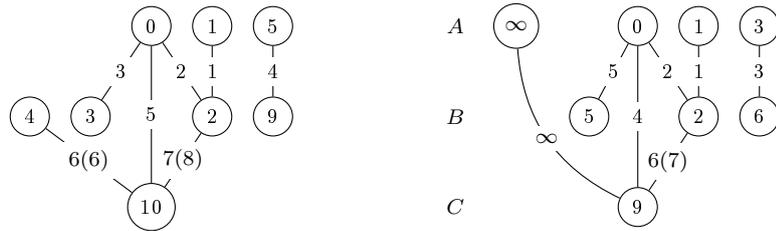


Fig. 34. G_7

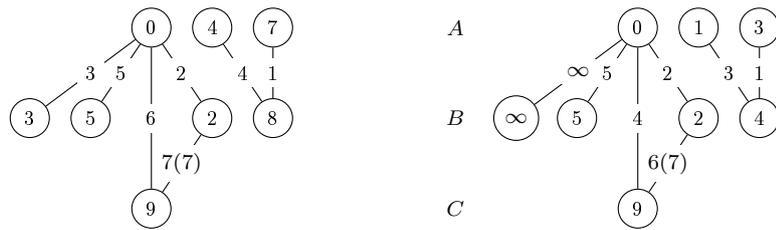


Fig. 35. G_8

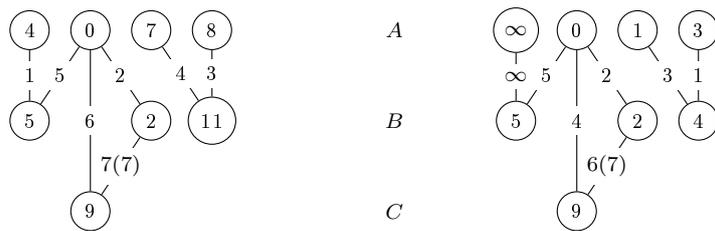


Fig. 36. G_9

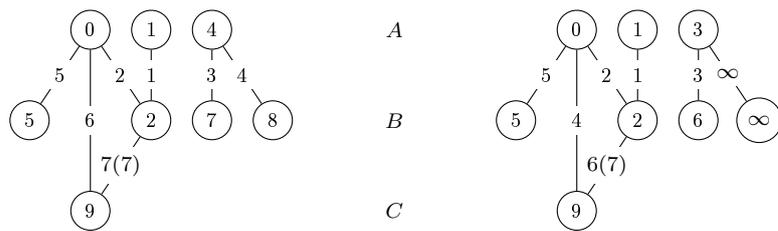


Fig. 37. G_{10}

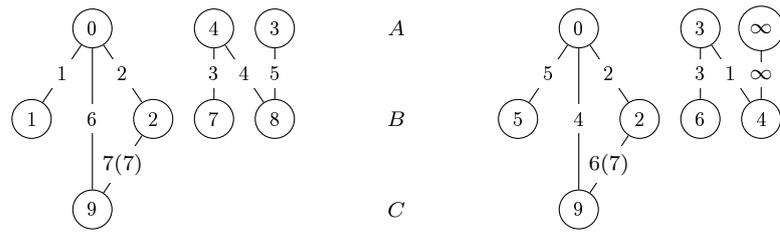


Fig. 38. G_{11}

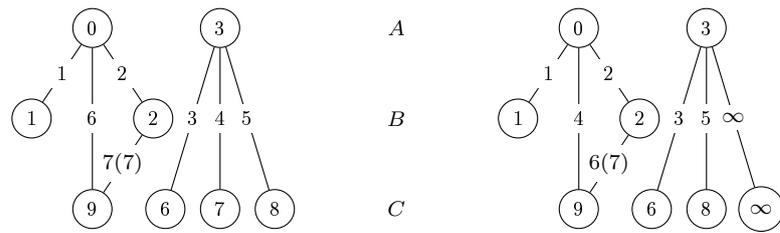


Fig. 39. G_{12}

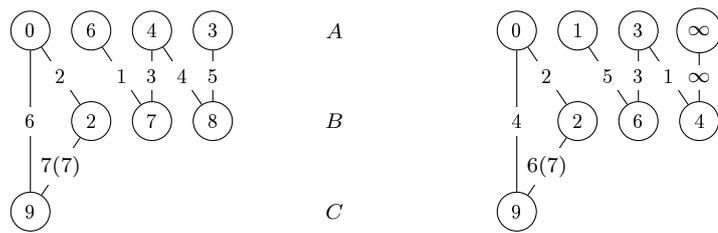


Fig. 40. G_{13}

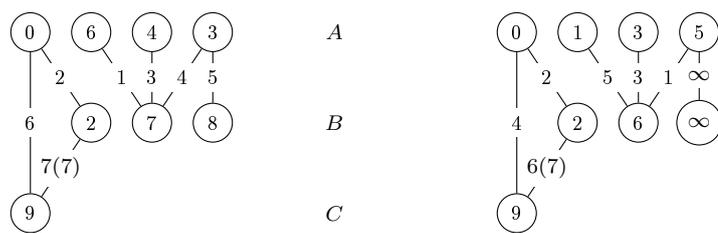


Fig. 41. G_{14}

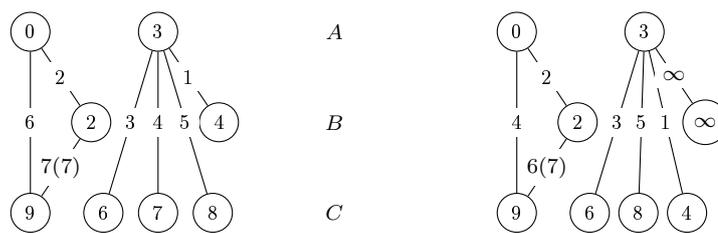


Fig. 42. G_{15}

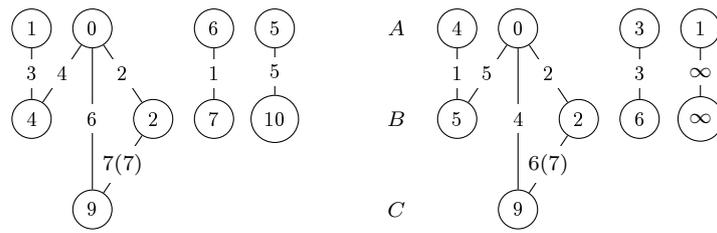


Fig. 43. G_{16}

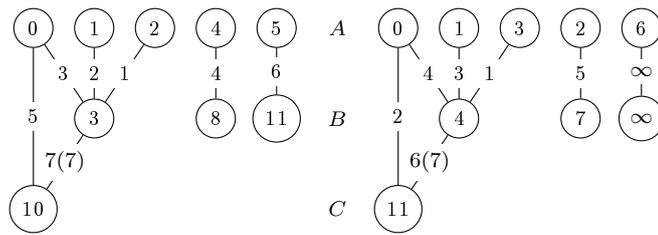


Fig. 44. G_{17}

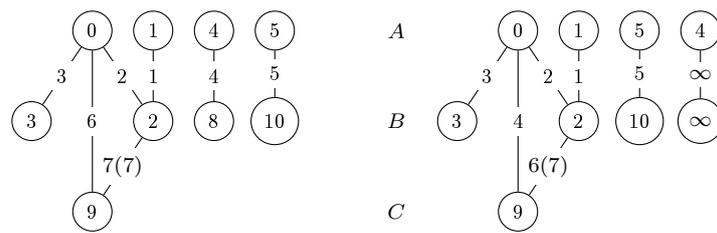


Fig. 45. G_{18}

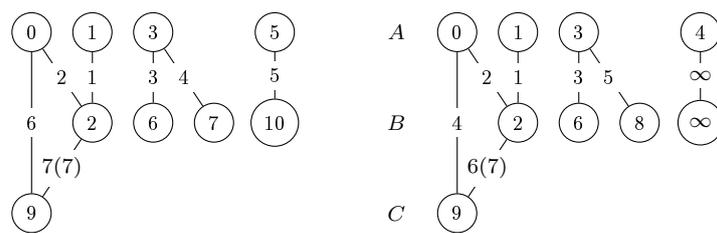


Fig. 46. G_{19}

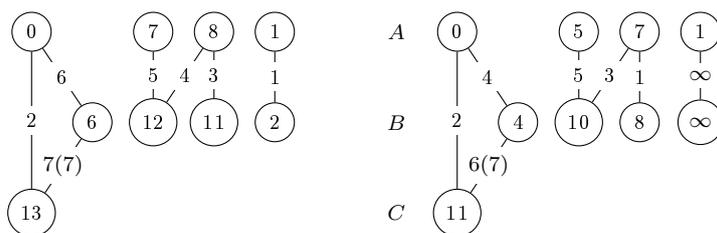


Fig. 47. G_{20}

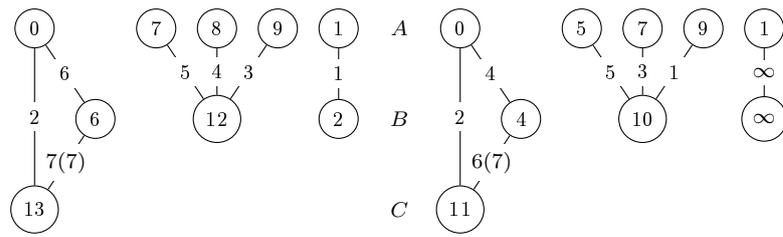


Fig. 48. G_{21}

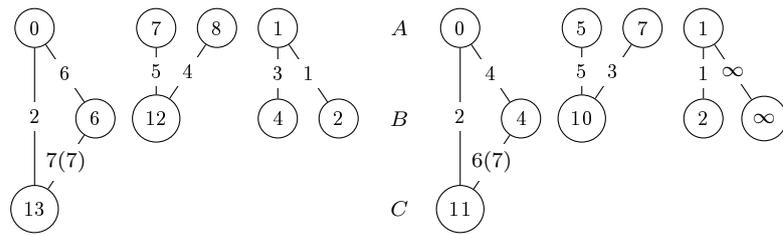


Fig. 49. G_{22}

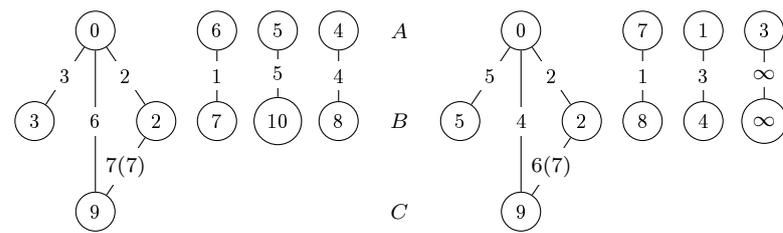


Fig. 50. G_{23}

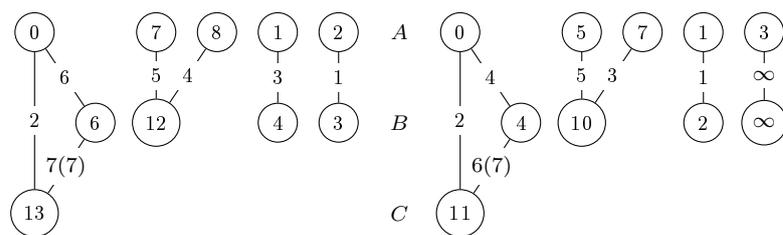


Fig. 51. G_{24}

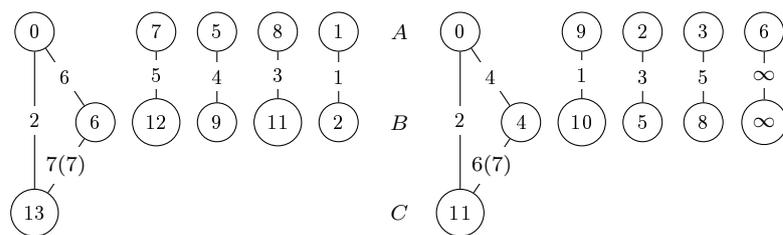


Fig. 52. G_{25}

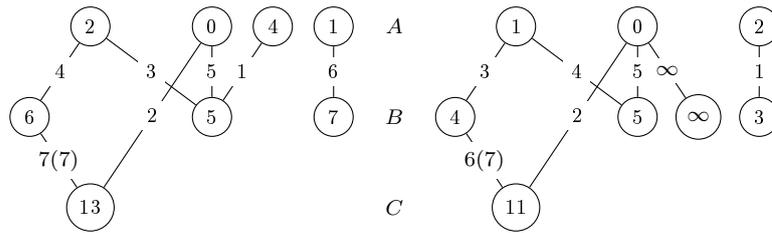


Fig. 53. G_{26}

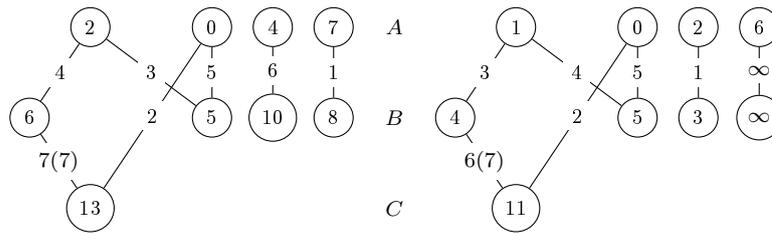


Fig. 54. G_{27}

6. Conclusion

Our result completely characterizes the existence of G -designs for disconnected unicyclic tripartite graphs for the complete graphs K_{14k} and K_{14k+1} . A similar result for connected unicyclic tripartite graphs is currently under preparation by a subset of the authors of this paper.

The case of the complete graphs K_{14k+7} and K_{14k+8} remains still open.

The results presented in this paper for the graphs with two components were obtained by M. Heck and Y. Hong under the supervision and with co-authorship of B. Freyberg. Y. Hong’s participation was partially funded by the University of Minnesota Office of Undergraduate Research.

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