J. COMBIN. MATH. COMBIN. COMPUT. 129 (2026) 17–31

Journal of Combinatorial Mathematics
and Combinatorial Computing
www.combinatorialpress.com/jcmcc

Multiple Ramsey numbers of disconnected graphs of size 3

Emma Jent and Ping Zhang

ABSTRACT

For a graph F and a positive integer t, the vertex-disjoint Ramsey number $VR_t(F)$ is the minimum positive integer n such that every red-blue coloring of the edges of the complete graph K_n of order n results in t pairwise vertex-disjoint monochromatic copies of subgraphs isomorphic to F, while the edge-disjoint Ramsey number $ER_t(F)$ is the corresponding number for edge-disjoint subgraphs. These numbers have been investigated for the three connected graphs K_3, P_4 and $K_{1,3}$ of size 3. For two vertex-disjoint graphs G and H, let $G + H$ denote the union of G and H. Here we study these numbers for the two disconnected graphs $3K_2$ and $P_3 + P_2$ of size 3. It is shown that $VR_t(3K_2) = 6t + 2$ and $VR_t(P_3 + P_2) = 5t + 1$ for every positive integer t. The numbers $ER_t(3K_2)$ and $ER_t(P_3 + P_2)$ are determined for $t \leq 4$ and bounds are established for $ER_t(3K_2)$ and $ER_t(P_3 + P_2)$ when $t \geq 5$. Other results and problems are presented as well.

Keywords: red-blue coloring, edge-disjoint and vertex-disjoint Ramsey numbers

2020 Mathematics Subject Classification: 05C55, 05C35, 05C15.

1. Introduction

In a *red-blue coloring* of a graph G, every edge of G is colored red or blue. For two graphs F and H, the well-known *Ramsey number* $R(F, H)$ is the minimum positive integer n such that for every red-blue coloring of the complete graph K_n of order n, there is either a subgraph of K_n isomorphic to F all of whose edges are colored red (a *red* F) or a subgraph of K_n isomorphic to H all of whose edges are colored blue (a *blue* H). Therefore, for a single graph F, the *Ramsey number* $R(F, F)$, also denoted by $R(F)$, is the minimum positive integer n such that for every red-blue coloring of K_n, there is a subgraph of K_n

Published Online 27 January 2026.

DOI: 10.61091/jcmcc129-02

isomorphic to F all of whose edges are colored the same (a *monochromatic* F). While these numbers exist for every graph F (see [8]), it is challenging in general to determine the exact value $R(F)$ for many graphs F. For example, $R(K_n)$ is only known when $n \leq 4$ (see [2, 6]).

Perhaps the best known Ramsey number is $R(K_3)$ which is 6. This result was established by Greenwood and Gleason [6] in 1955. In fact, this result essentially appeared as Problem A2 in the 1953 William Lowell Putnam Exam (see [6]). While every red-blue coloring of K_6 always results in a monochromatic triangle K_3, it turns out that every red-blue coloring of K_6 always results in at least two monochromatic triangles (see [5]). However, it is not true that every red-blue coloring of K_6 always results in two edge-disjoint monochromatic triangles. For example, in the red-blue coloring of K_6 in Figure 1 (where a bold edge represents a red edge and a thin edge represents a blue edge), there do not exist two edge-disjoint monochromatic triangles.

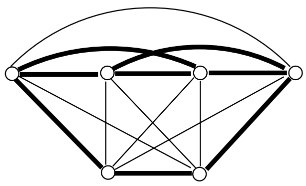

Fig. 1. A red-blue coloring of K_6

On the other hand, every red-blue coloring of K_7 always results in two edge-disjoint monochromatic triangles.

Proposition 1.1. [1] *Every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles.*

While every red-blue coloring of K_7 results in two edge-disjoint monochromatic triangles, it is not true that every red-blue coloring of K_7 always results in two vertex-disjoint monochromatic triangles. For example, the red-blue coloring of K_7 of Figure 2 with red subgraph $K_5 + K_2$ and blue subgraph $K_{2,5}$ does not contain two vertex-disjoint monochromatic copies of K_3. However, every red-blue coloring of K_8 results in two vertex-disjoint monochromatic triangles.

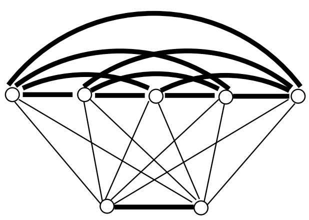

Fig. 2. A red-blue coloring of K_7

Proposition 1.2. [1] *Every red-blue coloring of K_8 results in two vertex-disjoint monochromatic triangles.*

Consequently, 6 is the smallest order n of a complete graph K_n for which every red-blue coloring results in a monochromatic triangle, 7 is the smallest order n of a complete graph K_n for which every red-blue coloring results in two edge-disjoint monochromatic triangles, and 8 is the smallest order n of a complete graph K_n for which every red-blue coloring results in two vertex-disjoint monochromatic triangles. These facts gave rise to the idea of extending Ramsey numbers to *multiple Ramsey numbers* in [1].

Let t be a positive integer and F a graph without isolated vertices. The *vertex-disjoint Ramsey number* $VR_t(F)$ is the minimum positive integer n such that for every red-blue coloring of K_n, there are at least t pairwise vertex-disjoint monochromatic copies of F. Thus, $VR_1(F) = R(F)$ for every graph F. Since the Ramsey number $R(tF)$ exists for the union tF of t pairwise vertex-disjoint copies of a graph F by a result of Ramsey [8], we have the following.

Observation 1.3. [1] *For every graph F without isolated vertices and every positive integer t, the number $VR_t(F)$ exists and $t|V(F)| \leq VR_t(F) \leq R(tF)$. Furthermore, $VR_t(F) \leq VR_{t+1}(F)$.*

Let t be a positive integer and let F be a graph without isolated vertices. The *edge-disjoint Ramsey number* $ER_t(F)$ of F is the minimum positive integer n such that for every red-blue coloring of K_n, there are at least t pairwise edge-disjoint monochromatic copies of F. Thus, $ER_1(F) = R(F)$ for every graph F.

Observation 1.4. [1] *For every graph F without isolated vertices and every positive integer t, the number $ER_t(F)$ exists and $ER_t(F) \leq VR_t(F)$. Furthermore, $ER_t(F) \leq ER_{t+1}(F)$.*

The concepts of vertex-disjoint and edge-disjoint Ramsey numbers of graphs were introduced and studied in [1] and studied further in [7]. In [1], the numbers $VR_t(F)$ and $ER_t(F)$ were investigated for the two connected graphs of order 3, namely K_3 and P_3. The following two results were obtained on vertex-disjoint Ramsey numbers of these two graphs.

Theorem 1.5. [1] *For an integer $t \geq 2$, $VR_t(K_3) = 3t + 2$.*

Theorem 1.6. [1] *For every positive integer t, $VR_t(P_3) = 3t$.*

Regarding edge-disjoint Ramsey numbers of K_3, it was also shown in [1] that $ER_2(K_3) = 7$, $ER_3(K_3) = 9$, and $ER_4(K_3) = 10$. The edge-disjoint Ramsey numbers of P_3 were obtained in [1] as well.

Theorem 1.7. [1] *For every positive integer t, $ER_t(P_3) = \lceil 2\sqrt{t} + 1 \rceil$.*

In [7], the numbers $VR_t(F)$ and $ER_t(F)$ were investigated for the connected graphs F of size 3 different from K_3, namely the star $K_{1,3}$ and the path P_4 of order 4. The following two results were obtained on vertex-disjoint Ramsey numbers of these two graphs.

Theorem 1.8. [7] *For each positive integer t, $VR_t(P_4) = 4t + 1$.*

Theorem 1.9. [7] $VR_1(K_{1,3}) = R(K_{1,3}) = 6$ *and for each integer $t \geq 2$, $VR_t(K_{1,3}) = 4t$.*

It was also shown in [7] that $ER_2(K_{1,3}) = 6$ and $ER_t(K_{1,3}) = 7$ for $t = 3, 4$, while $ER_t(P_4) = t + 3$ for $2 \leq t \leq 5$. Upper and lower bounds for these two numbers were established in [7] as well.

Theorem 1.10. [7] *For each integer $t \geq 4$, $\left\lceil \frac{3+\sqrt{9+24t}}{2} \right\rceil \leq ER_t(K_{1,3}) \leq t + 3$.*

Theorem 1.11. [7] *For each integer $t \geq 2$, $ER_t(P_4) \leq t + 3$.*
(1) *If $ER_t(P_4) \not\equiv 0 \pmod 3$, then $ER_t(P_4) \geq \left\lceil \frac{3+\sqrt{1+24t}}{2} \right\rceil$.*
(2) *If $ER_t(P_4) \equiv 0 \pmod 3$, then $ER_t(P_4) \geq \left\lceil \frac{3+\sqrt{9+24t}}{2} \right\rceil$.*

For two vertex-disjoint graphs G and H, let $G + H$ denote the *union* of G and H with $V(G + H) = V(G) \cup V(H)$ and $E(G + H) = E(G) \cup E(H)$. The *Cartesian product* of G and H is denoted by $G \,\square\, H$ with $V(G \,\square\, H) = V(G) \times V(H)$ where two distinct vertices (u, v) and (x, y) are adjacent in $G \,\square\, H$ if either $u = x$ and $vy \in E(H)$ or $v = y$ and $ux \in E(G)$. We refer to the book [2] for notation and terminology not defined here. The goal of this paper is to study $VR_t(F)$ and $ER_t(F)$ for the two disconnected connected graphs F of size 3, namely the matching $3K_2$ and the union $P_3 + P_2$ of P_3 and P_2.

2. On the multiple Ramsey numbers $VR_t(3K_2)$ and $ER_t(3K_2)$

We begin with the vertex-disjoint Ramsey numbers of the matching $3K_2$ of size 3. An upper bound for $VR_t(F)$ was established in [7] for every graph F of order at least 4 with no isolated vertices.

Proposition 2.1. [7] *Let F be a graph of order $n \geq 4$ without isolated vertices. If there is a positive integer t_0 such that $VR_{t_0}(F) \leq q$, then $VR_t(F) \leq q + (t - t_0)n$ for every integer $t \geq t_0$.*

With the aid of Proposition 2.1, we are able to determine $VR_t(3K_2)$ for every positive integer t.

Theorem 2.2. *For each positive integer t, $VR_t(3K_2) = 6t + 2$.*

Proof. Since $VR_1(3K_2) = R(3K_2) = 8$ and the order of $3K_2$ is 6, it follows by Proposition 2.1 that $VR_t(3K_2) \leq 8 + 6(t - 1) = 6t + 2$. Next, consider the red-blue coloring of $G = K_{6t+1}$ with red subgraph $G_r = K_2 \vee \overline{K_{6t-1}}$ (the join of K_2 and the empty graph $\overline{K_{6t-1}}$ of order $6t - 1$) and blue subgraph $G_b = K_{6t-1}$. Since the edge-independence number of G_r is $\alpha'(G_r) = 2$, there is no red $3K_2$ in G_r. The blue subgraph G_b has order $6t-1$ and so there are no t pairwise vertex-disjoint blue copies of $3K_2$. Since this red-blue coloring fails to have t pairwise vertex-disjoint monochromatic copies of $3K_2$, it follows that $VR_t(3K_2) \geq 6t+2$ and so $VR_t(3K_2) = 6t+2$. □

For the edge-disjoint Ramsey number $ER_t(3K_2)$, we begin by determining $ER_t(3K_2)$ for small values of t, namely $1 \leq t \leq 4$. First, we provide some preliminary information. The following two results are known.

Theorem 2.3. [4, 3] *For every positive integer n, $R(nK_2) = 3n-1$.*

Thus, $ER_1(3K_2) = R(3K_2) = 8$ by Theorem 2.3.

Theorem 2.4. [2] *For every integer $k \geq 1$, the complete graph K_{2k} can be factored into $k-1$ Hamiltonian cycles and a 1-factor.*

The *matching* (or *edge-independence*) *number* $\alpha'(G)$ of a graph G is the maximum number of edges in G, no two of which are adjacent. In order to determine $ER_t(3K_2)$ for $2 \leq t \leq 4$, the following lemma will be useful.

Lemma 2.5. *If G is a graph of order 8 and size 10 or more, then either*

$$\alpha'(G) \geq 3, \quad G \in \{K_5 + \overline{K}_3,\ (K_3 + \overline{K_4}) \vee K_1\},\ \text{or } G \subseteq K_2 \vee \overline{K}_6.$$

Proof. Let $P = (v_0, v_1, \ldots, v_k)$ be a path of greatest length k in the graph G. If $k \geq 5$, then $\{v_0v_1,\ v_2v_3,\ v_4v_5\}$ is an independent set of three edges in G and thus $\alpha'(G) \geq 3$. Hence, we may assume that $2 \leq k \leq 4$. We consider three cases according to the value of k.

Case 1. $k = 2$. Thus, every nonempty component of G is either K_3 or a star. Therefore, $|E(G)| \leq 7$, a contradiction.

Case 2. $k = 3$. Thus, $P = (v_0, v_1, v_2, v_3)$ is a path of greatest length 3 in G. Let v_4, v_5, v_6, v_7 be the remaining vertices of G. Then $v_0v_i, v_3v_i \notin E(G)$ for $i = 4, 5, 6, 7$, for otherwise G contains a path of length 4 or more. If any two of the four vertices v_4, v_5, v_6, v_7 are adjacent, then $\alpha'(G) \geq 3$. Therefore, we may assume that $\{v_4, v_5, v_6, v_7\}$ is an independent set of vertices in G. Let F be the bipartite graph with partite sets $\{v_0, v_1, v_2, v_3\}$ and $\{v_4, v_5, v_6, v_7\}$. Then $|E(F)| \leq 8$. Let $H = G[\{v_0, v_1, v_2, v_3\}]$ be the subgraph induced by $\{v_0, v_1, v_2, v_3\}$ in G. Then $E(G) = E(F) \cup E(H)$, see Figure 3.

If H is a Hamiltonian subgraph of G, then no vertex of H is adjacent to any vertex in $\{v_4,\ v_5,\ v_6,\ v_7\}$, for otherwise G contains a path of length 4 or more. Therefore, if H is Hamiltonian, then $|E(G)| = |E(H)| \leq 6$, a contradiction. Thus, we may assume that H is not Hamiltonian. Thus, $v_0v_3 \notin E(H)$ and at most one of v_0v_2 and v_1v_3 is an edge of H. Therefore, $|E(H)| \leq 4$. If both v_1 and v_2 are adjacent to a vertex v_i where $4 \leq i \leq 7$, then G contains a path of length 4, a contradiction. Thus, $|E(F)| \leq 4$ and so $|E(G)| = |E(H)| + |E(F)| \leq 4 + 4 = 8$, a contradiction.

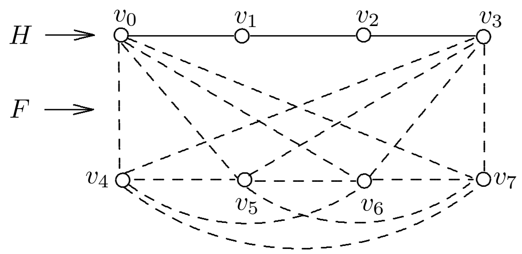

Fig. 3. A step in the proof of Case 2

Case 3. $k = 4$. Thus, $P = (v_0, v_1, v_2, v_3, v_4)$ is a path of greatest length 4 in G. Let v_5, v_6, v_7 be the remaining vertices of G. Then $v_0v_i, v_4v_i \notin E(G)$ for $i = 5, 6, 7$, for otherwise G contains a path of length 5 or more. If any two of the four vertices v_2, v_5, v_6, v_7 are adjacent, then $\alpha'(G) \geq 3$. Therefore, we may assume that $\{v_2, v_5, v_6, v_7\}$ is an independent set of vertices in G. Let F be the bipartite graph with partite sets $\{v_0, v_1, \ldots, v_4\}$ and $\{v_5, v_6, v_7\}$. Then $|E(F)|\leq 6$. Let $H = G[\{v_0, v_1, \ldots, v_4\}]$ be the subgraph induced by $\{v_0, v_1, \ldots, v_4\}$ in G. Then $E(G) = E(F) \cup E(H)$, see Figure 4.

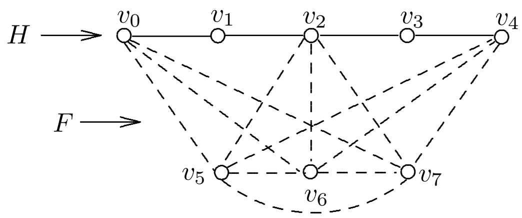

Fig. 4. A step in the proof of Case 3

If H is a Hamiltonian subgraph of G, then no vertex of H is adjacent to any vertex in $\{v_5,\ v_6,\ v_7\}$, for otherwise G contains a path of length 5 or more. Therefore, if H is Hamiltonian, then $|E(G)|= |E(H)|\leq 10$. Since $|E(G)|\geq 10$, it follows that $|E(G)|= 10$ and $G = K_5 + \overline{K}_3$ in which $G[\{v_0, v_1, \ldots, v_4\}] = K_5$ and $G[\{v_5, v_6, v_7\}] = \overline{K}_3$. Thus, we may assume that H is not Hamiltonian and so $v_0v_4 \notin E(G)$.

$\star$ First, suppose that both v_0v_2 and v_2v_4 belong to G. Then $v_1v_4, v_0v_3 \notin E(H)$ since H is not Hamiltonian and $v_1v_i, v_4v_i \notin E(G)$ for $i = 5, 6, 7$, for otherwise $\alpha'(G) \geq 3$. Thus, $|E(G)|\leq 7$, a contradiction.

$\star$ Next suppose that exactly one of v_0v_2 and v_2v_4 belongs to G, say $v_0v_2 \in E(G)$ and $v_2v_4 \notin E(G)$. Then $v_1v_4 \notin E(H)$ (since H is not Hamiltonian) and $v_1v_i \notin E(G)$ for $i = 5, 6, 7$, for otherwise $\alpha'(G) \geq 3$. Thus, $|E(G)|\leq |E(H)|+|E(F)|\leq 7 + 3 = 10$. Since $|E(G)|\geq 10$, it follows that $|E(G)|= 10$ and so $v_1v_3 \in E(H)$ and $G = (K_3 + \overline{K}_4) \vee K_1$ where $G[\{v_0, v_1, v_2\}] = K_3$, $G[v_4, v_5, v_6, v_7\}] = \overline{K}_4$, and $\deg_G v_3 = 7$.

$\star$ Finally, suppose that $v_0v_2, v_2v_4 \notin E(G)$. Thus, the edge v_1v_3 may or may not belong to G. Then G is a subgraph of $K_2 \vee \overline{K}_6$ where $G[\{v_1, v_3\}] = K_2$ and $G[\{v_0, v_2, v_4, v_5, v_6, v_7\}] = \overline{K}_6$.

Therefore, either $\alpha'(G) \geq 3$, or $G \in \{K_5 + \overline{K}_3, (K_3 + \overline{K_4}) \vee K_1\}$, or $G \subseteq K_2 \vee \overline{K}_6$. □

Theorem 2.6. *For* $1 \leq t \leq 4$, $ER_t(3K_2) = 8$.

Proof. Since $R(3K_2) = 8$ by Theorem 2.3, we may assume that $2 \leq t \leq 4$. First, we show that $ER_2(3K_2) = 8$. Since $ER_2(3K_2) \geq R(3K_2) = 8$, it remains to show that that $ER_2(3K_2) \leq 8$. Let c be a red-blue coloring of $G = K_8$. Since $R(3K_2) = 8$, there is a monochromatic subgraph $F = 3K_2$ of G. Let $H = G - E(F)$. Then H has order 8 and size $\binom{8}{2} - 3 = 25$. Let H_r and H_b be the red and blue subgraphs of H having sizes m_r and m_b respectively. We may assume that $m_r \geq m_b$. Thus, $m_r \geq 13$. Since H_r has order 8 and size 13 or greater, it follows by Lemma 2.5 that either $\alpha'(H_r) \geq 3$ or $H_r = K_2 \vee \overline{K}_6$, which contains no $3K_2$. If $\alpha'(H_r) \geq 3$, then H_r contains a subgraph isomorphic to $3K_2$ which is edge-disjoint from F. Thus, we may assume that $H_r = K_2 \vee \overline{K}_6$. Since the complement of H_r is $\overline{H}_r = K_6 + \overline{K}_2$, it follows that $H_b = (K_6 + \overline{K}_2) - E(F)$ where $F = 3K_2$ is a matching of size 3 in K_6. By Theorem 2.4, H_b is $3K_2$-decomposable into four copies of $3K_2$. Therefore, G contains two edge-disjoint copies of $3K_2$ and so $ER_2(3K_2) \leq 8$. Therefore, $ER_2(3K_2) = 8$.

Next, we show that $ER_3(3K_2) = 8$. Since $ER_3(3K_2) \geq ER_2(3K_2) = 8$, it remains to show that that $ER_3(3K_2) \leq 8$. Let there be given a red-blue coloring of $G = K_8$. Since $ER_2(3K_2) = 8$, there are two edge-disjoint monochromatic copies F_1 and F_2 of $3K_2$. Let $H = G - [E(F_1) \cup E(F_2)]$. Then H has order 8 and size $\binom{8}{2} - 6 = 22$. Let H_r and H_b be the red and blue subgraphs of H having sizes m_r and m_b, respectively. We may assume that $m_r \geq m_b$. Thus, $m_r \geq 11$. Since H_r has order 8 and size 11 or greater, it follows by Lemma 2.5 that either $\alpha'(H_r) \geq 3$ or H_r is a subgraph of $K_2 \vee \overline{K}_6$, which contains no $3K_2$. If $\alpha'(H_r) \geq 3$, then H_r contains a subgraph isomorphic to $3K_2$ which is edge-disjoint from F_1 and F_2. Thus, we may assume that $H_r \subseteq K_2 \vee \overline{K}_6$, which contains no $3K_2$. Since the complement of $K_2 \vee \overline{K}_6$ is $K_6 + \overline{K}_2$, it follows that $K_6 + \overline{K}_2 \subseteq \overline{H}_r$. Consequently,

$$H' = (K_6 + \overline{K}_2) - [E(F_1) \cup E(F_2)] \subseteq \overline{H}_r - [E(F_1) \cup E(F_2)] \subseteq H_b.$$

While it is possible that $E(F_1) \not\subseteq E(K_6 + \overline{K}_2)$ or $E(F_2) \not\subseteq E(K_6 + \overline{K}_2)$, if $E(F_1) \cup E(F_2) \subseteq E(K_6 + \overline{K}_2)$, then $(K_6 + \overline{K}_2) - [E(F_1) \cup E(F_2)] = K_3 \square K_2$. Therefore, $K_3 \square K_2 \subseteq H' \subseteq H_b$. Hence, H' contains a monochromatic copy of $3K_2$ edge-disjoint from F_1 and F_2. Therefore, $ER_3(3K_2) \leq 8$ and so $ER_3(3K_2) = 8$.

Finally, we show that $ER_4(3K_2) = 8$. Since $ER_4(3K_2) \geq ER_3(3K_2) = 8$, it remains to show that that $ER_4(3K_2) \leq 8$. Let there be given a red-blue coloring of $G = K_8$. Let G_r and G_b be the red and blue subgraphs of G, respectively. Since $ER_3(3K_2) = 8$, there are three pairwise edge-disjoint monochromatic copies F_1, F_2, F_3 of $3K_2$. Let $H = G - [E(F_1) \cup E(F_2) \cup E(F_3)]$. Then H has order 8 and size $\binom{8}{2} - 9 = 19$. Let H_r and H_b be the red and blue subgraphs of H having sizes m_r and m_b, respectively. We may assume that $m_r \geq m_b$. Thus, $m_r \geq 10$. Since H_r has order 8 and size 10 or greater, it follows by Lemma 2.5 that either $\alpha'(H_r) \geq 3$, $H_r \in \{K_5 + \overline{K}_3,\ (K_3 + \overline{K_4}) \vee K_1\}$, or H_r is a subgraph of $K_2 \vee \overline{K}_6$. If $\alpha'(H_r) \geq 3$, then H_r contains a subgraph isomorphic to $3K_2$ which is edge-disjoint from F_1, F_2, F_3. Thus, we may assume that either $H_r \in \{K_5 + \overline{K}_3,\ (K_3 + \overline{K_4}) \vee K_1\}$ or H_r is a subgraph of $K_2 \vee \overline{K}_6$. We consider these three cases.

Case 1. $H_r = K_5 + \overline{K}_3$. Then $\overline{H}_r = K_3 \vee \overline{K}_5$. If all subgraphs F_1, F_2, F_3 are blue, then $G_r = H_r$ and $G_b = \overline{H}_r = K_3 \vee \overline{K}_5$. Since the blue subgraph $G_b = K_3 \vee \overline{K}_5$ contains five pairwise edge-disjoint copies $F_1', F_2', F_3', F_4', F_5'$ of $3K_2$, it follows that F_1', F_2', F_3', F_4' are four pairwise edge-disjoint blue copies of $3K_2$ in G_b (and in G). Thus, we may assume that at least one of F_1, F_2, F_3 is red, say F_1 is red. Let $J_r = G[E(H_r) \cup E(F_1)]$. Then $J_r \subseteq G_r$ is a red subgraph of G and $E(J_r) \cap [E(F_2) \cup E(F_3)] = \emptyset$. We may assume that $V(G) = \{v_1, v_2, \ldots, v_8\}$, the vertex set of K_5 in H_r is $\{v_1, v_2, v_3, v_4, v_5\}$, and $E(F_1) = \{v_1v_6, v_2v_7, v_3v_8\}$. Then J_r contains three pairwise edge-disjoint copies F_1^*, F_2^*, F_3^* of $3K_2$ that are edge-disjoint from F_2 and F_3. Thus, G contains four pairwise edge-disjoint monochromatic copies of $3K_2$. For example, F_1^*, F_2^*, F_3^*, F_2 are four pairwise edge-disjoint monochromatic copies of $3K_2$ in G.

Case 2. $H_r = (K_3 + \overline{K_4}) \vee K_1$. Then $\overline{H}_r = (K_4 \vee \overline{K}_3) + K_1$. If all subgraphs F_1, F_2, F_3 are blue, then $G_r = H_r$ and $G_b = \overline{H}_r = (K_4 \vee \overline{K}_3) + K_1$. Since the subgraph $K_4 \vee \overline{K}_3$ of G_b contains four pairwise edge-disjoint copies F_1', F_2', F_3', F_4' of $3K_2$, it follows that F_1', F_2', F_3', F_4' are four pairwise edge-disjoint blue copies of $3K_2$ in G_b (and in G). Thus, we may assume that at least one of F_1, F_2, F_3 is red, say F_1 is red. Let $J_r = G[E(H_r) \cup E(F_1)]$. Then $J_r \subseteq G_r$ is a red subgraph of G and $E(J_r) \cap [E(F_2) \cup E(F_3)] = \emptyset$. Furthermore, J_r is isomorphic to one of the two graphs in Figure 5 where the three dashed edges belong to F_1 in each graph. In either case, J_r contains three pairwise edge-disjoint copies F_1^*, F_2^*, F_3^* of $3K_2$ that are edge-disjoint from F_2 and F_3. Thus, G contains four pairwise edge-disjoint monochromatic copies of $3K_2$. For example, F_1^*, F_2^*, F_2, F_3 are four pairwise edge-disjoint monochromatic copies of $3K_2$ in G.

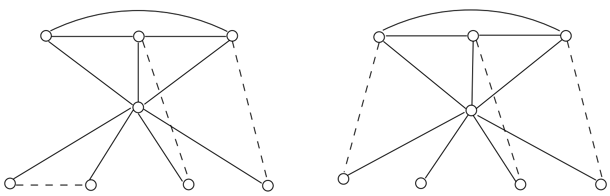

Fig. 5. Two possible graphs for J_r in Case 2

Case 3. $H_r \subseteq K_2 \vee \overline{K}_6$. We may assume that $V(\overline{K_6}) = \{v_1, v_2, \ldots, v_6\}$ and $V(K_2) = \{v_7, v_8\}$. Then $\overline{K_2 \vee \overline{K}_6} = K_6 + \overline{K}_2 \subseteq \overline{H}_r$, where $K_6 = G[\{v_1, v_2, \ldots, v_6\}]$ and $V(\overline{K}_2) = \{v_7, v_8\}$. We consider two subcases.

Subcase 3.1. All copies of F_1, F_2, F_3 of $3K_2$ are blue. Then $G_r = H_r$ and $G_b = \overline{H}_r$. Since $H_r \subseteq K_2 \vee \overline{K}_6$, it follows that $K_6 + \overline{K}_2 \subseteq \overline{H}_r = G_b$. The subgraph K_6 of G_b contains four pairwise edge-disjoint copies F_1', F_2', F_3', F_4' of $3K_2$ and so F_1', F_2', F_3', F_4' are four pairwise edge-disjoint blue copies of $3K_2$ in G_b (and in G).

Subcase 3.2. At least one of F_1, F_2, F_3, say F_1, is red. Since

$$E(G) = E(H_r) \cup E(H_b) \cup E(F_1) \cup E(F_2) \cup E(F_3),$$

it follows that $E(\overline{H}_r) = E(H_b) \cup E(F_1) \cup E(F_2) \cup E(F_3)$. Since $K_6 + \overline{K}_2 \subseteq \overline{H}_r$, it follows that

$$H' = (K_6 + \overline{K}_2) - [E(F_1) \cup E(F_2) \cup E(F_3)] \subseteq H_b.$$

Thus, H' contains a 2-regular graph of order 6 (since F_i may not belong to $K_6 + \overline{K}_2$ for some $i = 1, 2, 3$). If H' has a copy F_4 of $3K_2$, then F_4 is a blue $3K_2$ that is edge-disjoint from F_1, F_2, F_3. Therefore, F_1, F_2, F_3, F_4 are four pairwise edge-disjoint blue copies of $3K_2$ in G. Thus, we may assume that H' does not have a copy of $3K_2$. This implies that $H' = 2K_3$ and $E(F_1) \cup E(F_2) \cup E(F_3) \subseteq E(K_6)$. Hence, $E(K_6) = E(H') \cup E(F_1) \cup E(F_2) \cup E(F_3)$ where $V(K_6) = \{v_1, v_2, \ldots, v_6\}$, H' is blue, and $F_1 = 3K_2$ is red. We may assume that (v_1, v_2, v_3, v_1) and (v_3, v_4, v_5, v_3) are two copies of K_3 in H' and $E(F_1) = \{v_1v_4, v_2v_5, v_3v_6\}$.

First, suppose that there is a blue edge between $\{v_1, v_2, \ldots, v_6\}$ and $\{v_7, v_8\}$, say v_7v_1 is blue. Then there is a blue $F_4 = 3K_2$ with edge set $\{v_7v_1, v_2v_3, v_4v_5\}$ that is edge-disjoint from F_1, F_2, F_3 and so F_1, F_2, F_3, F_4 are four pairwise edge-disjoint monochromatic copies of $3K_2$ in G. Next, suppose that all edges between $\{v_1, v_2, \ldots, v_6\}$ and $\{v_7, v_8\}$ are red. Then there are two edge-disjoint red copies F' and F'' of $3K_2$ where $E(F') = \{v_7v_1, v_8v_2, v_3v_6\}$ and $E(F'') = \{v_7v_4, v_8v_6, v_2v_5\}$. Since F' and F'' are edge-disjoint from F_2 and F_3, it follows that F', F'', F_2, F_3 are four pairwise edge-disjoint monochromatic copies of $3K_2$ in G. □

While the exact value of $ER_t(3K_2)$ is not known when $t \geq 5$, we do have bounds for $ER_t(3K_2)$ for integers $t \geq 5$. First, we present some preliminary information. A vertex and an incident edge are said to *cover* each other. A *vertex cover* in a graph G is a set of vertices that covers all edges of G. The minimum number of vertices in a vertex cover of G is the *vertex covering number* $\beta(G)$ of G. A vertex cover of cardinality $\beta(G)$ is a *minimum vertex cover* in G. We mentioned that $ER_t(F) \leq ER_{t+1}(F)$ for every graph F and every positive integer t in Observation 1.4. In fact, $ER_{t+1}(F)$ can never exceed $ER_t(F)$ by more than $\beta(F)$.

Proposition 2.7. [7] *For every nonempty graph F and each positive integer t,*

$$ER_{t+1}(F) \leq ER_t(F) + \beta(F).$$

Since $\beta(3K_2) = 3$, it follows by Proposition 2.7 that $ER_{t+1}(3K_2) \leq ER_t(3K_2) + 3$ for each positive integer t. In fact, $ER_{t+1}(3K_2) \leq ER_t(3K_2) + 2$ for each positive integer t. In order to establish this fact, we first present two lemmas.

Lemma 2.8. *Let k and t be integers with $k \geq t + 4 \geq 8$. Then $\frac{1}{2}\left[\binom{k}{2} - 3t\right] \geq k$.*

Proof. Since $t + 4 \geq 8$, it follows that $t \geq 4$ and so $(t-4)(t+1) \geq 0$. Therefore,

$$(t-4)(t+1) = t^2 - 3t - 4 = t^2 + 3t - 4 - 6t = (t+4)(t-1) - 6t \geq 0.$$

Since $k \geq t+4$, it follows that $k-5 \geq t-1$ and so $k(k-5)-6t \geq (t+4)(t-1)-6t \geq 0$. Thus, $k^2 - 5k - 6t \geq 0$ and so $k^2 - k - 6t \geq 4k$. Therefore, $\frac{k^2-k}{2} - 3t \geq 2k$ and so $\frac{1}{2}\left[\binom{k}{2} - 3t\right] \geq k$. □

Lemma 2.9. *If G is a graph of size m, then $\alpha'(G) \geq \frac{m}{\Delta(G)}$ where $\Delta(G)$ is the maximum degree of G.*

Proof. Let $\{E_1, E_2, \dots, E_k\}$ be a partition of $E(G)$ into independent sets. Since $|E_i| \le \alpha'(G)$ for $1 \le i \le k$, it follows that $m = \sum_{i=1}^{k} |E_i| \le k\alpha'(G)$ and so $k \ge \frac{m}{\alpha'(G)}$. Hence, every partition of $E(G)$ into independent sets must contain at least $\frac{m}{\alpha'(G)}$ sets. If $v \in V(G)$ with $\deg v = \Delta(G)$, then each of the $\Delta(G)$ edges incident with v must belong to distinct independent sets of edges. Thus, $\Delta(G) \ge \frac{m}{\alpha'(G)}$ and so $\alpha'(G) \ge \frac{m}{\Delta(G)}$. □

We are now prepared to present the following.

Theorem 2.10. *Let t be a positive integer. If $ER_t(3K_2) \ge t + 4$, then $ER_{t+1}(3K_2) \le ER_t(3K_2) + 2$.*

Proof. Since the statement is true for $t = 1, 2, 3$, we may assume that $t \ge 4$. Let $ER_t(3K_2) = k \ge t + 4 \ge 8$. Let c be a red-blue coloring of $G = K_{k+2}$ with $V(G) = \{v_1, v_2, \dots, v_{k+2}\}$. We show that there are $t + 1$ pairwise edge-disjoint monochromatic copies of $3K_2$ in G. Let $F = G[\{v_1, v_2, \dots, v_k\}] = K_k$. Since $ER_t(3K_2) = k$, there are t pairwise edge-disjoint monochromatic copies $Q_1, Q_2, \dots, Q_t$ of $3K_2$ in F. Let H be the spanning subgraph of F whose edge set consists of all edges of G that do not belong to any of $Q_1, Q_2, \dots, Q_t$. That is,

$$H = F - [E(Q_1) \cup E(Q_2) \cup \cdots \cup E(Q_t)].$$

Thus, H is a graph of order k and size $\binom{k}{2} - 3t$. Let H_r be the red subgraph of H and H_b the blue subgraph of H. We may assume that $|E(H_r)| = m_r \ge m_b = |E(H_b)|$. If $\alpha'(H_r) \ge 3$ or $\alpha'(H_b) \ge 3$, then there is a monochromatic $3K_2$ that is edge-disjoint from $Q_1, Q_2, \dots, Q_t$. Thus, we may assume that $\alpha'(H_r) \le 2$ and $\alpha'(H_b) \le 2$.

Observe that if $\alpha'(H_r) \le 2$, then H_b is not empty, for otherwise, since $k \ge t + 4 \ge 8$, it follows by Lemma 2.8 that $m_r = \binom{k}{2} - 3t \ge 2k$. Furthermore $\Delta(H_r) \le k - 1$. Thus, $\alpha'(H_r) \ge \frac{m_r}{\Delta(H_r)} \ge \frac{2k}{k-1}$ by Lemma 2.9 and so $\alpha'(H_r) \ge 3$, a contradiction.

Since $m_r \ge m_b$, we have $m_r \ge \frac{1}{2}\left[\binom{k}{2} - 3t\right]$. Because $k \ge t + 4 \ge 8$, it follows by Lemma 2.8 that $\frac{1}{2}\left[\binom{k}{2} - 3t\right] \ge k$. Thus, H_r has order $k \ge 8$ and size at least k. Hence, H_r is neither a star nor K_3 and so $\alpha'(H_r) \ge 2$. Hence, $\alpha'(H_r) = 2$, say v_1v_2 and v_3v_4 are two edges in H_r. First, we make some observations.

(1) If there is a red edge in H_r that is incident with two vertices in $\{v_5, v_6, \dots, v_k\}$, say $v_5v_6 \in E(H_r)$, then v_1v_2, v_3v_4, v_5v_6 form a red $3K_2$ that is edge-disjoint from $Q_1, Q_2, \dots, Q_t$. Thus, no edge of H_r belongs to $G[\{v_5, v_6, \dots, v_k\}]$.

(2) If v_{k+1} or v_{k+2} is joined to a vertex v_i $(5 \le i \le k)$ by a red edge, say $v_{k+1}v_5$ is red, then v_1v_2, v_3v_4, $v_{k+1}v_5$ form a red $3K_2$ that is edge-disjoint from $Q_1, Q_2, \dots, Q_t$. Thus, $v_{k+1}v_i$ and $v_{k+2}v_i$ are blue for $5 \le i \le k$. Furthermore, $v_{k+1}v_{k+2}$ is blue, for otherwise, v_1v_2, v_3v_4, $v_{k+1}v_{k+2}$ form a red $3K_2$ that is edge-disjoint from $Q_1, Q_2, \dots, Q_t$.

Since $\alpha'(H_b) \le 2$, we consider these two cases.

Case 1. $\alpha'(H_b) = 2$. Let $e, f \in E(H_b)$ be two nonadjacent edges in H_b. Since $v_{k+1}v_{k+2}$ is blue by (2), it follows that e, f, $v_{k+1}v_{k+2}$ form a blue $3K_2$ that is edge-disjoint from $Q_1, Q_2, \dots, Q_t$.

Case 2. $\alpha'(H_b) = 1$. Let $v_pv_q \in E(H_b)$. Since $k \ge 8$, there are $i, j \in \{5, 6, \dots, k\} -$

$\{p,q\}$ and $i \neq j$. Then v_pv_q, $v_{k+1}v_i$, $v_{k+2}v_j$ form a blue $3K_2$ that is edge-disjoint from $Q_1, Q_2, \ldots, Q_t$ □ .

With the aid of Theorem 2.10, we now present bounds for $ER_t(3K_2)$ for each integer $t \geq 4$.

Theorem 2.11. *For each integer $t \geq 4$,*

$$\left\lceil \frac{5+\sqrt{1+24t}}{2} \right\rceil \leq ER_t(3K_2) \leq 2t.$$

Proof. To verify the upper bound, we proceed by induction on t. Since $ER_4(3K_2) = 8$ by Theorem 2.6, the result is true for $t = 4$. Assume that $ER_t(3K_2) \leq 2t$ for some integer $t \geq 4$. By Theorem 2.10 and the induction hypothesis, $ER_{t+1}(3K_2) \leq ER_t(3K_2) + 2 \leq 2t + 2 = 2(t+1)$.

To verify a lower bound, let $ER_t(3K_2) = k$. Then every red-blue coloring of K_k produces at least t pairwise edge-disjoint copies of $3K_2$. Consider the red-blue coloring of $G = K_k$ with red subgraph $G_r = K_2 \vee \overline{K}_{k-2}$ and blue subgraph $G_b = K_{k-2} + \overline{K}_2$. Since $\alpha'(G_r) = 2$, there is no red $3K_2$. Hence, G_b contains at least t pairwise edge-disjoint monochromatic copies of $3K_2$. This implies that $|E(G_b)| = \binom{k-2}{2} \geq 3t$ or $k^2 - 5k + 6 - 6t \geq 0$. Thus, $ER_t(3K_2) = k \geq \left\lceil \frac{5+\sqrt{1+24t}}{2} \right\rceil$. □

Note that the lower bound in Theorem 2.11 holds for every positive integer t. We saw that $ER_t(3K_2) = t + 4$ when $t = 4$ by Theorem 2.6 and for each integer $t \geq 2$, if $ER_t(3K_2) \geq t + 4$, then $ER_{t+1}(3K_2) \leq ER_t(3K_2) + 2$ by Theorem 2.10. Should it over occurs that $ER_t(3K_2) \geq t + 4$ for some $t \geq 10$, then $ER_{t+1}(3K_2) \leq ER_t(3K_2) + 1$. To show this, we first state a lemma whose proof is similar to that of Lemma 2.8.

Lemma 2.12. *Let k and t be integers with $k \geq t + 4 \geq 14$. Then $\frac{1}{2}\left[\binom{k}{2} - 3t\right] \geq 2k$.*

Theorem 2.13. *Let $t \geq 10$ be an integer. If $ER_t(3K_2) \geq t + 4$, then $ER_{t+1}(3K_2) \leq ER_t(3K_2) + 1$.*

Proof. Let $ER_t(3K_2) = k \geq t + 4 \geq 14$. Let c be a red-blue coloring of $G = K_{k+1}$ with $V(G) = \{v_1, v_2, \ldots, v_{k+1}\}$. We show that there are $t + 1$ pairwise edge-disjoint monochromatic copies of $3K_2$ in G. Let $F = G[\{v_1, v_2, \ldots, v_k\}] = K_k$. Since $ER_t(3K_2) = k$, there are t pairwise edge-disjoint monochromatic copies $Q_1, Q_2, \ldots, Q_t$ of $3K_2$ in F. Let

$$H = F - [E(Q_1) \cup E(Q_2) \cup \cdots \cup E(Q_t)].$$

Thus, H is a graph of order k and size $\binom{k}{2} - 3t$. Let H_r be the red subgraph of H and H_b the blue subgraph of H. We may assume that $|E(H_r)| = m_r \geq m_b = |E(H_b)|$. Hence, $m_r \geq \frac{1}{2}\left[\binom{k}{2} - 3t\right] \geq 2k$ by Lemma 2.12. Furthermore $\Delta(H_r) \leq k - 1$. Thus, $\alpha'(H_r) \geq \frac{m_r}{\Delta(H_r)} \geq \frac{2k}{k-1}$ by Lemma 2.9 and so $\alpha'(H_r) \geq 3$. Therefore, H_r contains a copy of $3K_2$ that is edge-disjoint from $Q_1, Q_2, \ldots, Q_t$. □

3. The multiple Ramsey numbers $VR_t(P_3 + P_2)$ and $ER_t(P_3 + P_2)$

We now turn our attention to the vertex-disjoint and edge-disjoint Ramsey numbers of $P_3 + P_2$. With the aid of Proposition 2.1, we are able to determine $VR_t(P_3 + P_2)$ for every positive integer t.

Theorem 3.1. *For each positive integer t, $VR_t(P_3 + P_2) = 5t + 1$.*

Proof. Since $VR_1(P_3 + P_2) = R(P_3 + P_2) = 6$ and the order of $P_3 + P_2$ is 5, it follows by Proposition 2.1 that $VR_t(P_3 + P_2) \le 6 + 5(t - 1) = 5t + 1$.

Next, consider the red-blue coloring of $G = K_{5t}$ with red subgraph $G_r = K_{1,5t-1}$ and blue subgraph $G_b = K_{5t-1}$. Then there is no red $P_3 + P_2$ in G_r. The blue subgraph G_b has order $5t - 1$ and so there are no t pairwise vertex-disjoint blue copies of $P_3 + P_2$. Since this red-blue coloring has no t pairwise vertex-disjoint monochromatic copies of $P_3 + P_2$, it follows that $VR_t(P_3 + P_2) \ge 5t + 1$ and so $VR_t(P_3 + P_2) = 5t + 1$. □

We now consider edge-disjoint Ramsey numbers of $P_3 + P_2$. We begin with $ER_t(P_3 + P_2)$ for $t = 2, 3, 4$. First, we present a useful lemma.

Lemma 3.2. *Let G be a graph of order $n \ge 6$ and size $m \ge 6$ with $\Delta(G) \ge 2$. If $G \notin \{K_4 + \overline{K}_{n-4}, K_{1,m} + \overline{K}_{n-m-1}\}$, then $P_3 + P_2 \subseteq G$.*

Proof. Let $\Delta(G) = \Delta$ and let $v \in V(G)$ such that $\deg v = \Delta$ where $N(v) = \{v_1, v_2, \ldots, v_\Delta\}$.

⋆ First, suppose that $\Delta(G) = 2$. Then each component of G is a cycle or a path. Since $m \ge 6$, it follows that $P_3 + P_2 \subseteq G$.

⋆ Next, suppose that $\Delta(G) = 3$. Since $G \ne K_4 + \overline{K}_{n-4}$ and $m \ge 6$, there is an edge not in the subgraph $G[N[v]]$ induced by the closed neighborhood $N[v]$ of v in G. Hence, $P_3 + P_2 \subseteq G$.

⋆ Finally, suppose that $\Delta(G) \ge 4$. Since $G \ne K_{1,\Delta} + \overline{K}_{n-\Delta-1}$, there is an edge that is not incident with v and so $P_3 + P_2 \subseteq G$. □

Theorem 3.3. $R(P_3 + P_2) = ER_2(P_3 + P_2) = 6$.

Proof. First, we show that $R(P_3 + P_2) = 6$. The red-blue coloring of K_5 with red subgraph $K_{1,4}$ and blue subgraph K_4 contains no monochromatic $P_3 + P_2$. Thus, $R(P_3 + P_2) \ge 6$. Let there be given an arbitrary red-blue coloring of $G = K_6$. Let G_r be the red subgraph of G and G_b the blue subgraph of G. Suppose that the size of G_r is m_r and the size of G_b is m_b. Thus, $m_r + m_6 = 15$. We may assume that $m_r \ge m_b$ and so $m_r \ge 8$. Since G_r has order 6 and size $m_r \ge 8$, it follows that $G_r \notin \{K_{1,m_r}, K_4 + \overline{K}_2\}$ and so $P_3 + P_2 \subseteq G_r$ by Lemma 3.2. Therefore, $R(P_3 + P_2) \le 6$ and so $R(P_3 + P_2) = 6$.

Next, we show that $ER_2(P_3 + P_2) = 6$. First, $ER_2(P_3 + P_2) \ge R(P_3 + P_2) = 6$. Let there be given an arbitrary red-blue coloring of $G = K_6$. Since $R(P_3 + P_2) = 6$, there is a monochromatic subgraph $F = P_3 + P_2$ of G. Let $H = G - E(F)$. Thus, H has order 6 and size $\binom{6}{2} - 3 = 12$. Let H_r be the red subgraph of size m_r and H_b the blue subgraph of size m_b. Thus, $m_r + m_6 = 12$. We may assume that $m_r \geq m_b$ and so $m_r \geq 6$.

If $m_r \geq 7$, then $H_r \notin \{K_{1,6}, K_4 + \overline{K}_2\}$ and so $P_3 + P_2 \subseteq H_r$ by Lemma 3.2. Hence, we may assume that $m_r = m_b = 6$.

Thus, H_r and H_b both have order 6 and size 6. So, neither H_r nor H_b is $K_{1,6}$. Since H_r and H_b are not both $K_4 + \overline{K}_2$, we may assume that $H_r \neq K_4 + \overline{K}_2$. Therefore, $H_r \notin \{K_{1,6}, K_4 + \overline{K}_2\}$ and so $P_3 + P_2 \subseteq H_r$ by Lemma 3.2. Therefore, G has two edge-disjoint copies of $P_3 + P_2$. Hence, $ER_2(P_3 + P_2) \leq 6$ and so $ER_2(P_3 + P_2) = 6$. □

Theorem 3.4. *$ER_3(P_3 + P_2) = ER_4(P_3 + P_2) = 7$.*

Proof. Let there be given a red-blue coloring of $H = K_6$ with red subgraph $H_r = K_2 \vee \overline{K}_4$ and blue subgraph $H_b = K_4 + \overline{K}_2$. The blue subgraph H_b fails to contain a copy of $P_3 + P_2$. Since there are not three pairwise edge-disjoint red copies of $P_3 + P_2$ in H, it follows that $ER_4(P_3 + P_2) \geq ER_3(P_3 + P_2) \geq 7$. First, we show that $ER_3(P_3 + P_2) = 7$. Let there be given an arbitrary red-blue coloring of $G = K_7$. Since $ER_2(P_3 + P_2) = 6$, there are edge-disjoint monochromatic copies F_1 and F_2 of $P_3 + P_2$ in G. Let $H = G - (E(F_1) \cup E(F_2))$. Then H has order 7 and size $\binom{7}{2} - 2 \cdot 3 = 15$. Let H_r be the red subgraph of H and H_b the blue subgraph of H. Suppose that the size of H_r is m_r and the size of H_b is m_b. Thus, $m_r + m_6 = 15$. We may assume that $m_r \geq m_b$ and so $m_r \geq 8$. Since H_r has order 7 and size $m_r \geq 8$, it follows that $H_r \notin \{K_{1,m_r}, K_4 + \overline{K}_3\}$ and so $P_3 + P_2 \subseteq H_r$ by Lemma 3.2. Therefore, $ER_3(P_3 + P_2) \leq 7$ and so $ER_3(P_3 + P_2) = 7$.

Next, we show that $ER_4(P_3+P_2) = 7$. Let there be given an arbitrary red-blue coloring of $G = K_7$. Since $ER_3(P_3+P_2) = 7$, there are three pairwise edge-disjoint monochromatic copies F_1, F_2, F_3 of $P_3 + P_2$ in G. Let $H = G - (E(F_1) \cup E(F_2)) \cup E(F_3)$. Then H has order 7 and size $\binom{7}{2} - 3 \cdot 3 = 12$. Let H_r of size m_r be the red subgraph of H and H_b of size m_b the blue subgraph of H. Thus, $m_r + m_b = 12$. We may assume that $m_r \geq m_b$ and so $m_r \geq 6$. Thus, H_r has order 7 and size $m_r \geq 6$. If $m_r \geq 7$, then $H_r \notin \{K_{1,7}, K_4 + \overline{K}_3\}$ and so $P_3 + P_2 \subseteq H_r$ by Lemma 3.2. Hence, we may assume that $m_r = m_b = 6$. Thus, both H_r and H_b both have order 7 and size 6. We verify the following claim.

Claim: At least one of H_r and H_b is neither $K_{1,6}$ nor $K_4 + \overline{K}_3$.

To verify the claim, suppose that $H_b \in \{K_{1,6}, K_4+\overline{K}_3\}$. We show that $H_r \notin \{K_{1,6}, K_4+\overline{K}_3\}$. Let $V(H) = \{v_1, v_2, \ldots, v_7\}$. First, suppose that $H_b = K_{1,6}$. Then $H_r \neq K_{1,6}$. Assume that $H_r = K_4+\overline{K}_3$. Let v_7 be the center of H_b and so v_7 is adjacent to $v_1, v_2, \ldots, v_6$ in H_b. For the subgraph K_4 in H_r, we may assume that $V(K_4) = \{v_3, v_4, v_5, v_6\}$. However then, $H' = G-(E(H_r) \cup E(H_b)) = G-E(H) = K_2 \vee \overline{K}_4$. Since $G-E(H)$ is decomposed into F_1, F_2, F_3 and $K_2 \vee \overline{K}_4$ cannot be decomposed into three copies of $P_3 + P_2$, this is a contradiction. Thus, $H_r \neq K_4 + \overline{K}_3$ and so $H_r \notin \{K_{1,6}, K_4 + \overline{K}_3\}$. By symmetry, we may assume that $\{H_r, H_b\} \neq \{K_{1,6}, K_4 + \overline{K}_3\}$. Next, suppose $H_b = K_4 + \overline{K}_3$ where say $V(K_4) = \{v_4, v_5, v_6, v_7\}$. Then $H_r \neq K_{1,6}$. Since v_1, v_2, v_3 are the only possible vertices in $H - E(H_b)$ having degree 3 or more and $H_r \subseteq H - E(H_b)$, it follows that $H_r \neq K_4 + \overline{K}_3$. Therefore, $H_r \notin \{K_{1,6}, K_4 + \overline{K}_3\}$. Thus, the claim holds. So, we may assume that $H_r \notin \{K_{1,6}, K_4 + \overline{K}_3\}$. It then follows by Lemma 3.2 that $P_3 + P_2 \subseteq H_r$. Hence, there is a monochromatic copy of $P_3 + P_2$ edge-disjoint from F_1, F_2, F_3. Therefore,

$ER_4(P_3+P_2) \leq 7$ and so $ER_4(P_3+P_2) = 7$. □

While the exact value of $ER_t(P_3+P_2)$ is not known when $t \geq 5$, we do have bounds for $ER_t(P_3+P_2)$ for every positive integer t. Since $\beta(P_3+P_2) = 2$, the following is a consequence of Proposition 2.7.

Corollary 3.5. *For each positive integer t, $ER_{t+1}(P_3+P_2) \leq ER_t(P_3+P_2) + 2$.*

With the aid of Theorem 3.4 and Corollary 3.5, we now present bounds for $ER_t(P_3+P_2)$ for each integer $t \geq 4$.

Theorem 3.6. *For each integer $t \geq 4$, $\left\lceil \frac{3+\sqrt{1+24t}}{2} \right\rceil \leq ER_t(P_3+P_2) \leq 2t-1$.*

Proof. To verify the upper bound, we proceed by induction on t. Since $ER_4(P_3+P_2) = 7$ by Theorem 3.4, the result is true for $t = 4$. Assume that $ER_t(P_3+P_2) \leq 2t-1$ for some integer $t \geq 4$. By Proposition 3.5 and the induction hypothesis, $ER_{t+1}(P_3+P_2) \leq ER_t(P_3+P_2) + 2 \leq (2t-1) + 2 = 2(t+1) - 1$.

To verify the lower bound, let $ER_t(P_3+P_2) = k$. Then every red-blue coloring of K_k produces at least t pairwise edge-disjoint monochromatic copies of P_3+P_2. Consider the red-blue coloring of $G = K_k$ with red subgraph $G_r = K_{1,k-1}$ and blue subgraph $G_b = K_{k-1}$. Since there is no red P_3+P_2, it follows that G_b contains at least t pairwise edge-disjoint monochromatic copies of P_3+P_2. This implies that $|E(G_b)| = \binom{k-1}{2} \geq 3t$ or $k^2 - 3k + (2-6t) \geq 0$. Consequently, $ER_t(P_3+P_2) = k \geq \left\lceil \frac{3+\sqrt{1+24t}}{2} \right\rceil$. □

Here too, the lower bound for $ER_t(P_3+P_2)$ in Theorem 3.6 holds for each positive integer t. We saw that $R(P_3+P_2) = ER_2(P_3+P_2) = 6$ and $ER_3(P_3+P_2) = ER_4(P_3+P_2) = 7$. Therefore, $ER_t(P_3+P_2) = t+3$ for $t = 4$. Should it ever occur that $ER_t(P_3+P_2) \geq t+3$ for some integer $t \geq 7$, then $ER_{t+1}(P_3+P_2) \leq ER_t(P_3+P_2) + 1$. To establish this, we first state a lemma whose proof is similar to that of Lemma 2.8.

Lemma 3.7. *Let k and t be integers with $k \geq t+3 \geq 10$. Then $\frac{1}{2}\left[\binom{k}{2} - 3t\right] \geq k+1$.*

Theorem 3.8. *Let $t \geq 7$ be an integer. If $ER_t(P_3+P_2) \geq t+3$, then $ER_{t+1}(P_3+P_2) \leq ER_t(P_3+P_2) + 1$.*

Proof. Let $ER_t(P_3+P_2) = k \geq t+3 \geq 10$. Let c be a red-blue coloring of $G = K_{k+1}$ with $V(G) = \{v_1, v_2, \ldots, v_{k+1}\}$. We show that there are $t+1$ pairwise edge-disjoint monochromatic copies of P_3+P_2 in G. Let $F = G[\{v_1, v_2, \ldots, v_k\}] = K_k$. Since $ER_t(P_3+P_2) = k$, there are t pairwise edge-disjoint monochromatic copies $Q_1, Q_2, \ldots, Q_t$ of P_3+P_2 in F. Let

$$H = F - [E(Q_1) \cup E(Q_2) \cup \cdots \cup E(Q_t)].$$

Thus, H is a graph of order k and size $\binom{k}{2} - 3t$. Let H_r be the red subgraph of H and H_b the blue subgraph of H. We may assume that $|E(H_r)| = m_r \geq m_b = |E(H_b)|$. Hence, $m_r \geq \frac{1}{2}\left[\binom{k}{2} - 3t\right] \geq k+1$ by Lemma 3.7. Since H_r is a graph of order $k \geq 10$

and size at least $k+1$, it follows that H_r is neither a star $K_{1,k-1}$ nor the unicyclic graph $(K_2 + \overline{K}_{k-3}) \vee K_1$. Therefore, H_r contains a copy of $P_3 + P_2$ that is edge-disjoint from $Q_1, Q_2, \ldots, Q_t$. □

Acknowledgment

We are grateful to Professor Gary Chartrand for suggesting the concepts and problems to us and kindly providing useful information on this topic. Furthermore, we thank the anonymous referee whose valuable suggestions resulted in an improved paper.

References

[1] G. Chartrand, E. Jent, and P. Zhang. Monochromatic subgraphs in graphs. *Contributions to Mathematics*, 9:62–68, 2024. https://doi.org/10.47443/cm.2024.036.

[2] G. Chartrand and P. Zhang. *Chromatic Graph Theory*. Chapman and Hall/CRC, Boca Raton, FL, USA, 2nd edition, 2020. https://doi.org/10.1201/9780429438868.

[3] E. J. Cockayne and P. J. Lorimer. On ramsey graph numbers for stars and stripes. *Canadian Mathematical Bulletin*, 18(1):31–34, 1975. https://doi.org/10.4153/CMB-1975-005-0.

[4] E. J. Cockayne and P. J. Lorimer. The ramsey number for stripes. *Journal of the Australian Mathematical Society (Series A)*, 19(2):252–256, 1975. https://doi.org/10.1017/S1446788700029554.

[5] A. W. Goodman. On sets of acquaintances and strangers at any party. *The American Mathematical Monthly*, 66(9):778–783, 1959. https://doi.org/10.1080/00029890.1959.11989408.

[6] R. E. Greenwood and A. M. Gleason. Combinatorial relations and chromatic graphs. *Canadian Journal of Mathematics*, 7:1–7, 1955. https://doi.org/10.4153/CJM-1955-001-4.

[7] E. Jent, S. Osborn, and P. Zhang. Extending ramsey numbers for connected graphs of size 3. *Symmetry*, 16(8), 2024. https://doi.org/10.3390/sym16081092.

[8] F. P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930. https://doi.org/10.1112/plms/s2-30.1.264.

Emma Jent
Department of Mathematics, Western Michigan University
Kalamazoo, Michigan 49008-5248, USA
E-mail jente65@hope.edu

Ping Zhang
Department of Mathematics, Western Michigan University
Kalamazoo, Michigan 49008-5248, USA
E-mail ping.zhang@wmich.edu