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Total balanced antimagic labeling

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ABSTRACT

Let G be a graph. We introduce the balanced antimagic labeling as an analogue to the antimagic labeling. A k-total balanced antimagic labelling is a map $c\colon V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that: the label classes differ in size by at most one, each vertex x is assigned the weight $w(x) = c(x) + \sum_{x \in e} c(e)$, and $w(x) \neq w(y)$ for $x \neq y$.

We present several properties of balanced antimagic labeling. We also derive such a labeling for complete graphs and complete bipartite graphs.

Keywords: antimagic labeling

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1. Introduction

For standard terms and notation in graph theory, the reader is referred to the textbook by Diestel [4]. An introduction to magic- and anti-magic-type labelings can be found in the monograph by Bača et al. [2].

An antimagic labeling of a graph $G = (V, E)$ is a one-to-one correspondence between $E(G)$ and $\{1, \ldots, |E(G)|\}$ such that the vertex-sum (i.e., the sum of the labels assigned to edges incident to a vertex) for distinct vertices are different. It was conjectured by Hartsfield and Ringel that every tree other than K_2 has an anti-magic labeling [5]. Similarly, a total antimagic labeling is a one-to-one correspondence between $V(G) \cup E(G)$ and $\{1, \ldots, |E(G) \cup V(G)|\}$ such that all vertex weights are pairwise distinct, where the *vertex weight* is the sum of the label of that vertex and labels of all edges incident with the vertex [1]. Miller, Phanalasy and Ryan proved that all graphs admit total antimagic labeling [6].

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A k-coloring of a graph G is called *balanced*, if the label classes differ in size by at most one. The best know balanced labeling is a cordial labeling introduced by Cahit [3].

In this paper we combine these two types of concepts and introduce a new labeling called *total balanced antimagic labeling*. A graph G is *k-total balanced antimagic* if there exists a labelling $c\colon V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ such that:

- the label classes differ in size by at most one,
- each vertex x is assigned the weight $w(x) = c(x) + \sum_{x \in e} c(e)$, and
- $w(x) \neq w(y)$ for $x \neq y$.

Note that although we call c antimagic it is not necessary a bijection. Moreover, for $k = |E(G) \cup V(G)|$, then k-total balanced antimagic labeling is just total antimagic labeling.

2. Main results

For $c\colon V(G) \cup E(G) \to \{1, 2, \ldots, k\}$ let $c_i = |c^{-1}(i)|$ for $i \in \{1, 2, \ldots, k\}$. We start this section with some basic observations.

Observation 2.1. *If $k \geq |V(G)|$, then a graph $G = (V, E)$ is k-total balanced antimagic.*

Proof. We will start by labeling the edges of G in a balanced way, i.e. the label classes differ in size by at most one. Without loss of generality, assume that the temporary weights of vertices fulfill $w'(v_i) = \sum_{v_i \in e} c(e) \leq w'(v_j) = \sum_{v_j \in e} c(e)$ for all $1 \leq i < j \leq n$ with $n = |V(G)|$. We next label vertices v_i, since the labeling of the edges is balanced and $k \geq n$ we can do that in such a way that $c(v_i) < c(v_j)$, for $1 \leq i < j \leq n$. Under this total labeling, it is clear that $w(v_i) = w'(v_i) + c(v_i) < w(v_j) = w'(v_j) + c(v_j)$, for $1 \leq i < j \leq n$. □

From the above observation, we obtain a corollary:

Corollary 2.2. *If a connected graph $G = (V, E)$ has an antimagic labeling, then it is k-total balanced antimagic for $k = |E(G)|$.*

Proof. By Observation 2.1 we can assume that $G = (V, E)$ is a tree, thus $k = |V(G)| - 1 = n - 1$. Let c be an antimagic labeling of G such that $w(v_i) = \sum_{v_i \in e} c(e) < w(v_j) = \sum_{v_j \in e} c(e)$ for all $1 \leq i < j \leq n$ with $n = |V(G)|$. We next label vertices v_i with i for $i \in \{1, 2, \ldots, n-1\}$ and $c(v_n) = n - 1$. Under this total labeling, it is clear that $w(v_i) = w'(v_i) + c(v_i) < w(v_j) = w'(v_j) + c(v_j)$, for $1 \leq i < j \leq n$. □

In this paper, we will focus on an open problem that was stated by the first author during IWOGL 2024:

Problem 2.3. *Characterize k-regular k-total balanced antimagic graphs.*

We will start with an easy observation:

Observation 2.4. *If a k-regular graph $G = (V, E)$ has a k-total balanced antimagic labeling, then $|V(G)| \leq k^2$.*

Proof. Assume that G admits such a labeling, then the $w(x) \in \{k+1, k+2, \ldots, (k+1)k\}$ for any $x \in V(G)$, as $|\{k+1, k+2, \ldots, (k+1)k\}| = k^2$ therefore $|V(G)| \leq k^2$. □

Theorem 2.5. *The complete graph K_{k+1} admits k-total balanced antimagic labeling if and only if $k \notin \{1, 2\}$.*

Proof. One can easily check that the graph K_{k+1} is not k-total balanced antimagic for $k \in 1, 2$. Suppose now, that $k \geq 3$. Note that $c_i \in \left\{\lfloor\frac{(k+1)(k+2)}{2k}\rfloor, \lceil\frac{(k+1)(k+2)}{2k}\rceil\right\}$, thus for $k = 3$, $c_i \in \{3, 4\}$ and for $k \geq 4$, $c_i < k$. Let $A = \left\{c^{-1}(i) : c_i = \lfloor\frac{(k+1)(k+2)}{2k}\rfloor\right\}$ and $B = \left\{c^{-1}(i) : c_i = \lceil\frac{(k+1)(k+2)}{2k}\rceil\right\}$. Observe that $A \neq \emptyset$ and $B \neq \emptyset$. We will label the graph K_{k+1} in such a way that $c^{-1}(k-1) \in A$ and $c^{-1}(k) \in B$. We will start by labeling the edges of K_{k+1}, where each color $i \in \{1, 2, \ldots, k-1\}$ will be used $(c_i - 1)$ times and color k will be used $(c_k - 2)$ times. Moreover, we pick a vertex $v_{k+1} \in V(K_{k+1})$, and all incident edges we label with the highest possible labels. Thus the temporary weights of the vertices can be put into a sequence: $w'(v_1) = \sum_{v_1 \in e} f(e) \leq w'(v_2) = \sum_{v_2 \in e} f(e) \leq \ldots \leq w'(v_k) = \sum_{v_k \in e} f(e) < w(v_{k+1}) = \sum_{v_{k+1} \in e} f(e)$. We next label vertices v_i with i for $i \in \{1, 2, \ldots, k\}$ and $v_{k+1} = k$. Under this total labeling, it is clear that $w(v_i) < w(v_j)$, for $1 \leq i < j \leq k+1$. An example of such labeling for K_4 is shown in Figure 1, where the vertex v_4 is black. □

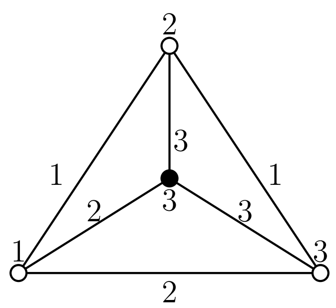

Fig. 1. 3-total balanced antimagic labeling of K_4

Theorem 2.6. *The complete graph $K_{k,k}$ admits k-total balanced antimagic labeling if and only if $k \neq 1$.*

Proof. One can easily check that a graph $K_{1,1}$ is not 1-total balanced antimagic. Let

$V = \{v_1, \ldots, v_k\}$ and $U = \{u_1, \ldots, u_k\}$ be partition sets for $G = K_{k,k}$.

For $k \geq 2$, we will show an explicit k-total balanced antimagic labeling of G, following the next steps:

1. We will give a k-balanced labeling of the edges of G , where each label is used exactly k times.
2. We will give a k-labeling for the vertices of G.
3. We will show that for each pair of vertices $u \neq v$ in G, $w(u) \neq w(v)$.

Case 1. $k = 2m$

Set

$$f(v_i u_j) = \begin{cases} \left\lfloor \frac{j+1}{2} \right\rfloor, & \text{for } \ i \in \{1, 3, \ldots, 2m-1\}, \\ 2m + 1 - \left\lfloor \frac{j+1}{2} \right\rfloor, & \text{for } \ i \in \{2, 4, \ldots, 2m\}. \end{cases}$$

Then:

1. For any $i \in \{1, 3, \ldots, 2m-1\}$, $\sum_{j=1}^{k} f(v_i u_j) = m(m+1)$;
2. for any $i \in \{2, 4, \ldots, 2m\}$,$\sum_{j=1}^{k} f(v_i u_j) = m(3m+1)$; and
3. for any $j \in \{1, 2 \ldots, k\}$, $\sum_{i=1}^{k} f(v_i u_j) = m(2m+1)$.

Let us put the following labels to the vertices of G.

$$f(v_i) = \begin{cases} \left\lfloor \frac{i+1}{2} \right\rfloor, & \text{for } \ i \in \{1, 3, \ldots, 2m-1\}, \\ m + \left\lfloor \frac{j+1}{2} \right\rfloor, & \text{for } \ i \in \{2, 4, \ldots, 2m\}, \end{cases}$$

and

$$f(u_j) = j \text{ for } j \in \{1, 2, \ldots, 2m\}.$$

Therefore,

1. For any $i \in \{1, 3, \ldots, 2m-1\}$, $m(m+1) + 1 \leq w(v_i) \leq m(m+2)$;
2. for any $i \in \{2, 4, \ldots, 2m\}$, $m(3m+2) + 1 \leq w(v_i) \leq 3m(m+1)$; and
3. for any $j \in \{1, 2 \ldots, k\}$, $m(2m+1) + 1 \leq w(u_j) \leq m(2m+3)$.

Case 2. $k = 2m + 1$

Set

$$f(v_i u_j) = \begin{cases} \left\lfloor \frac{j+1}{2} \right\rfloor, & \text{for } \ i \in \{1, 3, \ldots, 2m-1\}, \\ 2m + 2 - \left\lfloor \frac{j+1}{2} \right\rfloor, & \text{for } \ i \in \{2, 4, \ldots, m\}, \\ j, & \text{for } \ i = 2m+1. \end{cases}$$

Then

1. For any $i \in \{1, 3, \ldots, 2m-1\}$, $\sum_{j=1}^{k} f(v_i u_j) = (m+1)(m+2)$;
2. for any $i \in \{2, 4, \ldots, m\}$, $\sum_{j=1}^{k} f(v_i u_j) = (m+1)(3m+2)$;
3. for $i = 2m+1$, $\sum_{j=1}^{k} f(v_{2m+1} u_j) = (m+1)(2m+1)$; and

4. for any $j \in \{1, 2 \ldots, 2m+1\}$, $\sum_{i=1}^{k} f(v_i u_j) = m(2m+2) + j$.

Consider the following labels to the vertices of G.

$$f(v_i) = \begin{cases} \left\lfloor \frac{i+1}{2} \right\rfloor, & \text{for } i \in \{1, 3, \ldots, 2m-3\}, \\ m+1, & \text{for } i = 2m-1, \\ m, & \text{for } i = 2m+1, \\ m+1+\left\lfloor \frac{j+1}{2} \right\rfloor, & \text{for } i \in \{2, 4, \ldots, 2m\}, \end{cases},$$

and

$$f(u_j) = j \text{ for } j \in \{1, 2, \ldots, 2m+1\}.$$

Therefore,

1. For $i \in \{1, 3, \ldots, 2m-3\}$, $(m+1)(m+2)+1 \le w(v_i) \le (m+1)(m+2)+(m-1)$;
2. for $i = 2m-1$, $w(v_{2m-1}) = (m+1)(m+2)+(m+1)$;
3. for $i = 2m+1$, $(m+1)(2m+1)+m$;
4. for $i \in \{2, 4, \ldots, m\}$, $(m+1)(3m+2)+(m+2) \le w(v_i) \le (m+1)(3m+2)+(2m+1)$;
5. for $j \in \{1, 2, \ldots, 2m+1\}$, $w(u_j) = m(2m+2)+2j$. □

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