Hamilton Closures In Domination Critical Graphs

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Abstract. We define a closure operation on a particular family of graphs that has the property that the resulting graph is hamiltonian if and only if the original graph is hamiltonian.

1. Introduction

No successful characterization of hamiltonian graphs is known. Many results with sufficient conditions are known however. One of the earliest theorems to which many of the later results owe their success is the beautiful result of Dirac [2]: Let G be a graph on n vertices, if the minimum degree of G, $\delta = \delta(G) \geq n/2$ then G is hamiltonian.

Success has been achieved for some special graphs by considering so-called *neighbourhood conditions*. Ore, [4], extending Dirac's work, considered a neighbourhood condition, namely the sum of the degrees $d(x) + d(y)$ for any pair of independent vertices x and y. He showed that if this sum was at least n for all such pairs then G was hamiltonian. Recently Faudree, Gould, Jacobson and Schelp, [3], proved a number of interesting results by considering as the neighbourhood condition, the cardinality of $N(x) \cup N(y)$ for pairs of independent vertices x and y (where here and in what follows $N(x)$ (respectively, $N[x]$) denotes the set (closed set) of neighbours of the vertex x). For example they show that if $NC = |N(x) \cup N(y)|$ satisfies $NC \geq s$ for all such pairs in a 2-connected graph G then G contains a cycle of order at least $s + 2$ or (if $n < s + 2$) G is complete.

Bondy and Chvátal, [1], extended Ore's result by defining the *closure* of a graph G to be the graph $cl(G)$ obtained from G by recursively joining independent pairs of vertices whose degree sum is at least n. They showed that G is hamiltonian if and only $cl(G)$ is hamiltonian. This is particularly useful of course when for example $cl(G)$ is complete. In this paper we consider a different closure operator that, given satisfactory conditions on G, leads to the same result.

A *dominating set* in a graph G is a subset S of the vertices such that every vertex of $V(G)$-S is adjacent to at least one member of S. We consider a question of when so-called 3-domination-critical graphs are hamiltonian and give a partial answer. The graph G in Figure 1 has the property that it has no dominating set of size 2 but the addition of any new edge creates a new graph G' containing a dominating set of size 2, i.e. there exists vertices x and y such that the union of their closed neighbourhoods covers all vertices of G'. Such graphs G are known as *3-domination critical*, or simply, for purposes in this paper, critical. A nice discussion of such graphs appears in a paper of Sumner and Blitch, [5]. Sumner conjectured that every 3-domination critical graph has a hamiltonian path. This was recently proved by Wojcicka [7]. The problem of when they are hamiltonian remains open. Our results shed some light on this quesdon. We do not need quite as strong a condition as G being critical but only that the addition of any new edge creates a new graph G' containing a new dominating set of size 2. Sumner and Wojcicka [6] refer to such graphs as being 3-conservative. In light of this, we will simply call such graphs conservative. Clearly critical graphs are conservative.

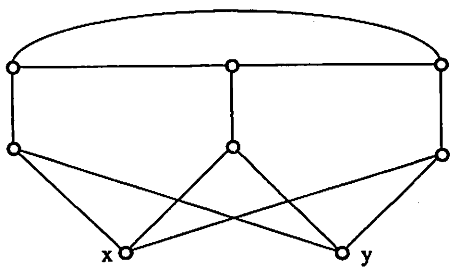

Figure 1

2. Domination Closure of Graphs

First consider the graph of Figure 1. It is easily seen to be hamiltonian but cannot be shown to be so by the previously mentioned theorems. The closure of this graph is itself, i.e. $cl(G) = G$, since all degrees are $3 < n/2$. The maximum allowable value of NC for this graph is 3, as seen by considering the vertices x and y, which again does not yield a hamiltonian circuit by the methods of [3].

For convenience we adapt some of the notation of [5] here. Let $e = ab \notin E(G)$; if there exists a vertex c such that $N[a] \cup N[c]$ covers all vertices of G except the vertex b, we will write $[a, c] \to b$.

We will define the *domination closure* of a graph G, $D^*(G)$, by means of a combination of a neighbourhood condition and a closure operation, namely, if $d(b) \geq 3$ then whenever $[a, c] \to b$ for independent vertices a and c *of* G, we add the edge ac. First we look at this concept and its relation to Hamilton circuits for conservative graphs.

Theorem 1. *Suppose that G is 2-connected and consevative. If $[a, c] \to b$ for some pair of independent vertices $\{a, c\} \in V(G)$, where $d(b) \geq 3$, then $G' = G + ac$ is hamiltonian if and only if G is hamiltonian.*

Proof: Clearly if G is hamiltonian then G' is also. Suppose for some pair of independent vertices $\{a, c\}$, where $[a, c] \to b$ and $d(b) \geq 3$, that G' is hamiltonian while G is not. Then G' contains a Hamilton path $P = \{v_1, v_2, \ldots, v_n\}$ from

$a = v_1$ to $c = v_n$ where $n = |V(G)|$. The vertices v_1 and v_n cover all the vertices of this path in G with the exception of $b = v_p$. As in the usual proof of Dirac's Theorem, we must have, in $\bar{G}$, that v_n is not adjacent to v_{i-1} if v_1 is adjacent to v_i otherwise G would be hamiltonian. Let $M = \max i$ such that $v_1 v_i \in E(G)$ and $m = \min j$ such that $v_j v_n \in E(G)$. We consider two possibilides, $p > M$ and $p < M$. We will consider only some of the resulting subcases in detail since other cases are similar and we leave them to the reader.

Case A. $p > M$.

Given $[v_1, v_n] \to v_p$ and $p > M$, then $v_1 v_i \in E(G)$ for $i = 2, 3, \ldots, M$ and $v_j v_n \in E(G)$ for $j = m, m+1, \ldots, n-1; j \neq p$. We consider three possibilities for m.

Case A1. $m = M + 1$.

Case A1.1 There exists an edge $v_i v_j \in E(G)$ with $1 \leq i < M, m < j \leq n$. Then if $v_{j-1} v_n \in E(G)$, G is hamiltonian, hence we may assume that if such a $v_i v_j$ exists then $v_{j-1} v_n \notin E(G)$ which in turn implies that $v_{j-1} = v_p$. (See Figure 2).

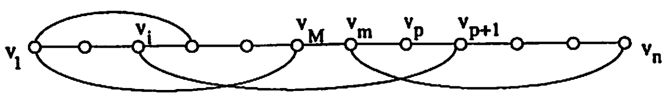

Figure 2

G being conservative and $v_i v_n \notin E(G)$ implies that $\exists x \in V(G)$ such that (*in* $\bar{G}$) either $[v_i, x] \to v_n$ or $[v_n, x] \to v_i$.

Case A1.1.1 $[v_i, x] \to v_n$. In this case we have ($x = v_p$ or $x v_p \in E(G)$) and ($x v_n \notin E(G)$).

If $x = v_p$, i.e. $[v_i, v_p] \to v_n$, we consider two cases. First suppose $j \neq n - 1$, then $v_p v_{j+1} \in E(G)$ and G has a Hamilton path as follows:

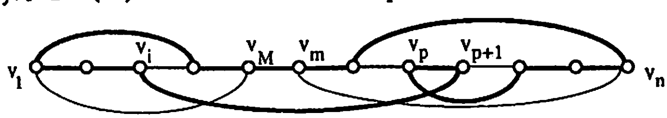

Figure 3

Suppose that $j = n - 1$. Then it is easy to show that G is hamiltonian if $p \neq m+1$ hence we may assume that $p = m+1$ and thus $m = n-3, p = n-2$ and $j = n-1$. Recall that, by assumption, $d(v_p) \geq 3$ which implies that $v_p v_t \in E(G)$ for some $1 \leq t \leq M$. Hamilton circuits in the cases $t = M$ and $t < M$ are illustrated in Figure 4, again a contradiction, thus $x \neq v_p$.

If $x v_p \in E(G)$, since $x v_n \notin E(G)$ (and $x \neq v_p$) then either x belongs to the set $\{v_1, \ldots, v_{M-1}\}$ or $x = v_M$. The first is impossible since this would imply the

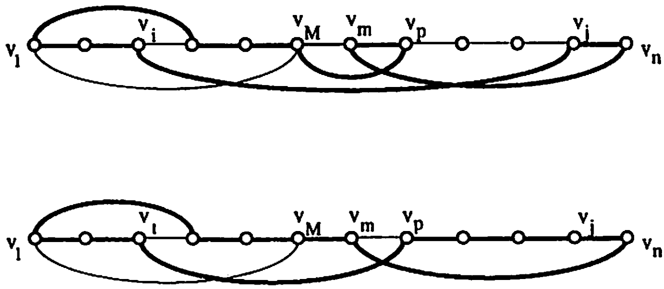

Figure 4

existence of edges $v_i v_p$, $v_{p-1} v_n \in E(G)$ for some $1 \leq i < M$ and $m < p < n$ and G is hamiltonian. Therefore we need $x = v_M$ but this also leads to a contradiction, i.e. the circuit $C = \{v_1, \ldots, v_i, v_j, \ldots, v_n, v_m, \ldots, v_p, v_M, \ldots v_{i+1}, v_1\}$ is hamiltonian. We thus have that Case A1.1.1 cannot hold.

Case A1.1.2 $[v_n, x] \rightarrow v_i$. In this case we have ($x = v_p$ or $xv_p \in E(G)$) and ($xv_i \notin E(G)$).

If $xv_p \in E(G)$ we have $x \notin \{v_1, \ldots, v_{M-1}\}$ or G is hamiltonian as before. If $x = v_k$ for some $k > M$ then neither v_n nor x is adjacent to v_1 contradicting the fact that $[v_n, x] \rightarrow v_i$ unless, in this last case, $i = 1$. Now the existence of the edge $v_1 v_j$ contradicts the choice of M. Thus we must have $x = v_M$, however, as we have seen before, this leads to the Hamilton circuit $C = \{v_1, \ldots, v_i, v_j, \ldots, v_n, v_m, \ldots, v_p, v_M, \ldots, v_{i+1}, v_1\}$ again a contradiction.

If $x = v_p$, i.e. $[v_n, v_p] \rightarrow v_i$, since $m > M$ we must have that v_p is adjacent to every vertex in $\{v_1, \ldots, v_M\}/v_i$, in particular $v_p v_M \in E(G)$ and we have the same contradiction as above. Therefore we must have,

Case A1.2 $v_i v_j \notin E(G)$ for all $1 \leq i < M, m < j \leq n$. Thus *any* $v_1 - v_n$ path must pass through at least one of v_M and v_m. G, by assumption, was 2-connected so that v_M and v_m are not cut points which implies there exists an i, $1 \leq i < M$, and j, $m < j \leq n$, such that $v_i v_m$ and $v_M v_j \in E(G)$.

The following cases that conclude Case A.1 assume the existence of these two edges.

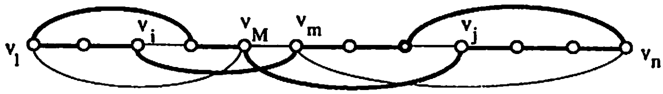

Figure 5

It is easy to see (Figure 5) that $v_{j-1} v_n \in E(G)$ would imply a Hamilton circuit thus we may assume that $v_{j-1} = v_p$. Since we are assuming in particular that $v_i v_n \notin E(G)$ we either have that $\exists x \in V(G)$ such that $[v_i, x] \rightarrow v_n$ or $[v_n, x] \rightarrow$

v_i (in G). In each of these cases we must allow for the possibility that $xv_p \in E(G)$ or $x = v_p$.

Case **A1.2.1** $[v_i, x] \to v_n$. In this case we have ($xv_p \in E(G)$ or $x = v_p$) and ($xv_n \notin E(G)$).

If $xv_p \in E(G)$ then $x \notin \{v_1, \ldots, v_{M-1}\}$ leaving $x = v_M$ as the only possibility, however as we can see (Figure 5 with v_j relabelled v_p) that this again contradicts the fact that G is non-hamiltonian.

If $x = v_p$, (i.e. $[v_i, v_p] \to v_n$), and also if $j \neq n-1$ then $v_pv_{j+1} \in E(G)$ and we have a contradiction similar to the case in Figure 3. Thus take $j = n-1$ and consider the possibilities of $p \neq m+1$ and $p = m+1$. The first case has a Hamilton circuit $C = \{v_1, \ldots, v_i, v_m, v_n, v_{m+1}, \ldots, v_j, v_M, \ldots, v_{i+1}, v_1\}$ while if $p = m+1$, $d(v_p) \geq 3$ implies $v_pv_t \in E(G)$ for some $1 \leq t \leq M$. If $t < M$ we are done as this is an earlier case. If $t = M$ we find the Hamilton circuit $C = \{v_1, \ldots, v_i, v_m, v_n, \ldots, v_{m+1}, v_M, \ldots, v_{i+1}, v_1\}$ and again we're done.

We are left with one final possibility under Case A1.

Case **A1.2.2** $[v_n, x] \to v_i$. In this case we have ($xv_p \in E(G)$ or $x = v_p$) and ($xv_i \notin E(G)$).

If $xv_p \in E(G)$ then $x \notin \{v_1, \ldots, v_{M-1}\}$ or we have a previous case. However we then must have $xv_1 \in E(G)$ leaving us with $x = v_M$ (note that $v_i \neq v_1$ by the maximality of M). Now consider the circuit $C = \{v_1, \ldots, v_i, v_m \ldots, v_{p-1}, v_n, \ldots, v_p, v_M, \ldots, v_{i+1}, v_1\}$ and we have a contradiction once more.

Finally, if $x = v_p$, (i.e. $[v_n, v_p] \to v_i$), we must have that v_p is adjacent to every vertex in $\{v_1, \ldots, v_{M-1}\}/v_i$ by the choice of m. If $i \neq 1$, then $v_1v_p \in E(G)$ contradicts the choice of M. If $i = 1$, then v_1v_m contradicts the choice of M.

Case **A2.** $M = m$.

In the event that $M = m$ most cases are similar to those in Case A1. We outline them for completeness but leave the details to the reader.

If $v_iv_j \in E(G)$ where $1 \leq i < M = m < j \leq n$ and $v_{j-1}v_n \in E(G)$ then G is hamiltonian thus we may assume if such a v_iv_j exists then $v_{j-1}v_n \notin E(G)$ which in turn implies $v_{j-1} = v_p$. We have $v_iv_n \notin E(G)$ and need to consider that there exists $x \in V(G)$ such that *in* G

Case**A2.1** $[v_i, x] \to v_n$. In this case we have ($xv_p \in E(G)$ or $x = v_p$) and ($xv_n \notin E(G)$).

Case **A2.2** $[v_n, x] \to v_i$. In this case we have ($xv_p \in E(G)$ or $x = v_p$) and ($xv_i \notin E(G)$).

In the first case we are led to $x = v_p$ and we need to consider subcases $j \neq n-1$; $j = n-1$ and $p \neq M+1$; and $j = n-1$ and $p = M+1$. In the last of these subcases we again appeal to the fact that $d(v_p) \geq 3$. In Case A2.2.2 the main division of the subcases centers on whether or not $v_p = v_{M+1}$. In the last of these subcases we also appeal to the fact that $d(v_p) \geq 3$. The final contradiction helps us conclude that no such edge $v_iv_j \in E(G)$ exists under Case A2 implying that

v_M is a cut point, however this contradicts the fact that G is 2-connected and we're done.

Case A3. $m = M + 2$.

Suppose that $M = m - 2$ so that $v_p = v_{M+1}$. As before we may assume there is no $v_i v_j \in E(G)$ where $1 \leq i < M$, $m < j < n$ and, since $d(v_p) \geq 3$, $\exists t$ such that $1 \leq t < M$ and $v_t v_p \in E(G)$. Since neither v_p nor v_m is a cut point we must have either that $\exists j$ such that $v_M v_j \in E(G)$ where $m < j \leq n$ and we have a Hamilton circuit (see Figure 6)

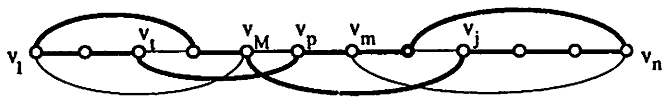

Figure 6

or otherwise $\exists q$ such that $v_p v_q \in E(G)$, $m < q \leq n$ and $v_M v_m \in E(G)$ and again we have a contradiction (see Figure 7).

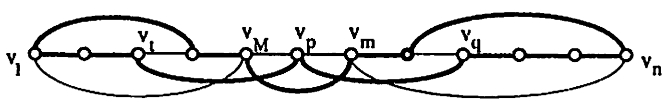

Figure 7

As $m < M$ is impossible under the assumption that $p > M$, Case A is finished.

Case B. $p < M$.

In this case we have the following possibilities: B1: $m = M + 1$; B2: $M = m$; B3: $m < p$. The first two cases are symmetrical to earlier cases dealt with under the assumption that $p > M$, leaving only the last case to consider. Again we outline the steps.

Case B3: $m < p$. The situation is illustrated in Figure 8.

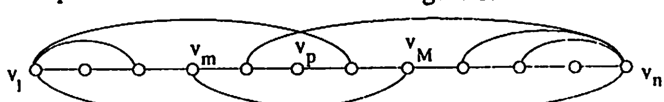

Figure 8

We have v_1 adjacent to all of $\{v_2, \ldots, v_{m-1}, v_{p+1}, \ldots, v_M\}$ while v_n is adjacent to the set $\{v_m, \ldots, v_{p-1}, v_{M+1}, \ldots, v_{n-1}\}$. v_p has degree at least 3 but cannot be adjacent to any of the vertices in the subpaths $\{v_1, \ldots, v_{m-2}\}$, $\{v_m, \ldots, v_{p-2}\}$, $\{v_{p+2}, \ldots, v_M\}$ and $(v_{M+2}, \ldots, v_n\}$ without creating a Hamilton circuit. For example, suppose $v_p v_{p-2} \in E(G)$, then $C = \{v_1, \ldots, v_{p-2}, v_p, v_{p-1}, v_n, v_{n-1}, \ldots, v_{p+1}, v_1\}$ is a Hamilton circuit in G. On the other hand, if for example $v_p v_{m-1} \in$

$E(G)$, then C $= \{v_1, \ldots, v_{m-1}, v_p, v_{p-1}, \ldots, v_m, v_n, v_{n-1}, \ldots, v_{p+1}, v_1\}$ is a Hamilton circuit in G. The edge $v_p v_{M+1} \in E(G)$ leads to a similar contradiction.

Thus G is hamiltonian if and only if G' is hamiltonian and we are done.

In the proof of Theorem 1 it is important to note that at each stage we only used edges *of* G (with the possible exception of the existence of the Hamilton path from v_1 to v_n). Suppose that G is a 2-connected conservative graph, we define its *domination closure*, $D^*(G)$, to be G together with all edges ac where $[a, c] \to b$ *in* G for some vertex b satisfying $d(b) \geq 3$. If a graph G contains a spanning 2-connected conservative subgraph G_0 then we take the domination closure of G to have edge set $E(D^*(G)) = E(G) \cup E(D^*(G_0))$. Theorem 1 then leads to

Theorem 2. *Suppose that G contains a spanning 2-connected conservative subgraph then the domination closure of G is hamiltonian if and only if G is hamiltonian.*

Corollary 1. *Suppose that G is 2-connected and critical, then the domination closure of G is hamiltonian if and only if G is hamiltonian.*

If we return to the graph of Figure 1 we in fact find that in this case $D^*(G)$ is almost complete (see Figure 9). The useful fact here is that in $D^*(G)$ all the degrees are now large enough so that the Bondy-Chvátal closure applies, i.e. $cl(D^*(G)) = K_8$, and G is thus seen to be hamiltonian.

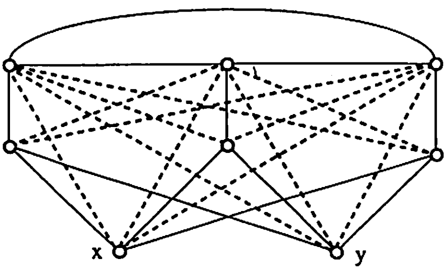

Figure 9

Note that the condition used by Faudree et al, [3], does not lead to the conclusion that G is hamiltonian in cases where there are a few points of small degree since NC would be small in this case. Further, the $cl(G)$ of Bondy and Chvátal will not be complete in such cases. The advantage of $D^*(G)$ is that there may well be a number of points of small degree in G which are no longer small in $D^*(G)$, and we can then apply a different neighbourhood condition. The next result is an example of such a condition.

Corollary 2. *Under the conditions of Theorem 2, G is hamiltonian if $cl(D^*(G))$ is complete.*

3. Concluding Remarks

The condition that $d(b) \geq 3$ is possibly not necessary. In most of the cases where it was used it can be avoided but we could not see how to do this in particular in case B3. It is unlikely that the conservative condition could be dropped; for example, the complete bipartite graph $K_{2,3}$ has the property that the addition of any new edge creates a new dominating set of size *at most* 2 but it is clearly non-hamiltonian. Replacing one of the '3' vertices here by a complete subgraph gives a new graph with similar properties and so on.

Acknowledgement

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References

1. Bondy, J.A. and Chvátal, V., *A method in graph theory*, Discrete Math. **15** (1976), 111–135.
2. Dirac, G.A., *Some theorems on abstract graphs*, Proc. London Math. Soc. **2** (1952), 69–81.
3. Faudree, R.J., Gould, R.J., Jacobson, M.S., Schelp, R.H., *Extremal Problems Involving Neighborhood Unions*, Journal of Graph Theory **11** No. 4 (1987), 555–564.
4. Ore, O., *A note on hamilton circuits*, Amer. Math. Monthly 77 (1960), 315–321.
5. Sumner, D.P and Blitch, P., *Domination Critical Graphs*, J. Comb. Theory. Ser. B **34** (1983), 65–76.
6. Sumner, D.P and Wojcicka, E., *Connectedness Properties of k-Domination Critical Graphs*, preprint.
7. Wojcicka, E., *Hamiltonian Properties of Domination-Critical Graphs*, Journal of Graph Theory **14, No.2** (1990), 205–215.