Some Data Concerning the Number of Latin Rectangles

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If $k \leq n$ by a $k \times n$ latin rectangle is meant a $k \times n$ array in which each row is a permutation of the set $\{1, 2, \ldots, n\}$ and no two elements are repeated in any column. Such a rectangle is said to be reduced if the first row and column are in standard order. If $L(k,n)$ and $R(k,n)$ denote the total number and number of reduced $k \times n$ latin rectangles, then we clearly have

$$L(k,n) = n!(n-1)\ldots(n-k+1)R(k,n), \quad (1)$$

so that for purposes of enumeration, it suffices to enumerate the reduced latin rectangles. Thus when $k = n$ we obtain the usual concepts from the theory of latin squares.

The literature [1–3,5,6,9,11,13–16] concerning latin square enumeration points out that a number of errors have been made in attempts to enumerate latin squares of various orders. For $n = 7$ Sade [14] found missing squares in Norton's count [11]. For $n = 8$ Wells reported the value of $R(8,8)$ in [15] but as pointed out in Kolesova, Lam, and Thiel [9], this number disagrees with calculations made in Brown [3] and Arlazarov, Baraev, Golfand, and Faradzhev [1]. Bammel and Rothstein [2] reported the value for $R(9,9)$ and in passing, verified the $R(8,8)$ value of Wells. Finally in Kolesova, Lam, and Thiel [9], the value for $R(8,8)$ of Wells was again independently verified. As stated in [9] this "points out the difficulty of performing an accurate enumeration. With the increasing use of computers in mathematics, the correctness of such "proofs" is very difficult to obtain. We should borrow an idea from the physical sciences, where a new result is accepted only after it has been independently verified."

With this in mind, for his master's paper the second author attempted to enumerate latin squares using a modified version of Sade's algorithm [13], see also Dénes and Keedwell [5, pp. 142–144]. He set out with the following goals in mind:

(1) To independently verify again that Well's value for $R(8,8)$ is correct.
(2) To independently verify that Bammel and Rothstein's value for $R(9,9)$ is correct.
(3) If lady luck is smiling, to obtain $R(10,10)$, which is currently unknown.

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As a result (1) and (2) were accomplished and while in spite of considerable effort we were unable to obtain $R(10, 10)$ for lack of sufficient computer memory, we have obtained several previously unreported values for $R(k, n)$ for $k \leq n \leq 8$. Consequently we felt it worthwhile to report these values in Table 1 as well as to verify both the result of Wells and that of Bammel and Rothstein.

Considerable efforts have been put forth to enumerate latin rectangles of various sizes, and as most results have been stated in terms of normalized (row one in standard order) rectangles, we will for the moment also adopt this convention. If $N(k, n)$ denotes the number of normalized $k \times n$ latin rectangles, then clearly from (1),

$$N(k,n) = (n-1)\dots(n-k+1)R(k,n). \tag{2}$$

For each $n \geq 3$, $N(2, n)$ is the number of derangements of $\{1, 2, \dots, n\}$ and $N(3, n)$ can be calculated in terms of derangements and ménage numbers, see [5, Sect. 4.4]. Light [10] gave the values of $N(4, n)$ for $n \leq 8$ and an explicit, though complicated, formula for $L(4, n)$ and thus for $N(4, n)$ is given in Pranesachar [12]. As indicated in Brualdi and Ryser [4, p. 285], several authors have obtained formulas for $L(k, n)$ in terms of the Möbius function for partitions of a set. For $k \geq 4$ however these formulas are difficult to evaluate. We refer the reader to [12] for additional references, to [7] for asymptotic results and [8] for tables for small values of n.

2. Sade's algorithm and reduced latin rectangles

In 1948 Sade [13] enumerated the reduced latin squares of order 7 and gave 16,942,080 as the value of $R(7, 7)$. His counting algorithm made use of an equivalence relation which we will denote by $\sim$. If $1 \leq k < n$ and X and Y are two $k \times n$ latin rectangles, then $X \sim Y$ if there exists some combination of permutation of rows, permutation of columns, and permutation of symbols which transforms X into Y. It is not too hard to prove the very useful fact that if $X \sim Y$, then the two latin rectangles can be extended or completed to $n \times n$ latin squares in the same number of ways. Thus when calculating $R(n, n)$ one need only keep track of the inequivalent reduced latin rectangles and the number of rectangles which they represent. We give the essence of Sade's algorithm, which basically says for a given k, to generate all $(k+1) \times n$ latin rectangles formed from the inequivalent $k \times n$ rectangles.

As indicated earlier using our own variation of Sade's algorithm, we confirmed that the values of $R(8, 8)$ and $R(9, 9)$ reported in [15] and [2] are indeed correct. As we were unable to obtain the value of $R(10, 10)$, i.e. the number of reduced latin squares of order 10, we will not describe our algorithm in detail except to say that it is another variation on Sade's method. It differs essentially from the previous computer adaptations in that we attempted to use "standard forms", somewhat

Let $\mathcal{L}(k,n)$ denote the set of all reduced $k \times n$ latin rectangles

1. Begin with $\mathcal{L}(1,n) = (1,2,\ldots,n)$. Here $k = 1$ and $|\mathcal{L}(1,n)| = |\mathcal{L}(1,n)/\sim| = 1$.
2. Form $\hat{\mathcal{L}}(k+1,n)$, which we define to be the set of all RLR's formed by extending the representatives of all equivalence classes of $k \times n$ RLR's to all possible $(k+1) \times n$ RLR's.
3. Form $\hat{\mathcal{L}}(k+1,n)/\sim$, and for each equivalence class formed record the size (the number of rectangles in the equivalence class), and a representative (say, the first) rectangle of each class. All other rectangles are discarded.
4. Repeat steps 2 and 3 until $k = n-1$.

Figure 1: Sade's Algorithm

analogous to Jordan canonical forms, for the incidence matrices of Wells associated to a latin rectangle. One difficulty with this approach is that computation of the inequivalent standard forms was very space and memory intensive.

As Sade's algorithm enumerates the total number of reduced latin squares of order n, it is easy to assume that it also in the process, enumerates the $k \times n$ reduced latin rectangles for each $k \leq n$. It is important to point out that Sade's algorithm as well as the adaptations of the algorithm used by Wells [15], [16] and Bammel and Rothstein [2] appear to enumerate $\mathcal{L}(k,n)$ only for $k = 1,2,n-1,n$, (this has been confirmed by machine for all $n \leq 7$). Thus for $n \leq 4$, $\hat{\mathcal{L}}(k,n) = \mathcal{L}(k,n)$ but $\hat{\mathcal{L}}(k,n) \subsetneq \mathcal{L}(k,n)$ when $5 \leq n \leq 7$ and $2 < k < n-1$. For example, by following Sade's algorithm we can see from Wells [16,p. 205] that $|\hat{\mathcal{L}}(3,5)| = 44$ but from [8] we know that $|\mathcal{L}(3,5)| = 46$. It seems likely that for $n \geq 5$, Sade's algorithm gives the correct value for $R(k,n)$ only for $k = 1,2,n-1,n$, (of course $R(n-1,n) = R(n,n)$). As illustrated by the above example, the counts from Sade's method appear to only provide a lower bound for $R(k,n)$.

In order to enumerate the reduced latin rectangles for $n \leq 8$, we devised yet another variation of Sade's method which can be used to count the total number of $k \times n$ reduced latin rectangles. It is essentially the same as that shown in Figure 1, except that at the beginning, the symbols 1 through k are filled in for the entire first column. The idea now is that if X' and Y' are two $j \times n$ reduced latin rectangles $(j \leq k)$ with symbols $1,\ldots,k$ in the first column and $X' \sim Y'$, then the number of ways that X' and Y' can be extended to $k \times n$ reduced latin rectangles is the same.

In Table 1 we list the values of $R(k,n)$ for $2 \leq k \leq n-1$ and $n \leq 8$.

n	k	$R(k,n)$
4	2	3
	3	4
5	2	11
	3	46
	4	56
6	2	53
	3	1,064
	4	6,552
	5	9,408
7	2	309
	3	35,792
	4	1,293,216
	5	11,270,400
	6	16,942,080
8	2	2,119
	3	1,673,792
	4	420,909,504
	5	27,206,658,048
	6	335,390,189,568
	7	535,281,401,856

Table 1: Number of Reduced Latin Rectangles

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