Existence of Indecomposable $B[4,6;v]$ without Repeated Blocks

Shen Hao[1]

Department of Applied Mathematics, Shanghai Jiao Tong University
Shanghai 200030, The People's Republic of China

Abstract. It is proved in this paper that there exists a simple $B[4,6;v]$ for every $v \geq 6$. It is also proved that there exists an indecomposable simple $B[4,6;v]$ for every $v \geq 6, v \notin \{12,13,16,17,20\}$.

1. Introduction

A *balanced incomplete block design* $B[k,\lambda;v]$ is a pair $(V,\mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of k-subsets (called *blocks*) of V such that each 2-subset of V is contained in exactly λ blocks. A $B[k,\lambda;v]$ is called *simple* and denoted $NB[k,\lambda;v]$ if it contains no repeated blocks.

Let $(V,\mathcal{B})$ be a $B[k,\lambda;v]$; if there is a subcollection $\mathcal{A}$ of $\mathcal{B}$; such that $(V,\mathcal{A})$ is a $B[k,\lambda';v]$ for some λ', $1 \leq \lambda' < \lambda$, then $(V,\mathcal{B})$ is called *decomposable*. Otherwise it is called *indecomposable*.

It is not difficult to verify that the following conditions are necessary for the existence of an $NB[k,\lambda;v]$ or an indecomposable $NB[k,\lambda;v]$:

$$\begin{aligned} \lambda(v-1) &\equiv 0 \pmod{(k-1)}; \\ \lambda v(v-1) &\equiv 0 \pmod{k(k-1)}; \\ \lambda &\leq \binom{v-2}{k-2}. \end{aligned} \tag{1}$$

For given k and λ, any positive integer v satisfying (1) is called *admissible*.

A $B[3,\lambda;v]$ is also known as a λ*-fold triple system*. The existence of simple triple systems and indecomposable simple triple systems has been studied by several authors. The existence of simple triple systems for arbitrary λ was completely determined by Dehon [2]: There exists an $NB[3,\lambda;v]$ for every admissible v. Much less is known concerning the existence of indecomposable $NB[3,\lambda;v]$. Kramer [5] showed that there exists an indecomposable $NB[3,2;v]$ if and only if $v \equiv 0,1 \pmod 3$, $v > 3$ and $v \neq 7$, and there exists an indecomposable $NB[3,3;v]$ if and only if $v \equiv 1 \pmod 2$, $v \geq 5$. The case of $\lambda = 4$ was solved by Colbourn and Rosa [1] who showed that an indecomposable $NB[3,4;v]$ exists if and only if $v \equiv 0,1 \pmod 3$ and $v \geq 10$. For $\lambda = 6$, it was proved independently by Dinitz [3], Milici [7] and Shen [9] that there exists an indecomposable $NB[3,6;v]$ for every $v \geq 8$, $v \neq 9$ with the possible exceptions $v = 10, 11, 12, 13, 15$ and 16.

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For $k = 4$, it was proved by Guo [4] and Phelps and Rosa [8] that there exists an indecomposable $NB[4,2;v]$ if and only if $v \equiv 1 \pmod 3$ and $v \geq 7$. Guo [4] also proved that there exists an indecomposable $NB[4,3;v]$ if and only if $v \equiv 0,1 \pmod 4$, $v \geq 5$. For $\lambda = 4,5$, the present author [10] proved that there exists an $NB[4,\lambda;v]$ for every admissible v, and there exists an indecomposable $NB[4,\lambda;v]$ for every admissible v, with a few possible exceptions.

The purpose of this paper is to study the existence of $NB[4,6;v]$ and indecomposable $NB[4,6;v]$, and prove the following theorems;

Theorem 1. *There exists an* $NB[4,6;v]$ *for every* $v \geq 6$.

Theorem 2. *There exists an indecomposable* $NB[4,6;v]$ *for every* $v \geq 6$, $v \notin \{12,13,16,17,20\}$.

2. Existence of $NB[4,6;v]$

If there exists an $NB[4,6;v]$, then obviously $v \geq 6$. We shall prove in this section that there exists an $NB[4,6;v]$ for every $v \geq 6$.

Let $V = Z_v$ and $(Z_v, \mathcal{B})$ be a $B[k,\lambda;v]$. If for any $B \in \mathcal{B}$, $B = \{a_1, a_2, \ldots, a_k\}$, we have $\{a_1+1, a_2+1, \ldots, a_k+1\} \in \mathcal{B}$, then $(Z_v, \mathcal{B})$ is called a *cyclic design*.

Let $B = \{a_1, a_2, \ldots, a_k\}$ be any subset of Z_v. For $t \in Z_v$, $(t,v) = 1$, let tB denote the subset $\{ta_1, ta_2, \ldots, ta_k\}$.

Lemma 1. *If $v = p$ is a prime and $p \geq 7$, then there exists a cyclic* $NB[4,6;v]$.

Proof: Let $B = \{0,1,2,3\}$ and $tB = \{0,t,2t,3t\} t = 1,2,\ldots,(p-1)/2$. Then it can be easily checked that with $V = Z_p$ as point set and $B, 2B, \ldots, (p-1)/2 \cdot B$ as base blocks, we obtain a cyclic $NB[4,6;v]$.

Let $(V,\mathcal{B})$ be a $B[k,\lambda;v]$ and $(W,\mathcal{A})$ be a $B[k,\lambda;w]$. If $W \subset V$ and $\mathcal{A}$ is a subcollection of $\mathcal{B}$, then $(W,\mathcal{A})$ is called a *subdesign* of $(V,\mathcal{B})$ or $(W,\mathcal{A})$ is said to be *embedded* in $(V,\mathcal{B})$.

Lemma 2. *If p is a prime and there exists an $NB[4,6;v]$, $2v+1 \leq p$, then any $NB[4,6;v]$ can be embedded in an $NB[4,6;p+v]$.*

Proof: Let $V = \{\infty_1, \infty_2, \ldots, \infty_v\}$ and $(V,\mathcal{A})$ be an $NB[4,6;v]$. By Lemma 1, there exists a cyclic $NB[4,6;p]$ on Z_p with $B_t = tB = \{0,t,2t,3t\}$, $t = 1,2,\ldots,(p-1)/2$ as base blocks. As $2v+1 \leq p$, we have $v \leq (p-1)/2$. Now for each $t = 1,2,\ldots,v$, substitute for the base block B_t the following two base blocks

$$B_{t'} = \{\infty_t, 0, t, 2t\}, \qquad B_{t''} = \{\infty_t, 0, t, 3t\}.$$

Let $\mathcal{B}$ denote the set of blocks obtained by developing the base blocks $B_{1'}, B_{1''}, B_{2'}, B_{2''} \ldots, B_{v'}, B_{v''}, B_{v+1}, \ldots, B_{(p-1)/2}$. Let $X = V \cup Z_p$, then $(X, \mathcal{A} \cup \mathcal{B})$ is an $NB[4,6;p+v]$ containing $(V,\mathcal{A})$ as a subdesign.

Lemma 3. *There exists an* $NB[4,6;v]$ *for* $v = 6, 9, 10, 15$ *and* 16.

Proof: We prove the lemma by direct constructions.

$v = 6$ Taking all the 4-subsets of Z_6 as blocks gives an $NB[4,6;6]$.

$v = 9$ $V = Z_9$, Base blocks

$\{0,1,2,4\},\{0,1,3,5\},\{0,1,2,6\},\{0,1,3,6\}$

$v = 10$ $V = Z_9 \cup \{\infty\}$. Base blocks

$\{0,1,2,4\},\{0,1,3,5\},\{0,1,3,6\},\{\infty,0,1,3\},\{\infty,0,1,5\}$.

$v = 15$ $V = Z_{15}$. Base blocks

$\{0,1,3,10\},\{0,1,3,11\},\{0,1,4,6\},\{0,1,4,9\},$
$\{0,1,5,7\},\{0,2,8,12\},\{0,1,3,7\}$.

$v = 16$ $V = Z_{15} \cup \{\infty\}$. Base blocks

$\{0,1,3,10\},\{0,1,3,11\},\{0,1,4,6\},\{0,1,4,9\},$
$\{0,1,5,7\},\{0,2,8,12\},\{\infty,0,1,4\},\{\infty,0,2,8\}$.

Lemma 4. *There exists an* $NB[4,6;21]$ *containing an* $NB[4,6;6]$ *and there exists an* $NB[4,6;22]$ *containing an* $NB[4,6;7]$.

Proof: Let $V_1 = \{\infty_1,\infty_2,\ldots,\infty_6\}$ and $(V_1,\mathcal{A}_1)$ be an $NB[4,6;6]$. Let $X_1 = V_1 \cup Z_{15}$, $\mathcal{B}_1 = \mathcal{A}_1 \cup \mathcal{B}_1'$ where $\mathcal{B}_1'$ is obtained by developing the following base blocks

$\{0,1,3,7\},\{\infty,0,2,5\},\{\infty_1,0,1,7\},\{\infty_2,0,1,4\},\{\infty_2,0,2,7\},$
$\{\infty_3,0,1,3\},\{\infty_3,0,4,10\},\{\infty_4,0,1,6\},\{\infty_4,0,3,7\},$
$\{\infty_5,0,1,5\},\{\infty_5,0,2,9\},\{\infty_6,0,2,6\},\{\infty_6,0,3,8\}$.

Then it can be checked that $(X_1,\mathcal{B}_1)$ is an $NB[4,6;21]$ containing $(V_1,\mathcal{A}_1)$.

Let $V_2 = \{\infty_1,\infty_2,\ldots,\infty_7\}$ and $(V_2,\mathcal{A}_2)$ be an $NB[4,6;7]$. Let $X_2 = V_2 \cup Z_{15}$, $\mathcal{B}_2 = \mathcal{A}_2 \cup \mathcal{B}_2'$ where $\mathcal{B}_2'$ is obtained by developing the following base blocks

$\{\infty_1,0,1,4\},\{\infty_1,0,2,9\},\{\infty_2,0,2,5\},\{\infty_2,0,1,7\},$
$\{\infty_3,0,3,4\},\{\infty_3,0,2,7\},\{\infty_4,0,1,3\},\{\infty_4,0,4,10\},$
$\{\infty_5,0,1,6\},\{\infty_5,0,3,7\},\{\infty_6,0,1,5\},\{\infty_6,0,2,8\},$
$\{\infty_7,0,2,6\},\{\infty_7,0,3,10\}$.

Then $(X_2,\mathcal{B}_2)$ is an $NB[4,6;22]$ containing $(V_2,\mathcal{A}_2)$ as a subdesign.

Lemma 5. *For each* $6 \le v \le 72$, *there exists an* $NB[4,6;v]$. *Further, for every* v *such that* $19 \le v \le 72, v \ne 20$, *there exists an* $NB[4,6;v]$ *containing an* $NB[4,6;w]$ *where* $w = 6,7,8,9,10$ *or* 11.

Proof: For $v \in \{7,11,13,17,19\}$, the existence of an $NB[4,6;v]$ follows from Lemma 1. For $v \in \{8,12,14,18,20\}$, the existence of an $NB[4,6;v]$ follows by the method of Lemma 2. For $v \in \{6,9,10,15,16\}$, the existence of an $NB[4,6;v]$ follows from Lemma 3. By Lemma 4, there exists an $NB[4,6;21]$

containing an$NB[4,6;6]$ and there exists an $NB[4,6;22]$ containing an $NB[4,6;7]$. If $v = 19$ or $23 \leq v \leq 72$, then there exists a prime p and an integer w, $w \in \{6,7,8,9,10,11\}$ and $p \geq 2w+1$, such that $v = p + w$. Thus by Lemma 2, for every $19 \leq v \leq 72$, $v \neq 20$, there exists an $NB[4,6;v]$ containing an $NB[4,6;w]$ where $w = 6,7,8,9,10$ or 11. This completes the proof.

Let v be a positive integer and K be a set of positive integers, A *pairwise balanced design* $B[K,1;v]$ is an ordered pair $(V,\mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a set of subsets (called blocks) of V such that $|B| \in K$ for every $B \in \mathcal{B}$, and each pair of distinct elements of V is contained in a unique block of $\mathcal{B}$.

Let K be a given set of positive integers and

$$B(K) = \{v/\text{there exists a} B[K,1;v]\}$$

K is called a *PBD-closed set* if $B(K) = K$.

For given positive integers k and λ, let

$$NB(k,\lambda) = \{v/\text{there exists an} NB[k,\lambda;v]$$

The following lemma can be easily proved.

Lemma 6. *$NB[k,\lambda]$ is a PBD-closed set.*

A *transversal design* $TD[k,n]$ is an ordered triple $(V,\mathcal{G},\mathcal{A})$ where V is a v-set, $v = kn$, $\mathcal{G}$ is a set of n-subsets (called groups) of V, $\mathcal{G}$ partitions V, and $\mathcal{A}$ is a set of k-subsets (called blocks) of V such that any block intersects each group in a unique element, and each pair of elements from distinct groups is contained in a unique block. It is well known that the existence of a $TD[k,n]$ is equivalent to the existence of k-2 mutually orthogonal Latin squares of order n.

Now we are in a position to prove our fundamental lemma:

Lemma 7. *For every $v \geq 72$, there exists an $NB[4,6;v]$ containing an $NB[4,6;6]$.*

Proof: Let $k,n \geq 7$ and $(X,\mathcal{G},\mathcal{A})$ be a $TD[k,n]$, where $\mathcal{G} = \{G_1, G_2, \ldots, G_k\}$. Let $1 \leq s_i \leq n-6$ for $i = 7,8,\ldots,k$. Deleting s_i points from $G_i, 7 \leq i \leq k$, we obtain a $B[K,1;v]$ with $v = kn - \Sigma_{i=7}^{k} s_i$, block sizes ≥ 6 and at least one block of size 6. Let $k = 8, n = 11$ and then let $k = 9, n = 9$,11 13,16,19,25,31 and 43, we obtain such a $B[K,1;v]$ for every v $72 \leq v \leq 384$. As there is a $TD[7,n]$ for every $n \geq 63$[5], there exists a $B[K,1;v]$ with block sizes ≥ 6 and at least one block of size 6, for every $v \geq 384$. Thus, by Lemma 5 and Lemma 6, we have proved that there exists an $NB[4,6;v]$ containing an $NB[4,6;6]$, by induction.

As a consequence of Lemma 5 and Lemma 7, we have proved the following theorem:

Theorem 1. *There exists an $NB[4,6;v]$ for every $v \geq 6$.*

3. Indecomposability

In this section, we give an almost complete solution to the existence of indecomposable $NB[4,6;v]$.

The following lemma is useful in the construction of indecomposable designs and the proof is obvious:

Lemma 8. *If an $NB[k,\lambda;v]$ contains an indecomposable subdesign $NB[k,\lambda;v']$ for some $v' < v$, then it is also indecomposable.*

Lemma 9. *There exists an indecomposable $NB[4,6;v]$ for every $v \equiv 2,3,6,11$ (mod 12), $v \geq 6$.*

Proof: The existence of an $NB[4,6;v]$ for every $v \geq 6$ is proved in Theorem 1. The necessary conditions for the existence of a $B[4,\lambda;v]$ are

$$\lambda(v-1) \equiv 0(\mod 3),$$
$$\lambda v(v-1) \equiv 0(\mod 12).$$

It is easy to see that for $v \equiv 2,3,6$ or 11 (mod 12), there does not exist any $B[4,\lambda;v]$ with $\lambda < 6$, Thus, for $v \equiv 2,3,6$ or 11 (mod 12), every $NB[4,6;v]$ must be indecomposable.

Lemma 10. *There exists an indecomposable $NB[4,6;v]$ for $v \in \{7,8,9,10\}$.*

Proof: It can be checked that the $NB[4,6;9]$ and $NB[4,6;10]$ constructed in Lemma 3 are in fact indecomposable. Now we construct an indecomposable $NB[4,6;7]$ and an indecomposable $NB[4,6;8]$ as follows:

$v = 7$ $\quad V = Z_7$. $\quad$ Base blocks

$\{0,1,2,3\},\{0,1,3,4\},\{0,1,,3,5\}$.

$v = 8$ $\quad V = Z_7 \cup \{\infty\}$. $\quad$ Base blocks

$\{0,1,3,4\},\{0,1,3,5\},\{0,1,2,\infty\},\{0,1,3,\infty\}$.

With the above preparation, we can prove the following theorem:

Theorem 2. *There exists an indecomposable $NB[4,6;v]$ for every $v \geq 6, v \notin \{12,13,16,17,20\}$.*

Proof: For $v = 6,11,14,15$ and 18, the existence of an indecomposable $NB[4,6;v]$ follows from Lemma 9. For $v = 7,8,9$ and 10, the existence of an indecomposable $NB[4,6;v]$ follows from Lemma 10. By Lemma 5, for every $19 \leq v \leq 72, v \neq 20$, there exists an $NB[4,6;v]$ containing an indecomposable $NB[4,6;w]$ where $w \in \{,7,8,9,10,11\}$, and by Lemma 7, for every $v \geq 72$, there exists an $NB[4,6;v]$ containing an indecomposable $NB[4,6;6]$. Thus, by Lemma 8, for every $v \geq 19, v \neq 20$, there exists an indecomposable $NB[4,6;v]$. This completes the proof.

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