# A characterisation of well-covered cubic graphs

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**Abstract.** A graph is said to be *well-covered* if all maximal independent sets of vertices in the graph have the same cardinality. Determining whether a graph is well-covered has recently been shown (independently by Chvátal and Slater and by Sankaranarayana and Stewart) to be a co-NP-complete problem. In this paper we characterise all well-covered cubic (3-regular) graphs. Our characterisation yields a polynomial time algorithm for recognising well-covered cubic graphs.

## 1. Introduction

All graphs in this paper are finite, with no loops or multiple edges. If two vertices $u$ and $v$ are adjacent in a graph, we shall write $u \sim v$; otherwise we shall write $u \not\sim v$. An *independent set* in a graph is a set of mutually nonadjacent vertices. The cardinality of a maximum independent set in $G$ will be denoted $\alpha(G)$. A *vertex cover* in a graph is a set of vertices such that every edge is incident with at least one vertex in the set.

In 1970 Plummer [12] introduced the idea of a *well-covered* graph, a graph in which all maximal (with respect to inclusion) independent sets are maximum. A graph is well-covered if and only if all maximal independent sets have the same cardinality. Algorithmically a graph is well-covered if and only if the greedy algorithm for constructing independent sets is always guaranteed to find a maximum independent set. Well-covered graphs can also be described as graphs in which every minimal vertex cover is minimum.

Plummer [12] investigated the relationship between well-covered graphs and some other covering concepts for graphs. Berge [1] studied the relationship between well-covered graphs and various other properties related to independent sets. Lewin [11] investigated graphs which have the edge analogue of well-coveredness: every maximal matching is maximum. Other authors have studied various subclasses of well-covered graphs [6,13,15,16]. Of particular interest to us will be

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the result of Finbow, Hartnell and Nowakowski [7], who characterised the well-covered graphs of girth at least 5; their result is stated in Section 3. The same authors have also investigated well-covered graphs with no 4- or 5-cycles [8]. Campbell [3] characterised the well-covered cubic graphs of connectivity 1 or 2, and Campbell and Plummer [3,4] found all 3-connected planar cubic graphs; their results will be stated in Section 2. The aim of the present paper is to characterise all cubic well-covered graphs; we do not make use of the main results of [3,4] except as a check on our final characterisation.

While the maximum independent set problem is NP-complete in general, as shown by Karp [10], a maximum independent set in a well-covered graph can be found very easily. This is counterbalanced by the recent result, due independently to Chvátal and Slater [5] and Sankaranarayana and Stewart [14], that the recognition problem for well-covered graphs is co-NP-complete. It is therefore interesting that while the independent set problem is NP-complete for cubic (or even cubic planar) graphs, as shown by Garey, Johnson and Stockmeyer [9], our results here imply that the recognition problem for cubic well-covered graphs can be solved in polynomial time.

## 2. Some well-covered cubic graphs

In this section we describe an infinite family of well-covered cubic graphs, and six exceptional well-covered cubic graphs which do not belong to this family. The main result of this paper will be that all connected well-covered cubic graphs belong to this family or these six exceptions.

We denote the vertex and edge sets of a graph $G$ by $VG$ and $EG$ respectively. We shall say that a vertex $v$ in a graph *covers* a vertex $u$ if either $v = u$ or $v$ is adjacent to $u$. A set of vertices $S$ is said to cover $u$ if some $v$ in $S$ covers $u$. An independent set is maximal precisely if it covers all vertices of the graph.

**Lemma 2.1.** *Suppose that the graph $A$ of Figure 2.1 is a subgraph of a cubic graph $G$. Let $aa'$, $bb'$, $ee'$ and $ff'$ denote the edges not in $A$ incident with $a$, $b$, $e$ and $f$. If $a'b' \in EG$ and $e'f' \in EG$ then every maximal independent set in $G$ uses exactly two vertices of $A$.*

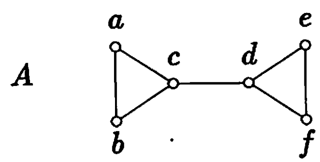



Figure 2.1

Proof: Let $I$ be a maximal independent set in $G$. By maximality of $I$, $I$ covers every vertex of $G$. Let $I_A = I \cap VA$. Obviously $I$ can use at most one element of $\{a, b, c\}$ and at most one element of $\{d, e, f\}$, so $|I_A| \leq 2$. For $I$ to cover $c$ (or $d$), we must have $|I_A| \geq 1$.

Suppose that $|I_A| = 1$. Then $I_A$ must be either $\{c\}$ or $\{d\}$, because otherwise one of $c$ or $d$ is not covered by $I$. Without loss of generality we may suppose that $I_A = \{c\}$. Since $I$ must cover $e$, and $d, e, f \notin I$, we must have $e' \in I$. Similarly, we must have $f' \in I$. But this is a contradiction because $e'$ and $f'$ are adjacent and $I$ is supposed to be an independent set.

Therefore, we conclude that $|I_A| = 2$. ∎

Note that in Lemma 2.1 we do not require that $a'$, $b'$, $e'$ and $f'$ be distinct from each other, or from the vertices of $A$ itself.

**Lemma 2.2.** *Suppose that the graph $B$ of Figure 2.2 is a subgraph of a cubic graph $G$. Let $aa'$, $bb'$, $gg'$ and $hh'$ denote the edges not in $B$ incident with $a, b, g$ and $h$. If $a'b' \in EG$ and $g'h' \in EG$ then every maximal independent set in $G$ uses exactly three vertices of $B$.*

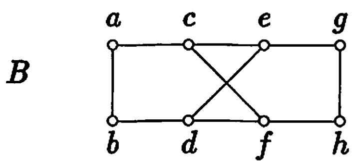



Figure 2.2

Proof: Let $I$ be a maximal independent set in $G$. By maximality of $I$, $I$ covers every vertex of $G$. Let $I_B = I \cap VB$. The set $I_B$ is independent in $B$, although it may not be a maximal independent set in $B$.

Suppose that $|I_B| \geq 4$. Considering the 8-cycle $aceghfdba$ we see that $|I_B| = 4$, and moreover that $I_B$ is either $\{a, e, h, d\}$ or $\{b, c, g, f\}$. However, each of these sets contains adjacent vertices ($e$ and $d$ or $c$ and $f$), a contradiction.

Now suppose that $|I_B| \leq 2$. Divide up the vertices of $B$ into the following complementary sets: left $\{a, b, c, d\}$ and right $\{e, f, g, h\}$, and inner $\{c, d, e, f\}$ and outer $\{a, b, g, h\}$.

The fact that $a'b'$ is an edge of $G$ means that at most one of $a$ and $b$ is covered by a vertex of $I$ outside $B$. To cover the other one, $I_B$ must contain a left vertex. Similarly, $I_B$ must contain a right vertex. Thus, $I_B$ contains exactly one left vertex and one right vertex.

Notice that the four inner vertices of $B$ must be covered by vertices of $I_B$. Thus, $I_B$ cannot contain two outer vertices, because each outer vertex covers only one inner vertex. Also, $I_B$ cannot contain two inner vertices, because the two pairs of nonadjacent inner vertices, $\{c, d\}$ and $\{e, f\}$, do not contain one left and one right vertex. Thus, $I_B$ contains an outer vertex and an inner vertex. Without loss of generality, suppose the outer vertex is $a$. Since the only inner vertex covered by $a$ is $c$, the inner vertex of $I_B$ must cover all of $d$, $e$ and $f$; therefore the inner vertex of $I_B$ must be $d$. But now $I_B$ contains two left vertices, a contradiction.

Since we cannot have $|I_B| \geq 4$ or $|I_B| \leq 2$, we conclude that $|I_B| = 3$. ∎

Again, in Lemma 2.2 we do not require that $a'$, $b'$, $g'$ and $h'$ be distinct from each other, or from the vertices of $B$ itself.

**Lemma 2.3.** *Suppose that the graph $C$ of Figure 2.3 is a subgraph of a cubic graph $G$. Let $aa'$ and $bb'$ denote the edges not in $C$ incident with $a$ and $b$. If $a'b' \in EG$ then every maximal independent set in $G$ uses exactly two vertices of $C$.*

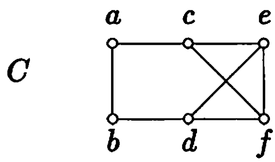



Figure 2.3

Proof: Let $I$ be a maximal independent set in $G$. By maximality of $I$, $I$ must cover every vertex of $C$. Let $I_C = I \cap VC$. Since $a'b' \in EG$, the vertices of $I$ outside of $C$ cover at most one vertex of $C$ ($a$ or $b$), so $I_C$ covers at least five vertices of $C$. Therefore, $|I_C| \geq 2$, because a single vertex of $C$ covers at most four vertices of $C$.

Suppose that $|I_C| \geq 3$. Considering the 6-cycle $acefdba$, we see that $|I_C| = 3$, and moreover that $I_C$ is either $\{a, e, d\}$ or $\{b, c, f\}$. However, both of these sets contain adjacent vertices ($e$ and $d$ or $c$ and $f$), so this is impossible.

Therefore, we conclude that $|I_C| = 2$. ∎

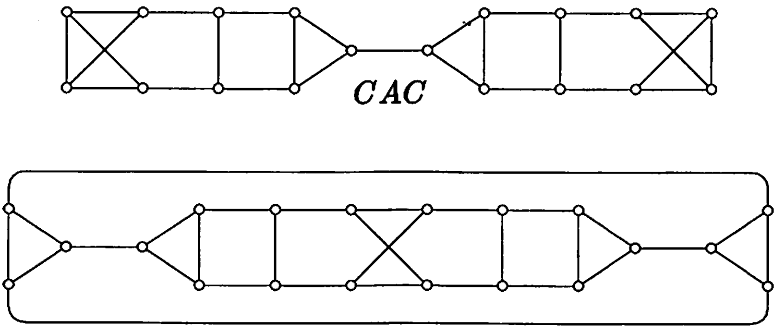
Lemmas 2.1, 2.2 and 2.3 give us the following procedure for constructing an infinite family of well-covered cubic graphs. Define a *terminal pair* to be a pair of adjacent degree two vertices.

**Theorem 2.4.** *Let $W$ denote the class of cubic graphs constructed as follows. Given a collection of copies of $A$, $B$ and $C$, join every terminal pair by two edges to a terminal pair in another (possibly the same) graph, so that the result is cubic. Then every graph in $W$ is well-covered.*

Proof: Let $G \in W$. Each individual copy of $A$, $B$ or $C$ from which $G$ was constructed satisfies the conditions of Lemma 2.1, 2.2 or 2.3, as appropriate. Thus, the size of any maximal independent set in $G$ depends only on the numbers of $A$'s, $B$'s and $C$'s. ∎

Since each copy of $A$ or $B$ has two terminal pairs, and each copy of $C$ has one, this construction amounts to stringing together copies of $A$ and $B$ in cycles, or in paths with a copy of $C$ at each end (so that the number of copies of $C$ must be even). Because $A$ and $B$ have automorphisms exchanging the two vertices in one terminal pair while fixing the vertices in the other terminal pair, the isomorphism class of the resulting graph is not affected by the exact way we join up each pair of terminal pairs. Thus, a connected graph in this family can be described by giving the sequence in which we join up the $A$'s, $B$'s and $C$'s. For example, a graph we might describe as $CAC$ and a graph we might describe as $-ABA-$ (the dashes indicating that the graphs are to be joined in a cycle) are shown in Figure 2.4. Descriptions of this type are unique up to reversal and (in the cyclic case) rotation.

Notice that when forming "cyclic" elements of $W$ we can use just a single $A$ or $B$. We can form $-A-$, which turns out to be $C_3 \times K_2$, and we can form $-B-$,which turns out to be the Möbius ladder with eight vertices.




$-ABA-$

Figure 2.4

There also exist well-covered cubic graphs which do not belong to $W$. Figure 2.5 shows six such graphs. Proof that these graphs are well-covered requires checking all maximal independent sets to verify that they have the same cardinality. We omit the details.

The names of $K^*_{3,3}$ and $Q^{**}$ come from the fact that $K^*_{3,3}$ can be obtained by replacing one vertex of $K_{3,3}$ by a triangle, and $Q^{**}$ can be obtained by replacing two opposite vertices of the cube $Q$ by triangles.

As mentioned in Section 1, Campbell [3] characterised all well-covered cubic graphs of connectivity 2. His characterisation can be shown to be equivalent to the statement that all such graphs are elements of the family $W$ described above. Campbell and Plummer [3,4] also characterised the well-covered cubic planar graphs of connectivity 3. There are only four such graphs, namely $K_4$, $C_3 \times K_2 = -A-$, $C_5 \times K_2$ and $Q^{**}$. One of these graphs belongs to $W$, and the other three are among the graphs shown in Figure 2.5.

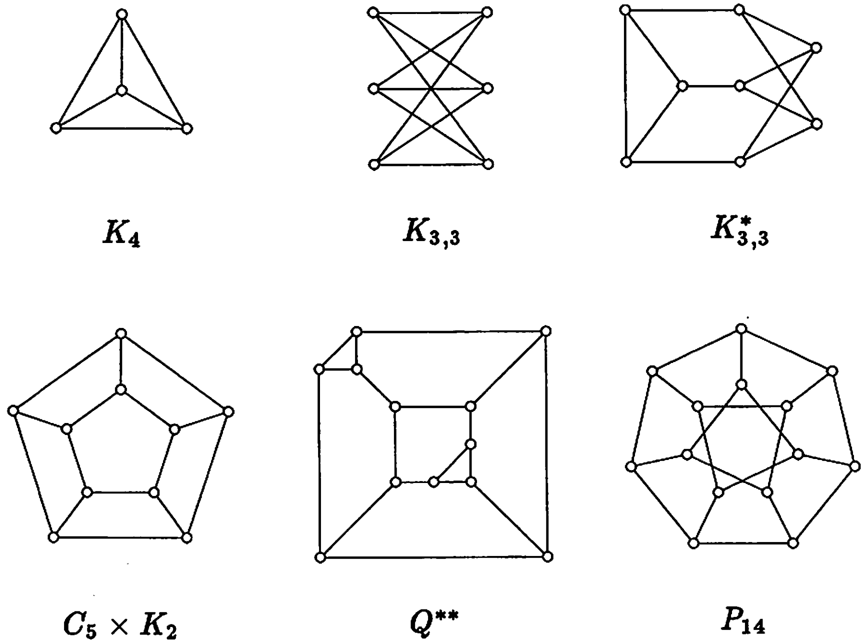


Figure 2.5

## 3. General results

In this section we give some general results which will be useful later. The proofs of our first three results are trivial and we omit them.

Let $S$ be an independent set of vertices in a graph $G$. Let $G_S$ be the subgraph of $G$ obtained by deleting all vertices either in $S$ or adjacent to a vertex of $S$. If some component of $G_S$ is isomorphic to a graph $F$, then we shall say that $S$ *produces a copy of* $F$.

**Lemma 3.1.** *$G$ is well-covered if and only if every component of $G$ is well-covered.* ∎

**Lemma 3.2.** *(Campbell [3, Corollary 1.5]). If $G$ is well-covered and $S$ is a set of independent vertices in $G$, then $G_S$ is well-covered.* ∎

**Corollary 3.3.** *If there exists an independent set $S$ of vertices in a graph $G$ which produces a copy of a non-well-covered graph, then $G$ is not well-covered.* ∎

We shall use Corollary 3.3 very often, without explicit reference. Figure 3.1 shows some small non-well-covered cubic graphs which occur frequently in applying this result.

The following lemma will also be useful for eliminating possibilities in later sections.

**Lemma 3.4.** *Let $G$ be a cubic well-covered graph. Then $G$ contains no induced subgraph isomorphic to $J_6$, the graph obtained by deleting an edge from $K_{3,3}$.*

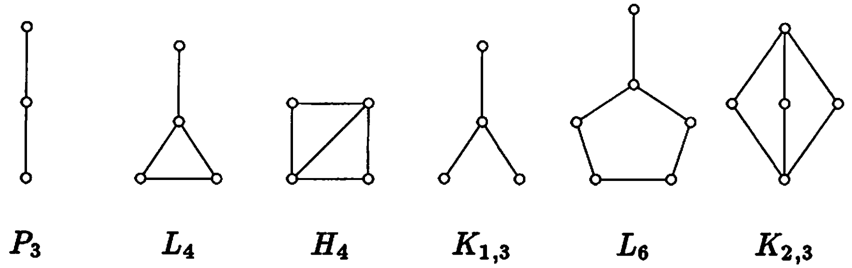



Figure 3.1

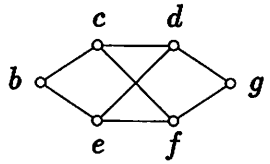



Figure 3.2

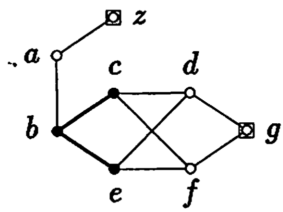



Figure 3.3

Proof: Suppose to the contrary that $G$ does contain an induced copy of $J_6$. Label $J_6$ as shown in Figure 3.2. Let $a$ be a neighbour of $b$, where $a \neq c, e$. Let $y$ and $z$ be the neighbours of $a$ other than $b$. At most one of $y$ or $z$ is adjacent to $g$, so we may assume that $z \not\sim g$ (possibly $z = g$). Then the independent set $\{z, g\}$ produces a copy of $P_3$ as shown in Figure 3.3 (where elements of $S$ are shown with squares around them, vertices covered by $S$ are shown as open circles, and vertices of $G_S$ are shown as dark circles). This is a contradiction, and therefore $G$ contains no induced copies of $J_6$. ∎

The following theorem will allow us to restrict our inquiry to cubic graphs of girth 3 or 4.

**Theorem 3.5.** *(Finbow, Hartnell and Nowakowski [7]). Let $G$ be a connected graph of girth at least 5. Then $G$ is well-covered if and only if $G$ is one of six exceptional graphs or $G$ is constructed in the following manner: take a collection of vertex-disjoint 5-cycles and edges, and join them up so that at least one vertex in each original edge still has degree one, and each of the original 5-cycles has no two adjacent vertices of degree three or more.* ∎

Of the six exceptional graphs, the only one which is cubic is the graph $P_{14}$ shown in Figure 2.5.

**Corollary 3.6.** *The only well-covered connected cubic graph of girth 5 or more is the graph $P_{14}$ shown in Figure 2.5.* ∎

## 4. Structure of well-covered cubic graphs

In this section we show that, with a small number of exceptions, every connected cubic well-covered graph of girth 3 or 4 contains an induced subgraph isomorphic to $A$ or $B$.

The theorems in this section and the next form the heart of our argument. Unfortunately, most of their proofs consist of tedious case-by-case analyses. We shall illustrate the techniques used by examining one or two cases, but we shall replace most cases with summaries of our findings. Full details of all of these proofs are available upon request from one of the authors (Ellingham).

We first consider cubic graphs of girth 3. By a *neighbour* of a subgraph in a graph we mean a vertex not in the subgraph, but adjacent to a vertex of the subgraph. Let $d_{max}$ denote the maximum number of neighbours of any triangle, and for the triangles with $d_{max}$ neighbours, let $n_{max}$ denote the maximum number of edges induced by the neighbours.

**Theorem 4.1.** *Suppose $G$ is a connected well-covered cubic graph with girth 3. Then one of the following is true.*

(i) *$G$ is one of $K_4$, $K^*_{3,3}$ or $Q^{**}$;*
(ii) *$G$ is $C_3 \times K_2 = -A-$ or $CC$, so that $G \in W$; or*
(iii) *$G$ contans an induced subgraph isomorphic to $A$ or $B$.*

Proof: The proof consists of a case-by-case analysis.

(1) Suppose that $d_{max} = 3$ and $n_{max} \geq 1$. We shall show that either $G$ is $C_3 \times K_2 = -A-$, or $G$ contains an induced subgraph isomorphic to $A$.

Since $d_{max} = 3$, there is a triangle $T = abca$ with three neighbours which induce $n_{max}$ edges. Thus, $G$ contains edges $ad$, $be$ and $cf$, where $d$, $e$ and $f$ are distinct from each other and from $a$, $b$ and $c$. This situation is shown in Figure 4.1.

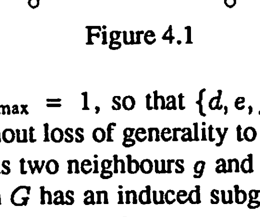



Figure 4.1

(1.1) Now suppose that $n_{max} = 1$, so that $\{d, e, f\}$ induces one edge, which we may assume without loss of generality to be $de$. Since $f$ is adjacent to neither $d$ nor $e$, $f$ has two neighbours $g$ and $h$ distinct from $a$, $b$, $c$, $d$ and $e$. If $gh \in EG$ then $G$ has an induced subgraph $A$. So, we suppose that $gh \notin EG$. We consider some subcases.

(1.1.1) Suppose there is at least one edge from $\{d, e\}$ to $\{g, h\}$. Without loss of generality we may assume that $d \sim g$. Now $h$ must have at least one neighbour $k$ where $k \neq e, f$. The independent set $S = \{d, k\}$ produces a copy of $P_3$, as shown in Figure 4.2, a contradiction.
Therefore, we may now assume that neither $d$ nor $e$ is adjacent to $g$ or $h$.

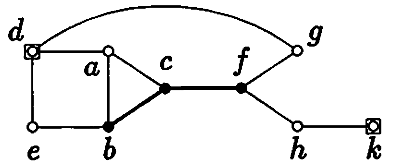



Figure 4.2

(1.1.2) Suppose $g$ and $h$ have a common neighbour $k$, where $k \neq f$. Then $k$ is not adjacent to at least one of $d$ and $e$; without loss of generality we may assume that $k \not\sim d$. Now, the independent set $\{d, k\}$ produces a copy of $P_3$ as shown in Figure 4.3, a contradiction.

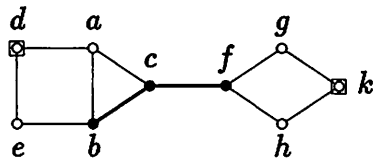



Figure 4.3

(1.1.3) Therefore, we may now suppose that $G$ contains edges $gk$, $gl$, $hm$ and $hn$ where $k, l, m$ and $n$ are distinct from each other and from $a, \dots, h$. By Lemma 3.4, we cannot have both $k$ and $l$ adjacent to both $m$ and $n$. Without loss of generality we may assume that $k \not\sim m$.

(1.1.3.1) If both of $d$ and $e$ have a neighbour in $\{k, m\}$ then $\{k, m\}$ produces a copy of $L_4$ as shown in Figure 4.4.

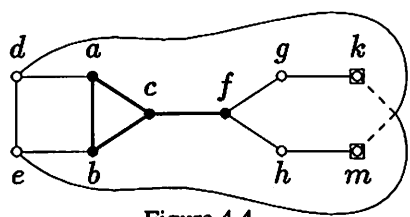



Figure 4.4

(1.1.3.2) Now suppose that one of $d$ and $e$ does not have a neighbour in $\{k, m\}$. Without loss of generality we may assume that $d$ is adjacent to neither

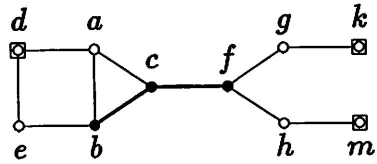



Figure 4.5

$k$ nor $m$. Then the independent set $\{d, k, m\}$ produces a copy of $P_3$ as shown in Figure 4.5, a contradiction.

(1.2) Now suppose that $n_{max} = 2$, so that there are two edges induced by $d$, $e$ and $f$. Without loss of generality we may assume that $d \sim e$ and $e \sim f$. Suppose that $g \sim d$ where $g \neq a, e$. Since at most one neighbour of $g$ is adjacent to $f$, $g$ has a neighbour $h$ with $h \neq d$ and $h \not\sim f$ (possibly $h = f$). Now the independent set $\{h, f\}$ produces a copy of $P_3$ as shown in Figure 4.6.

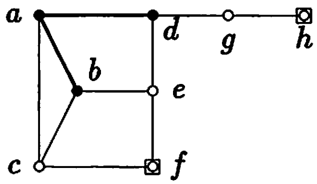



Figure 4.6

(1.3) Now suppose that $n_{max} = 3$. In this case $d$, $e$ and $f$ are all mutually adjacent, and $G$ is the graph $C_3 \times K_2 = -A-$ shown in Figure 4.7.

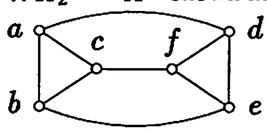



Figure 4.7

(2) Suppose that $d_{max} = 3$ and $n_{max} = 0$. Since $d_{max} = 3$, $G$ contains a triangle $abca$ with three neighbours $d$, $e$ and $f$: we may assume that $d \sim a$, $e \sim b$ and $c \sim f$. Also, since $n_{max} = 0$, we know that $\{d, e, f\}$ is an independent set of vertices in $G$.

There are six further edges from $\{d, e, f\}$, so there can be from two to six vertices adjacent to $\{d, e, f\}$. Suppose that the neighbours of $\{d, e, f\}$ (other than $a$, $b$ or $c$) form the set $X = \{x_1, x_2, \ldots, x_k\}$ and that each vertex $x_i$ is joined to $d_i$ of the vertices in $\{d, e, f\}$. Then $D = \{d_1, d_2, \ldots, d_k\}$ is a partition of 6. We divide the cases according to these partitions; there are seven cases. We summarise the results in each case.

(2.1) If $D = \{3, 3\}$ then $G$ is $K^*_{3,3}$.

(2.2) If $D = \{2, 2, 2\}$ then either $G$ contains an induced copy of $A$, or $G$ is $Q^{**}$.

(2.3) The situation $D = \{3, 2, 1\}$ is impossible.
(2.4) The situation $D = \{3, 1, 1, 1\}$ is impossible.
(2.5) If $D = \{2, 2, 1, 1\}$ there are two inequivalent ways for the vertices of $X$ to be joined to $\{d, e, f\}$. One way is impossible, and for the other way $G$ contains an induced copy of $A$.
(2.6) The situation $D = \{2, 1, 1, 1, 1\}$ is impossible.
(2.7) If $D = \{1, 1, 1, 1, 1, 1\}$ then $G$ contains an induced copy of $A$.

(3) If $d_{max} = 2$ then we can show that $G$ must either be the graph $CC$, or else contain an induced copy of $B$. We omit all details.
(4) If $d_{max} = 1$ then $G$ is $K_4$.

This concludes the proof of the theorem. ∎

Now we consider well-covered cubic graphs of girth 4. We shall use the graph parameters $d_{min}$ and $n_{max}$ (which now means something different from what it meant in Theorem 4.1). Here $d_{min}$ refers to the minimum number of neighbours of any 4-cycle, and $n_{max}$ refers to the maximum number of edges induced by the neighbours of a 4-cycle with $d_{min}$ neighbours. Since we will be discussing cubic graphs of girth 4, we know that $2 \leq d_{min} \leq 4$.

**Theorem 4.2.** *Suppose that $G$ is a connected well-covered cubic graph of girth 4. Then one of the following is true.*

(i) *$G$ is one of $K_{3,3}$ or $C_5 \times K_2$;*
(ii) *$G$ is $-B-$, so that $G \in W$; or*
(iii) *$G$ contains an induced subgraph isomorphic to $B$.*

Proof: We shall divide the proof into cases and subcases according to the values of $d_{min}$ and $n_{max}$. In all cases, we shall let $Q = cdefc$ denote a 4-cycle with $d_{min}$ neighbours which induce $n_{max}$ edges. We omit details, and merely summarise most of the cases here.

(1) If $d_{min} = 2$ then $G$ is $K_{3,3}$.
(2) The case $d_{min} = 3$ is impossible.
(3) Suppose that $d_{min} = 4$. Divide into subcases according to the value of $n_{max}$.

(3.1) If $n_{max} = 4$ then there are only two possible girth 4 cubic graphs, namely the cube and the eight-vertex Möbius ladder. The cube is not well-covered, and the eight-vertex Möbius ladder is actually the graph $-B-$, so the conclusion of the theorem is satisfied.
(3.2) The case $n_{max} = 3$ is impossible.
(3.3) If $n_{max} = 2$ then the neighbours of $Q$ cannot induce a subgraph isomorphic to $P_3 \cup K_1$. There are two inequivalent ways in which they can induce a copy of $2K_2$; for one of these ways $G$ is $C_5 \times K_2$, and for the other way $G$ contains an induced copy of $B$.
(3.4) The case $n_{max} = 1$ is impossible.

(3.5) The case $n_{max} = 0$ contains arguments which are slightly more interesting than the cases above, so we examine it in full. If $n_{max} = 0$ then $X = \{a, b, g, h\}$ induces no edges. Thus, $G$ has the induced subgraph shown in Figure 4.8.

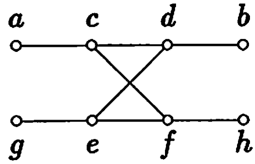



Figure 4.8

(3.5.1) Suppose there exists a vertex $x$ adjacent to three vertices of $X$. Without loss of generality we may assume that $x$ is adjacent to $a$, $g$ and $b$. Then $\{x, h\}$ produces a copy of $P_3$ as shown in Figure 4.9.

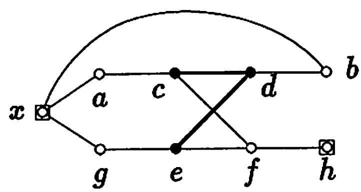



Figure 4.9

(3.5.2) Suppose there exists a vertex $x$ which is adjacent to both $a$ and $g$, or both $b$ and $h$. Without loss of generality we may assume that $x$ is adjacent to $a$ and $g$.

(3.5.2.1) Suppose that there exists a vertex $i$ which is either (i) adjacent to $h$, and not adjacent to $b$ or $x$; or (ii) adjacent to $b$, and not adjacent to $h$ or $x$. These two situations are equivalent, so we shall assume that (i) occurs. Then $\{b, i, x\}$ produces a copy of $P_3$ as shown in Figure 4.10.

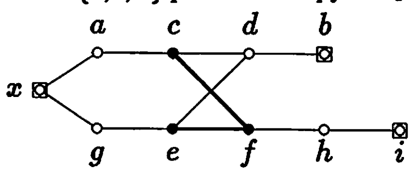



Figure 4.10

(3.5.2.2) Now we know that case (3.5.2.1) does not happen, so every new neighbour of $b$ and $h$ is either a neighbour of both of them, or is adjacent to $x$. Let $i$ and $j$ be the new neighbours of $b$. If $i \not\sim h$, then $h$ has a new neighbour $k$, $k \neq f, j$. Both $i$ and $k$ must be adjacent to $x$, giving $x$ a degree greater than 3, which cannot happen. Therefore, $i \sim h$, and

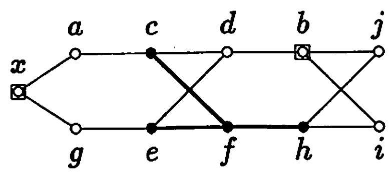



Figure 4.11

similarly $j \sim h$. Possibly $i \sim x$ or $j \sim x$. Now $\{b, x\}$ produces a copy of $K_{1,3}$ as shown in Figure 4.11, so this is impossible.

(3.5.3) Suppose that there exists a vertex adjacent to $a$ and $b$ or a vertex adjacent to $g$ and $h$. Without loss of generality we may assume that there exists $y$ for which $y \sim a$ and $y \sim b$. From (3.5.1) above we know that $y \not\sim g$ and $y \not\sim h$. We claim that there exists a new vertex $i$ such that either (i) $i \sim g, i \not\sim h, i \not\sim y$ or (ii) $i \sim h, i \not\sim g, i \not\sim y$.
Suppose that no new neighbours of $g$ or $h$ satisfy (i) or (ii). Then this must mean that $g$ and $h$ have two new common neighbours. If not, then $g$ must have some neighbour $j$ not adjacent to $h$, and $h$ must have some neighbour $k$ not adjacent to $g$; since $j$ does not satisfy (i) and $k$ does not satisfy (ii) we must have both $j$ and $k$ adjacent to $y$, making the degree of $y$ greater than 3, which is impossible. So, $g$ and $h$ have two common neighbours $j$ and $k$. Then $gjhkg$ is a 4-cycle with either less than 4 neighbours (if $j$ and $k$ have a common new neighbour) or with more than 0 edges induced by its neighbours (since $ef$ is induced by the neighbours of this 4-cycle). This contradicts $d_{\min} = 4$ and $n_{\max} = 0$. We conclude that there must be a vertex satisfying (i) or (ii).
Now, (i) and (ii) are equivalent, so without loss of generality we may assume that (i) occurs. Then $\{h, i, y\}$ produces a copy of $P_3$ as shown in Figure 4.12.

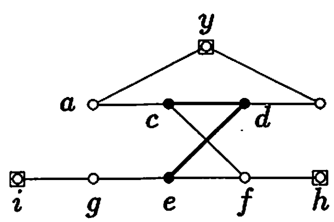



Figure 4.12

(3.5.4) Now suppose that no pair of vertices in $X = \{a, b, g, h\}$ has a common neighbour. For each $x$ in $X$ let the two new neighbours of $x$ be $x_1$ and $x_2$. Let $Y = \{a_1, a_2, b_1, b_2, g_1, g_2, h_1, h_2\}$. Let $F$ be the subgraph of $G$ induced by $Y$, and let $H = F \cup \{a_1a_2, b_1b_2, g_1g_2, h_1h_2\}$. Notice that the new edges added to $F$ to form $H$ were not present in $G$, for if $x_1x_2 \in EG$ then $G$ would have a triangle $xx_1x_2x$.

Now the degree of any vertex in $F$ must be at most 2, and therefore the degree of any vertex in $H$ must be at most 3.

(3.5.4.1) Suppose $H$ contains a complete subgraph of order 4. This subgraph must consist of four vertices $x_1$, $x_2$, $y_1$ and $y_2$ for some $x$ and $y$ in $X$. Consider the subgraph of $G$ induced by $\{x, x_1, x_2, y, y_1, y_2\}$. As shown in Figure 4.13, it is isomorphic to the graph $J_6$ obtained by deleting an edge from $K_{3,3}$. But this is impossible by Lemma 3.4.

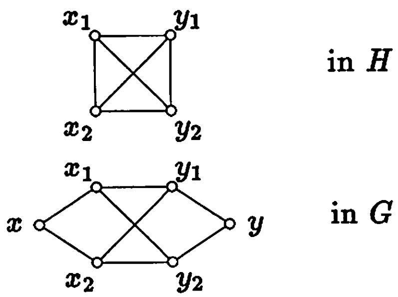



Figure 4.13

(3.5.4.2) Now suppose that $H$ contains no complete subgraphs of order 4. Then $H$ is a graph on 8 vertices, with maximum degree 3 or less, and with no complete subgraph of order 4. Therefore, it follows from Turán's Theorem [17,2 Theorem 7.9] applied to $\overline{H}$ that $H$ contains an independent set $S$ of size 3. $S$ must have the form $\{x_i, y_j, z_k\}$ where $x$, $y$ and $z$ are distinct vertices of $X$. Without any loss of generality we may suppose that $S = \{a_1, b_1, g_1\}$. Then $S \cup \{h\}$ produces a copy of $P_3$ in $G$ as shown in Figure 4.14.

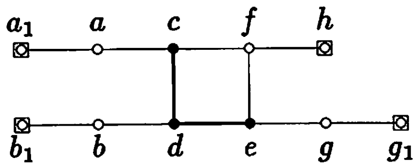



Figure 4.14

This completes the proof of the theorem. ∎

## 5. Embeddings of induced subgraphs $A$ or $B$

In this section we show that any copies of $A$ or $B$ in a well-covered cubic graph must be embedded in such a way that they satisfy Lemma 2.1 or Lemma 2.2. Theorems 5.1 and 5.2 below will enable us to use an inductive argument to prove our main result in the next section.

**Theorem 5.1.** *Suppose $G$ is a well-covered cubic graph, with an induced subgraph $A$ as shown in Figure 5.1. Then $G$ has a subgraph $A'$, as shown in Figure 5.1, which contains $A$.*

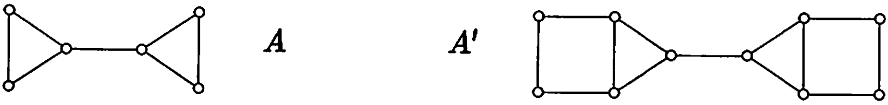



Figure 5.1

Proof: A long case-by-case analysis. We omit the details. ∎

**Theorem 5.2.** *Suppose $G$ is a well-covered cubic graph with an induced subgraph $B$ as shown in Figure 5.2. Then $G$ has a subgraph $B'$, as shown in Figure 5.2, which contains $B$.*

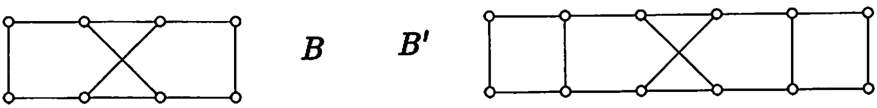



Figure 5.2

Proof: Label the vertices of $B$ as shown in Figure 5.3. Notice that $B$ has an automorphism $(ef)(gh)$ which fixes $a$ and $b$ and swaps $g$ and $h$: this will be used to assume certain situations occur without losing generality.

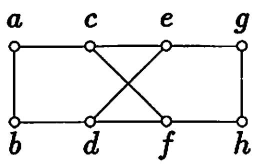



Figure 5.3

We divide the proof into three cases, according to the existence of common neighbours of $a$, $b$, $g$ and $h$.

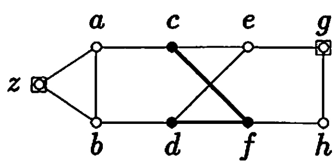



Figure 5.4

(1) Suppose that $a$ and $b$ have a common neighbour, or that $g$ and $h$ have a common neighbour. Without loss of generality we may assume that $a$ and $b$ have a common neighbour $z$. Then $z$ is not adjacent to at least one of $g$ or $h$; without loss of generality we may assume that $z \not\sim g$. However, $\{z, g\}$ then produces a copy of $P_3$, as shown in Figure 5.4.

(2) Suppose that at least one of $a$ and $b$ has a neighbour in common with one of $g$ and $h$. Without loss of generality we may assume that $a$ and $g$ have a common neighbour $x$. Since case (1) does not occur, $x$ is adjacent to neither $b$ nor $h$. Suppose that $u \sim x$, where $u \neq a, g$.

(2.1) Suppose that $u$ is adjacent to one or both of $b$ or $h$. Without loss of generality we may assume that $u \sim b$. Then $\{b, f\}$ produces a copy of $P_3$, as shown in Figure 5.5.

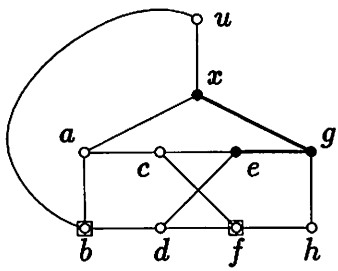


Figure 5.5

(2.2) Now suppose that $u$ is adjacent to neither $b$ nor $h$. Then $u$ has at least one neighbour different from $x$ which is not adjacent to $b$. Call such a neighbour $v$. Then $\{v, b, f\}$ induces a copy of $P_3$ as shown in Figure 5.6.

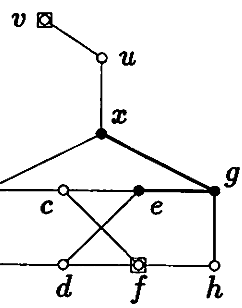


Figure 5.6

(3) Finally, after eliminating cases (1) and (2) we may suppose that $a$, $b$, $g$ and $h$ have no common neighbours. Therefore $G$ contains edges $ay, bz, gm, hn$

where $y, z, m, n$ are vertices distinct from each other and from all vertices of $B$.

Now if $G$ contains both edges $yz$ and $mn$, then we have the required subgraph $B'$. Assume that $G$ does not contain both of these edges. Without loss of generality we may assume that $yz \notin EG$. But now $\{g, y, z\}$ is an independent set which produces a copy of $P_3$, as shown in Figure 5.7. This is impossible.

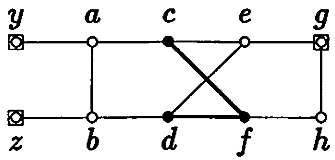



Figure 5.7

Thus, we conclude that $yz, mn \in EG$, giving the subgraph $B'$ shown in Figure 5.8.

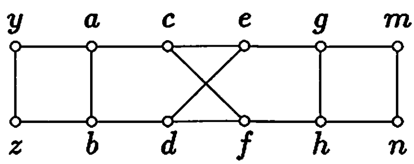



Figure 5.8

This concludes the proof of the theorem. ∎

## 6. Main theorem

In this section we state and prove our main result, a characterisation of all cubic well-covered graphs.

**Theorem 6.1.** *Let G be a connected cubic graph. Then G is well-covered if and only if one of the following is true.*

(i) $G \in W$; *or*
(ii) *G is one of* $K_4, K_{3,3}, K_{3,3}^*, C_5 \times K_2, Q^{**}$ *or* $P_{14}$ *(as shown in Figure 2.5).*

Proof: The 'if' part of this theorem was dealt with in section 2. For the 'only if' part, let $i(F, G)$ denote the number of induced subgraphs of a graph $G$ which are isomorphic to a graph $F$. The proof will be by the induction on $i(G) = i(A, G) + i(B, G)$.

First suppose that $i(G) = 0$. By Corollary 3.6 and Theorems 4.1 and 4.2 this means that $G$ is one of $-A-$, $-B-$, $CC$ or the six graphs mentioned in (ii) above. Thus, the theorem is satisfied.

Now suppose that $i(G) > 0$, and that any connected cubic well-covered graph $H$ with $i(H) < i(G)$ satisfies (i) or (ii) above.

If $i(A, G) > 0$, take an induced copy of $A$ in $G$. By Theorem 5.1, this copy of $A$ is contained in a subgraph $A'$ as shown in Figure 5.1. Replace the copy of $A$ by two copies $C_1$ and $C_2$ of $C$, as shown in Figure 6.1, to obtain a new graph $G'$.

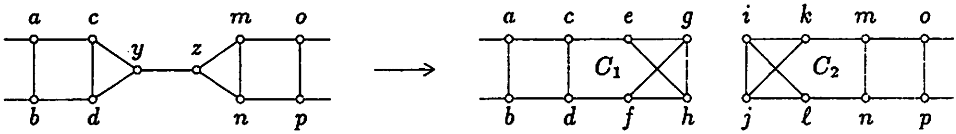



Figure 6.1

Now $G'$ is well-covered. To prove this, let $I'$ be an arbitrary maximal independent set in $G'$. Now $I' - \{e, f, g, h, i, j, k, l\}$ is an independent set in $G$, which can be extended to a maximal independent set $I$ of $G$. Clearly $I - VA = I' - (VC_1 \cup VC_2)$. Also, by Lemma 2.1 we have $|I \cap VA| = 2$, and by Lemma 2.3 we have $|I' \cap (VC_1 \cup VC_2)| = 4$. Therefore,

$$\begin{aligned} |I'| &= |I' \cap (VC_1 \cup VC_2)| + |I' - (VC_1 \cup VC_2)| \\ &= 4 + |I - VA| \\ &= 2 + |I \cap VA| + |I - VA| \\ &= 2 + |I| = 2 + \alpha(G) \end{aligned}$$

where $|I| = \alpha(G)$ because $G$ is well-covered and $I$ is a maximal independent set in $G$. Therefore, all maximal independent sets in $G'$ have the same size, and $G'$ is well-covered.

Thus, $G'$ is a well-covered cubic graph containing (at least) two induced copies of $C$. Moreover, $i(G') < i(G)$ because we have destroyed one induced copy of $A$ and we have not created any new copies of $A$ or $B$ in constructing $G'$ from $G$. Also, $G'$ has either one or two components.

Suppose $G'$ has only one component, or, in other words, suppose that $G'$ is connected. By the induction hypothesis $G'$ satisfies either (i) or (ii); but since none of the graphs in (ii) contain an induced copy of $C$, $G'$ must satisfy (i), or in other words $G' \in W$. Thus $G' = CX_1X_2 \ldots X_kC$, where each $X_i$ is either $A$ or $B$, and $k \geq 0$. The only induced copies of $C$ in $G'$ are the two $C$'s at the ends, and so we can recreate $G$ from $G'$ by removing these two $C$'s and replacing them by an $A$; in other words, $G = -AX_1X_2 \ldots X_k-$ and $G \in W$, as required.

If $G'$ has two components $G'_1$ and $G'_2$, then both satisfy $i(G'_j) < i(G)$, and both contain an induced copy of $C$. Therefore, from the induction hypothesis both are elements of $W$, where we may suppose that $G'_1 = CX_1X_2 \ldots X_kC$ and $G'_2 = CY_1Y_2 \ldots Y_lC$, each $X_j$ or $Y_j$ being $A$ or $B$, and $k, l \geq 0$. Without loss of

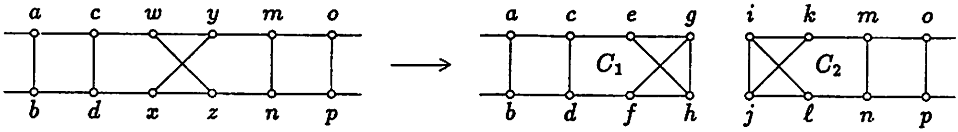



Figure 6.2

generality we may assume that $C_1$ is the last $C$ of $G'_1$ and $C_2$ is the first $C$ of $G'_2$. Then we can recreate $G$ from $G'$ by removing $C_1$ and $C_2$ and replacing them by an $A$; in other words, $G = CX_1X_2 \ldots X_kAY_1Y_2 \ldots Y_lC$ and $G \in W$, as required.

If $i(B, G) > 0$ we proceed in a similar fashion, replacing an induced copy of $B$ embedded in a subgraph $B'$ by two copies of $C$ as shown in Figure 6.2, to form $G'$. Once again we can show that $G'$ is well-covered, that its components belong to $W$, and that therefore $G \in W$. ∎

We note here that the above characterisation has been confirmed for all cubic graphs with twenty or fewer vertices by computer testing. Also, it agrees with the results, obtained by Campbell [3] and Campbell and Plummer [4], which were mentioned in Section 2.

Theorem 6.1 makes it easy to develop a polynomial time algorithm to recognise well-covered cubic graphs. The algorithm searches for an induced copy of $A$, $B$ or $C$, and, having found one, tries to follow one of the paths or cycles of subgraphs isomorphic to $A$, $B$ and $C$ which make up a graph in $W$.

**Corollary 6.2.** *The problem of recognising well-covered cubic graphs is solvable in polynomial time.* ∎

Having characterised well-covered cubic graphs in a polynomial-time fashion, there are obvious related questions. Can we characterise graphs of maximum degree three or less? Can we recognise well-covered graphs of bounded degree, or well-covered regular graphs of fixed degree, in polynomial time?

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