Distance-Hereditary Graphs and Multidestination Message-Routing in Multicomputers

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Abstract. A graph G is *distance-hereditary* if for every connected induced subgraph H of G and every pair u, v of vertices of H, we have $d_H(u,v) = d_G(u,v)$. A frequently occurring communication problem in a multicomputer is to determine the most efficient way of routing a message from a processor (called the *source*) to a number of other processors (called the *destinations*). When devising a routing from a source to several destinations it is important that each destination receives the source message in a minimum number of time steps and that the total number of messages generated be minimized. Suppose G is the graph that models a multicomputer and let $M = \{s, v_1, v_2, \ldots, v_k\}$ be a subset of $V(G)$ such that s corresponds to the source node and the nodes $v_1, v_2, \ldots, v_k$ correspond to the destinations nodes. Then an optimal communication tree (OCT) T for M is a tree that satisfies the following conditions:

(a) $M \subseteq V(T)$,
(b) $d_T(s, v_i) = d_G(s, v_i)$ for $1 \leq i \leq k$,
(c) no tree T' satisfying (a) and (b) has fewer vertices than T.

It is known that the problem of finding an OCT is NP-hard for graphs G in general, and even in the case where G is the n-cube, or a graph whose maximum degree is at most three. In this article, it is shown that an OCT for a given set M in a distance-hereditary graph can be found in polynomial time. Moreover, the problem of finding the minimum number of edges in a distance-hereditary graph H that contains a given graph G as spanning subgraph is considered, where H is isomorphic to the n-cycle, the n-cube or the grid.

1. Introduction.

A *multicomputer* (MC) consists of a collection of processors in which each processor has its own local memory. In an MC each processor is connected directly to a number of other processors called *neighboring processors*. Neighboring processors are also said to be *adjacent*. Adjacent processors can communicate directly, whereas nonadjacent processors have to communicate indirectly through other processors. Much attention has been given to interprocessor communication since it is an important factor affecting the efficiency of MCs (see [Farl79], [BhJa83], [Fox83], [ChEs92]).

[1] This research has been supported in part by a grant fr— the Foundation for Research Development, South Africa.

A frequently occurring problem in MC communication is that of finding the most "efficient" way of sending a message from some processor, called the *source*, to $k \geq 1$ other processors called the *destinations*. We call this 1*-to-k communication*. The main issue here is that of determining which paths should be used to deliver the source message to its destinations. This path selection process is commonly referred to as *routing*. In general there are many paths joining pairs of processors, but when routing a message we will require that the following two criteria are met.

(A) Each individual destination must receive the source message in a minimum number of time steps, that is, for each destination node, the message should be delivered through a shortest path, from the source to that destination.

(B) The total number of messages generated in the MC should be minimized, that is, we wish to minimize the number of links used to deliver the source message to all destinations.

The underlying topology of an MC can be modeled by a graph whose nodes (vertices) correspond to the processors in the MC and where two nodes are joined by an edge if the corresponding processors can communicate directly. Graph theory terminology not presented here can be found in [Hara69] or [ChLe86].

Suppose now that a graph G models some MC and that $M = \{s, v_1, v_2, \ldots, v_k\}$ is a subset of $V(G)$ where s corresponds to the source node and $v_1, v_2, \ldots, v_k$ correspond to k destination nodes in a 1-to-k communication. To implement a routing that satisfies conditions (A) and (B) above, one needs to find a subtree T of G called an *optimal communication tree* (OCT), such that

(a) $M \subseteq V(T)$,

(b) $d_T(s, v_i) = d_G(s, v_i)$ for $1 \leq i \leq k$,

(c) no tree T' satisfying (a) and (b) has fewer vertices than T.

The problem of finding an OCT in a general graph is NP-hard, and, in fact, it was shown in [ChEs92] that the problem of finding an OCT in the n-cube or a graph with maximum degree at most 3 is NP-hard. This motivates the development of (i) heuristics that find suboptimal communication trees (see [ChEs92]), or (ii) polynomial algorithms for finding OCTs in certain special classes of graphs.

In 1977 Howorka [Howo77] defined a graph to be *distance-hereditary* if each connected induced subgraph F of G has the property that $d_F(u, v) = d_G(u, v)$ for every pair of vertices $u, v \in V(F)$. To be able to state the characterizations of distance-hereditary graphs given by Howorka, we need the following terminology. An *induced path* of G is a path that is an induced subgraph of G. Let $u, v \in V(G)$. Then a $u - v$ *geodesic* is a shortest $u - v$ path. Let C be a cycle of G. A path P is an *essential part* of C if P is a subgraph of C and

$$\frac{1}{2}|E(C)| < |E(P)| < |E(C)|.$$

An edge that joins two vertices of C that are not adjacent in C is called a *diagonal* of C. We say two diagonals e_1, e_2 are *skew diagonals* if the graph $C + e_1 + e_2$ is homeomorphic with K_4.

Theorem 1.1 (Howorka). *The following statements are eqivalent.*

(a) *G is distance-hereditary.*
(b) *Every induced path in G is a geodesic.*
(c) *No essential part of a cycle is induced.*
(d) *Each cycle of length at least 5 has at least two diagonals and each 5-cycle has a pair of skew diagonals.*
(e) *Each cycle of G of length at least 5 has a pair of skew diagonals.*

Since these characteristics of distance-hereditary graphs were established, a number of other characterizations of distance-hereditary graphs which lend themselves to the development of polynomial algorithms that determine whether a graph is distance-hereditary, have been obtained (see **[BaMu86]**, **[HaMa91]**, and **[DaOS92]**). In particular, the characterization of distance-hereditary graphs from **[DaOS92]** which is stated in the following theorem will prove to be useful. Before stating this result we need the following terminology. For a connected graph G and vertex u of G, let $V_i(u)$ be the set of vertices at distance i from u in G. If $v \in V_i(u)$, then $V_{i-1}(u, v)$ is $N(v) \cap V_{i-1}(u)$, that is, $V_{i-1}(u, v)$ is the intersection of the neighborhood of v with the set of vertices at distance $i - 1$ from u.

Theorem 1.2. *Let G be a connected graph. Then G is not distance-hereditary if and only if there exists a vertex u such that*

(a) *for some $i \geq 2$ there exist two vertices $x, y \in V_i(u)$ such that $xy \in E(G)$ and $V_{i-1}(u, x) \neq V_{i-1}(u, y)$; or*
(b) *for some $i \geq 2$, there exist two vertices $x, y \in V_i(u)$ and a vertex $z \in V_{i+1}(u)$ such that $xy \notin E(G)$ and $xz, yz \in E(G)$ but $V_{i-1}(u, x) \neq V_{i-1}(u, y)$.*

2. Distance-hereditary graphs and OCTs.

Distance-hereditary graphs have been shown to have a number of interesting properties. We will describe one of these properties here because of its close ties with the OCT problem. Suppose G is a connected graph and S is a nonempty subset of $V(G)$. Then the *Steiner distance* $d_G(S)$ of S is the smallest number of edges in a connected subgraph of G that contains S. Such a subgraph is necessarily a tree, and is called a *Steiner tree* for S. The problem of finding a Steiner tree for a given set S of vertices in a connected graph G is known to be NP-hard (see [GaJo79]). However, it was shown in **[DAMo88]** and **[DaOS92]** that if G is a distance-hereditary graph, then there is an efficient algorithm for finding $d_G(S)$, and a Steiner tree for S. The algorithm described in [DaOS92] and whose validity is established in the same article proceeds as follows.

Algorithm 2.1. *Let M be a set of m vertices in a connected graph G on $p \geq m$ vertices and let $V(G) - M = \{v_1, v_2, \ldots, v_{p-m}\}$.*

1. *Let $G_0 = G$.*
2. *For $i = 1, 2, \ldots, p-m$, if M is contained in a component of $G_{i-1} - v_i$, then let $G_i \leftarrow G_{i-1} - v_i$; otherwise let $G_i \leftarrow G_{i-1}$.*
3. *Let T_M be any spanning tree of G_{p-m}. Then T_M is a Steiner tree for M, and $d_G(M) = |E(T_M)| = |V(G_{p-m})| - 1$.*

If T is any OCT for a set $M = \{s, v_1, v_2, \ldots, v_k\}$ of vertices in a graph G on p vertices and T_M is a Steiner tree for M, then $|E(T_M)| \leq |E(T)|$. If in addition G is distance-hereditary, then it follows, if we apply the above algorithm to G, that the subgraph $G_{p-(k+1)} = H$ produced by the algorithm is induced. Thus, $d_H(s, v_i) = d_G(s, v_i)$ for all i. If T' is any spanning tree in H which is distance-preserving from s, then $d_{T'}(s, v_i) = d_H(s, v_i) = d_G(s, v_i)$ for all i. Moreover,

$$|E(T)| \leq |E(T')| \leq |V(H)| - 1 = d_G(M) = |E(T_M)| \leq |E(T)|.$$

Hence, equality must hold throughout this string of inequalities and, therefore, T' is an OCT for M. This establishes the existence of a polynomial algorithm for finding an OCT in a connected distance-hereditary graph.

3. Distance-hereditary graphs with the smallest number of edges that have a given graph as spanning subgraph.

Both the n-cube and the $n \times m$ grid (that is, $P_n \times P_m$) are popular topologies for MCs, and are currently receiving a great deal of attention due to their many applications in parallel processing. As mentioned earlier, an important factor affecting the efficiency of an MC is interprocessor communication. In this article, we are particularly concerned with the 1-to-k communication. As pointed out in the introduction, if efficiency is defined by criteria (A) and (B) given in the introduction, then an OCT for a set $M = \{s, v_1, v_2, \ldots, v_k\}$ in the graph that models the MC will provide the basis for routing. Unfortunately, however, no efficient algorithm is known for finding OCTs in either of these classes of graphs. However, by adding edges to an n-cube or a grid it is possible to produce a graph that is distance-hereditary and still has the desirable substructure. For these graphs we have described a polynomial algorithm for finding an OCT for a given set M of nodes. Of course, it is desirable that the number of links (that is edges) be minimized in any MC. This leads naturally to the following problem: For a given graph G, what is the smallest number of edges that need to be added to G to produce a distance-hereditary graph G' having G as a spanning subgraph? We will denote the quantity $|E(G')| - |E(G)|$ by $DH(G)$ and call it the *distance-hereditary number* of G.

If T is a tree then clearly $DH(T) = 0$. To ensure reliability of an MC it is desirable that their topologies have no cut-nodes, that is, nodes whose failure would disrupt communication between certain pairs of nodes. For this reason the graphs that model them should be 2-connected. The 2-connected graph on k nodes with the fewest number of edges is the k-cycle C_k, $k \geq 3$. We now establish the DH number of these cycles.

Theorem 3.1. *$DH(C_{2n}) = 2(n-2)$ and $DH(C_{2n+1}) = 2(n-1)$ for all $n \geq 2$.*

Proof: Let C_p be a cycle of length at least 4. Let v be a vertex in C_p and let V_i denote all the vertices at distance i from v for $0 \leq i \leq \lfloor \frac{p}{2} \rfloor$, say $V_i = \{u_i, v_i\}$ for all i, $1 \leq i < \lfloor \frac{p}{2} \rfloor$. Further, if p is odd, then $V_{\lfloor \frac{p}{2} \rfloor} = \{u_{\lfloor \frac{p}{2} \rfloor}, v_{\lfloor \frac{p}{2} \rfloor}\}$, and if p is even, then $V_{\lfloor \frac{p}{2} \rfloor} = \{v_{\lfloor \frac{p}{2} \rfloor}\}$. Clearly $V_0 = \{v\}$. For each i, $1 \leq i < \lfloor \frac{p}{2} \rfloor$, add the edges $u_{i-1}\ v_i$ and $v_{i-1}\ u_i$. Moreover, if p is odd, then add the edges $u_{\lfloor \frac{p}{2} \rfloor - 1}\ v_{\lfloor \frac{p}{2} \rfloor}$ and $v_{\lfloor \frac{p}{2} \rfloor - 1}\ u_{\lfloor \frac{p}{2} \rfloor}$ as well. Now let H be the resulting graph. Then

$$|E(H)| - |E(C_p)| = \begin{cases} 2\left(\lfloor \frac{p}{2} \rfloor - 1\right) & \text{if } p \text{ is odd} \\ 2\left(\lfloor \frac{p}{2} \rfloor - 2\right) & \text{if } p \text{ is even.} \end{cases}$$

We show next, using Theorem 1.2, that H is distance-hereditary. Let u be a vertex of H. We show that neither (a) nor (b) is satisfied for u. If p is even and u is v, or $v_{\lfloor \frac{p}{2} \rfloor}$, or if p is odd and u is v, $v_{\lfloor \frac{p}{2} \rfloor}$ or $u_{\lfloor \frac{p}{2} \rfloor}$, then it is easy to see that neither condition (a) nor (b) holds for v. Suppose now that $v = u_i$ or v_i for some i, $1 \leq i < \lfloor \frac{p}{2} \rfloor$. By symmetry we may assume $v = v_i$, (see Figure 1).

Figure 1: Graph H

Observe that the set of vertices at distance d from u is

$$V_d(u) = \begin{cases} V_{i-d} \cup V_{i+d} & \text{if } d \neq 2 \text{ and } i-d \geq 0 \text{ and } i+d \leq \lfloor \frac{p}{2} \rfloor \\ V_{i-d} \cup V_{i+d} \cup \{u_i\} & \text{if } d = 2 \\ V_{i-d} & \text{if } i-d \geq 0 \text{ and } i+d > \lfloor \frac{p}{2} \rfloor \\ V_{i+d} & \text{if } i-d < 0 \text{ and } i+d \leq \lfloor \frac{p}{2} \rfloor. \end{cases}$$

Since $V_d(u)$ is an independent set of vertices we only need to check that condition (b) of Theorem 1.2 does not hold; which is seen to be the case. Hence, H is a distance-hereditary graph. Thus,

$$DH(C_p) \leq \begin{cases} 2\left(\lfloor \frac{p}{2} \rfloor - 1\right) & \text{if } p \text{ is odd} \\ 2\left(\lfloor \frac{p}{2} \rfloor - 2\right) & \text{if } p \text{ is even.} \end{cases}$$

We now prove by induction on p that $DH(C_p) \geq 2\left(\lfloor \frac{p}{2} \rfloor - 2\right)$ for even $p \geq 4$, and that $DH(C_p) \geq 2\left(\lfloor \frac{p}{2} \rfloor - 1\right)$ for odd $p \geq 5$. If $p = 4$, then $C_p = C_4$ which is a distance-hereditary graph. So $DH(C_4) = 0 = 2\left(\lfloor \frac{4}{2} \rfloor - 2\right)$. If $p = 5$, then by Theorem 1.1, $DH(C_5) = 2 = 2\left(\lfloor \frac{5}{2} \rfloor - 1\right)$. So the results hold in these two cases. Suppose now that $p > 5$ and that $DH(C_k) \geq 2\left(\lfloor \frac{k}{2} \rfloor - 2\right)$ for all even k, $4 \leq k < p$, and that $DH(C_k) \geq 2\left(\lfloor \frac{k}{2} \rfloor - 1\right)$ for all odd k, $5 \leq k < p$. Let H be a distance-hereditary graph with the smallest number of edges that has C_p as spanning subgraph. By Theorem 1.1, since C_p is a cycle in H it must have two skew diagonals e_1 and e_2. Say $e_1 = uv$ and $e_2 = xy$. Choose e_1 and e_2 in such a way that $d_{C_p}(u,v)$ is as small as possible. Then $d_{C_p}(u,v) = 2$ or 3; otherwise H has a cycle of length at least 5 without skew diagonals. Suppose first that $d_{C_p}(u,v) = 2$ and let u, w, v be the shortest uv path on C_p. Then it follows necessarily since e_1 and e_2 are skew diagonals that $w = x$ or y, say x. Let P be the uv path of C_p that does not contain w. Then P together with the edge uv is a cycle C of length $p-1$. Moreover, since the subgraph H' of H induced by the vertices of C is also distance-hereditary and as it contains C as spanning subgraph, it follows from the induction hypothesis that

$$|E(H')| - |E(C)| \geq \begin{cases} 2\left(\lfloor \frac{p-1}{2} \rfloor - 1\right) & \text{if } p \text{ is even} \\ 2\left(\lfloor \frac{p-1}{2} \rfloor - 2\right) & \text{if } p \text{ is odd.} \end{cases}$$

Since H contains at least two more edges, namely, e_1 and e_2, that are neither in

$E(H') - E(C)$ or C_p, it follows in this case if p is even that

$$\begin{aligned}|E(H)| - |E(C_p)| &\geq 2\left(\lfloor\frac{p-1}{2}\rfloor - 1\right) + 2\\ &= 2\left(\lfloor\frac{p-1}{2}\rfloor + 1 - 1\right)\\ &= 2\left(\lfloor\frac{p+1}{2}\rfloor - 1\right)\\ &= 2\left(\lfloor\frac{p}{2}\rfloor - 1\right)\\ &> 2\left(\lfloor\frac{p}{2}\rfloor - 2\right)\end{aligned}$$

and if p is odd

$$\begin{aligned}|E(H)| - |E(C_p)| &\geq 2\left(\lfloor\frac{p-1}{2}\rfloor - 2\right) + 2\\ &= 2\left(\lfloor\frac{p-1}{2}\rfloor - 1\right)\\ &= 2\left(\lfloor\frac{p}{2}\rfloor - 1\right).\end{aligned}$$

Suppose now that $d_{C_p}(u,v) = 3$. Let u, w_1, w_2, v be a shortest uv path on C_p. Then again x or y is w_1 or w_2. Let C be the cycle obtained from C_p by deleting w_1 and w_2 and adding the edge uv. Then, as in the previous case, it follows from the induction hypothesis if $H' = \langle V(C)\rangle$ that

$$|E(H')| - |E(C)| \geq \begin{cases} 2\left(\lfloor\frac{p-2}{2}\rfloor - 1\right) & \text{if } p \text{ is odd}\\ 2\left(\lfloor\frac{p-2}{2}\rfloor - 2\right) & \text{if } p \text{ is even.}\end{cases}$$

So if p is odd, then it follows in this case that

$$\begin{aligned}|E(H)| - |E(C_p)| &\geq 2\left(\lfloor\frac{p-2}{2}\rfloor - 1\right) + 2\\ &= 2\left(\lfloor\frac{p}{2}\rfloor - 1\right)\end{aligned}$$

and if p is even

$$\begin{aligned}|E(H)| - |E(C_p)| &\geq 2\left(\lfloor\frac{p-2}{2}\rfloor - 2\right) + 2\\ &= 2\left(\lfloor\frac{p}{2}\rfloor - 2\right).\end{aligned}$$

It now follows that

$$DH(C_p) \geq \begin{cases} 2\left(\lfloor \frac{p}{2} \rfloor - 1\right) & \text{if } p \text{ is odd} \\ 2\left(\lfloor \frac{p}{2} \rfloor - 2\right) & \text{if } p \text{ is even.} \end{cases}$$

This concludes the proof of the Theorem. ∎

Corollary 3.1. *Suppose G is a graph on p vertices having a hamiltonian cycle. Then*

$$DH(G) \geq \begin{cases} 2\left(\lfloor \frac{p}{2} \rfloor - 1\right) + p - |E(G)| & \text{if } p \text{ is odd} \\ 2\left(\lfloor \frac{p}{2} \rfloor - 2\right) + p - |E(G)| & \text{if } p \text{ is even.} \end{cases}$$

We now turn our attention to the n-cube Q_n.

Theorem 3.2. *If $n \geq 2$, then*

$$DH(Q_n) \leq \sum_{i=1}^{n} \binom{n}{i}\binom{n}{i-1} - n2^{n-1}.$$

Proof: Let v be any vertex of Q_n and let V_i be the vertices at distance i from v in Q_n, for $0 \leq i \leq n$. Add an edge between every vertex $u \in V_i$ and every vertex $w \in V_{i-1}$ if they are not already joined by an edge in Q_n, for $1 \leq i \leq n$, and let H be the resulting graph. Using arguments similar to those employed in Theorem 3.1, one can show that H is distance-hereditary. Since there are $\binom{n}{i}$ vertices at distance i from v in Q_n for $0 \leq i \leq n$ and as Q_n has $n2^{n-1}$ edges, the result now follows. ∎

Corollary 3.2. *If $n \geq 2$, and H is a distance-hereditary graph with diameter n and a smallest number of edges such that Q_n is a spanning subgraph of H, then $|E(H)| = \sum_{i=1}^{n} \binom{n}{i}\binom{n}{i-1}$.*

Proof: Since H and Q_n both have diameter n, and Q_n is a spanning subgraph of H, there exist two vertices u and v in Q_n such that $d_H(u,v) = d_{Q_n}(u,v)$. Since every vertex of $Q_n - \{u,v\}$ lies on a shortest uv path, it follows that $d_H(u,w) = d_{Q_n}(u,w)$ for all $w \in V(Q_n)\{u,v\}$. Hence, if V_i is the set of all vertices at distance i from u in Q_n, $0 \leq i \leq n$, then V_i is also the set of vertices at distance i from u in H. Since $V_n = \{v\}$ and since every vertex in V_{n-1} is adjacent with v (in Q_n and, hence, in H) and as every vertex in V_{n-2} is adjacent in Q_n with some vertex in V_{n-1}, it follows that every vertex in V_{n-1} is adjacent in H with every vertex in V_{n-2}. Using the same argument it can be shown that every vertex in V_{n-2} is adjacent in H with every vertex in V_{n-3}. Continuing in this fashion it can be shown that every vertex in V_i is adjacent in H with every vertex in V_{i-1} for $1 \leq i \leq n$. Thus, $|E(H)| \geq \sum_{i=1}^{n} \binom{n}{i}\binom{n}{i-1}$. However, the proof of Theorem 3.2 shows that $|E(H)| \leq \sum_{i=1}^{n} \binom{n}{i}\binom{n}{i-1}$. The corollary now follows. ∎

The next result provides an upper bound on the DH number of the $m \times n$ grid where $m \leq n$.

Theorem 3.3. *If G is an $m \times n$ grid, $2 \leq m \leq n$, then*

$$DH(G) \leq 2 \sum_{i=1}^{m-1} i(i+1) + (n-m)m^2 - 2mn + m + n$$
$$= \frac{2m(m^2-1)}{3} + (n-m)m^2 - 2mn + m + n.$$

Proof: Let v be a vertex of G which has maximum eccentricity in G, namely, $m+n-2$. Let V_i be the vertices of distance i from v for $0 \leq i \leq m+n-2$. Then $|V_i| = i+1$ for $0 \leq i \leq m-1$ and $|V_i| = m$ for $m1 \leq i \leq n-1$, and $|V_i| = m + m - 1 - i$ for $n \leq i \leq m+n-2$. Join every vertex in V_i with every vertex in V_{i+1} for $0 \leq i \leq m+n-2$ if the vertices are not already joined by an edge in G, and let H be the resulting graph. Then it can be shown as in the proof of Theorem 3.1 that H is distance-hereditary. Since H has $2\sum_{i=1}^{m-1} i(i+1)$ $+(n-m)m^2$ edges and as G has $m(n-1) + (m-1)n = 2mn - m - n$ edges, the result now follows. ∎

Corollary 3.3. *If $n \geq m \geq 2$ and H is a distance-hereditary graph with diameter $n + m2$ and a smallest number of edges such that $P_n \times P_m$ is a spanning subgraph of H, then $|E(H)| = \frac{2m(m^2-1)}{3} + (n-m)n^2$.*

Proof: Using arguments similar to those employed in the proof of Corollary 3.2, it can be shown that the Corollary holds. ∎

We conclude the paper with the following conjecture.

Conjecture. *We conjecture that inequalities given in Theorem 3.2 and Theorem 3.3 are equalities.*

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