A characterization of block graphs that are well-k-dominated

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Abstract. Let $k \geq 1$ be an integer and let G be a graph. A set D of vertices of G is a k-dominating set if every vertex of $V(G) - D$ is within distance k of some vertex of D. The graph G is called well-k-dominated if every minimal k-dominating set of G is of the same cardinality. A characterization of block graphs that are well-k-dominated is presented, where a block graph is a graph in which each of its blocks is complete

Introduction

For graph theory terminology not presented here we follow [1]. Specifically $p(G)$ and $q(G)$ will denote, respectively, the number of vertices (also called the order) and number of edges (also called the size) of a graph G with vertex set $V(G)$ and edge set $E(G)$. If S is a set of vertices of G and v is a vertex of G, then the *distance from v to S*, denoted by $d_G(v, S)$, is the shortest distance from v to a vertex of S.

A set D of vertices of a graph G is a *dominating set* of G if every vertex of $V(G) - D$ is adjacent to some vertex of D. Finbow, Hartnell and Nowakowski [3] introduced the concept of a well-dominated graph. In [3], a graph is defined to be *well-dominated* if every minimal dominating set has the same cardinality.

In this paper we extend the definition of well-dominated graphs. Let $k \geq 1$ be an integer and let G be a graph. A set D of vertices of G is a *k-dominating set* if every vertex not in D is within distance k from some vertex in D. Thus D is a 1-dominating set if and only if D is a dominating set. The *k-domination number*, denoted by $\gamma_k(G)$, and the *upper k-domination number*, denoted by $\Gamma_k(G)$, are respectively the minimum and the maximum cardinalities taken over all minimal k-dominating sets of G. We say that a graph is *well-k-dominated* if every minimal k-dominating set of the graph has the same cardinality. Hence G is well-k-dominated if and only if $\gamma_k(G) = \Gamma_k(G)$.

We characterize block graphs that are well-k-dominated, where a *block graph* is a graph in which each block is complete. A tree is a block graph where each block is K_2, the complete graph on two vertices.

JCMCC **13** (1993), pp. 33-38

Known results

A parameter of interest here is the k-packing number defined by Meir and Moon [4]. A set I of vertices of a graph G is a *k-packing* of G if $d_G(x, y) > k$ for all pairs of distinct vertices x and y in I. The *k-packing number* of G, denoted by $\beta_k(G)$, is the maximum cardinality of a k-packing set in G. A major result relating $\beta_{2k}(G)$ and $\gamma_k(G)$ where G is a connected block graph is the following theorem due to Domke, Hedetniemi and Laskar [2].

Theorem 1. *For any connected block graph G and $k \geq 1$*

$$\beta_{2k}(G) = \gamma_k(G).$$

∎

The following result is due to Topp and Volkmann [5].

Theorem 2. *If T is a tree, then $\gamma_k(T) = \beta_k(T) = n$ if and only if one of the following statements holds:*

(1) *T is a tree of diameter at most k.*

(2) *There exists a decomposition of T into n subgraphs $T_1, T_2, \ldots, T_n$ in such a way that*

(a) *T_i is a tree of diameter k ($i = 1, 2, \ldots, n$), and*

(b) *for each $i \in \{1, 2, \ldots, n\}$, there exists $u_i \in V(T_i) - V(T_0)$ such that $d_T(u_i, V(T_0)) = k$, where T_0 is the subgraph of T generated by the edges which do not belong to any of the trees $T_1, \ldots, T_n$.*

∎

A characterization of block graphs that are well-k-dominated

Since a graph is well-k-dominated if and only if each of its components is well-k-dominated, we restrict ourselves to connected graphs. We begin this section with the following result:

Proposition 1. *For any graph G and integer $k \geq 1$, $\gamma_k(G) \leq \beta_k(G) \leq \Gamma_k(G)$.*

Proof: The proof follows immediately from the observation that every maximal k-packing of G is a minimal k-dominating set of G. ∎

The following result extends Theorem 2 to connected block graphs. The proof is along similar lines to that of Theorem 2.

Theorem 3. *Let G be a connected block graph. Then the following statements are equivalent:*

(i) $\gamma_k(G) = \beta_k(G) = n$

(ii) One of the following statements holds:

(1) *G has diameter at most k and $n = 1$.*

(2) *There exists a decomposition of G into n subgraphs $G_1, G_2, \ldots, G_n$ in such a way that*

  (a) *G_i is a connected block graph of diameter k ($i = 1, 2, \ldots, n$),*

  (b) *for each $i \in \{1, 2, \ldots, n\}$, there exists $u_i \in V(G_i) - V(G_0)$ such that $d_G(u_i, V(G_0)) = k$, where G_0 is the subgraph of G generated by the edges which do not belong to any of the subgraphs $G_1, G_2, \ldots, G_n$, and*

  (c) *there is at most one edge with one end in $V(G_i)$ and the other end in $V(G_j)$ for $1 \leq i < j \leq n$;*

(iii) G is well-k-dominated. ∎

Proof: (i) ⇒ (ii): Assume that (i) holds. If $n = 1$, then the diameter of G is at most k and G satisfies (1) of (ii). In what follows, we assume $n \geq 2$. Let d denote the diameter of G, and consider a longest path $P : v_0, v_1, \ldots, v_d$ in G. If $d \leq 2k$, then $\{v_k\}$ is a k-dominating set of G and so $\gamma_k(G) = 1$, which contradicts our assumption that $n \geq 2$. Hence $d \geq 2k + 1$.

Let H be the block which contains the vertices v_k and v_{k+1}. Then, since all blocks are complete, H is isomorphic to the complete graph on $m \geq 2$ vertices, say. Let $V(H) = \{w_1, w_2, \ldots, w_m\}$, where $w_1 = v_k$ and $w_m = v_{k+1}$. We consider the graph $G' = G - E(H)$. Since H is a block, there is no $w_i - w_j$-path in $G'(1 \leq i < j \leq m)$. Hence no two vertices w_i and w_j belong to the same component of G'. Thus G' is disconnected with m components. Let $G_1, G_2, \ldots, G_{m-1}, H_m$ denote the components of G' that contain the vertices $w_1, w_2, \ldots, w_{m-1}, w_m$ repectively. Since G is a block graph, G_i is a connected block graph ($i = 1, 2, \ldots, m-1$) as is H_m. It follows from our choice of P that $\{w_i\}$ is a k-dominating set of G_i for $i = 1, 2, \ldots, m-1$. Before proceeding further, we prove four claims.

Claim 1. *For each $i \in \{1, 2, \ldots, m-1\}$, there exists a vertex $u_i \in V(G_i)$ such that $d(u_i, w_i) = k$.*

Proof: Since $d(v_0, v_k) = k$, the claim is true for $i = 1$. Suppose that, if $m > 2$, there is a subgraph G_i in which every vertex is within distance $k - 1$ from w_i for some $i \in \{2, \ldots, m-1\}$. Let $G_{1,i} =< V(G_1) \cup V(G_i) >$ and let I_i be any maximum k-packing set of $G_{1,i}$. Since v_0 is at distance $k+1$ from w_i, the diameter of $G_{1,i}$ is at least $k + 1$ and so $|I_i| \geq 2$. Now let I be a maximal k-packing set of G that properly contains I_i. Then $|I| \leq \beta_k(G) = \gamma_k(G) = n$. Also, I is a minimal k-dominating set of G and so $n = \gamma_k(G) \leq |I|$. Thus $|I| = n$. However, since $w_1 w_i$ is an edge, it follows that $w_1 (= v_k)$ is within distance k from every vertex of G_i. This, together with the fact that $\{w_1\}$ is a k-dominating set of G_1 implies that $(I - I_i) \cup \{w_1\}$ is a k-dominating set of G of cardinality less than

$|I| = n$, which produces a contradiction. This completes the proof of the claim. •

Claim 2. *For each* $i \in \{1,2,\ldots,m-1\}, d_G(u_i, V(G) - V(G_i)) > k$.

Proof: The proof follows immediately from Claim 1 and the observation that every path which starts at a vertex of G_i and ends at a vertex not in G_i must pass through w_i $(i = 1,2,\ldots,m-1)$. •

Claim 3. *For each* $i \in \{1,2,\ldots,m-1\}$, *the diameter of* G_i *is* k.

Proof: By Claim 1, it follows that the diameter of G_i is at least k $(i = 1,2,\ldots,m-1)$. Suppose that the diameter of G_i exceeds k for some $i \in \{1,2,\ldots,m-1\}$. Let I' be any maximum k-packing set of G_i. Necessarily, $|I| \geq 2$ and $w_i \notin I'$. Further, let I be a maximal k-packing set of G such that I' is a proper subset of I. Then I is a minimum dominating set of G and $|I| = n$. On the other hand, it is seen at once that $(I - I') \cup \{w_i\}$ is a k-dominating set of G of cardinality less than $|I| = n$, which produces a contradiction. Hence the diameter of G_i is k $(i = 1,2,\ldots,m-1)$. •

Claim 4. $\gamma_k(H_m) = \beta_k(H_m) = n - m + 1$.

Proof: We show firstly that $\gamma_k(H_m) \geq n - m + 1$. If this is not the case, then let D_1 be a minimum dominating set of H_m and consider the set $D = D_1 \cup \{w_1, w_2, \ldots, w_{m-1}\}$. Necessarily, D is a k-dominating set of G with $|D| = |D_1| + (m-1) < n - m + 1 + (m-1) = n = \gamma_k(G)$, which is impossible. Hence $\gamma_k(H_m) \geq n - m + 1$. Furthermore, $\beta_k(H_m) \leq n - m + 1$, for if $\beta_k(H_m) > n - m + 1$, then for any maximum k-packing J_1 of H_m, the set $J_1 \cup \{u_1, u_2, \ldots, u_{m-1}\}$ is (cf. Claim 2) a k-packing set of G of cardinality at least $n + 1 > \beta_k(G)$, which is impossible. Hence $\beta_k(H_m) \leq n - m + 1$. However (cf. Proposition 1) $\gamma_k(H_m) \leq \beta_k(H_m)$; consequently, $\gamma_k(H_m) = \beta_k(H_m) = n - m + 1$. •

We are now in a position to prove, by induction on n, that G has property (2) of condition (ii). First, assume that $n = 2$ (by Claim 4). Then $\gamma_k(H_m) = \beta_k(H_m) = 3 - m$ with $m \geq 2$ and $3 - m \geq 1$, so that $m = 2$. Since H_2 contains the $v_{k+1} - v_d$ path of length at least k, we have that H_2 has diameter k and $d_G(v_d, v_{k+1}) = k$. One sees immediately that the decomposition $G_1, G_2 = H_2$ of G with $u_2 = v_d$ satisfies (2).

Assume that every connected block graph with k-domination number less than $n \geq 3$ and equal to the k-packing number satisfies condition (ii) (with n replaced by the k-domination number). We consider the connected block graph H_m. By the inductive hypothesis, H_m has diameter at most k or it satisfies (2). If H_m has diameter at most k, then H_m has diameter k and $d_G(v_d, v_{k+1}) = k$ and $n = m$. One sees immediately that the decomposition $G_1, \ldots, G_{n-1}, G_n = H_n$ of G with $u_n = v_d$ satisfies (2).

Suppose H_m has diameter greater than k. Then there exists a decomposition $G_m, G_{m+1}, \ldots, G_n$ into $n-m+1$ connected block graphs with property (2). For convenience, let G'_0 (G_0, resp.) denote the subgraph of H_m (G resp.) induced by the edges which do not belong to any of the subgraphs $G_m, G_{m+1}, \ldots, G_n$ ($G_1, G_2, \ldots, G_n$ resp.). We shall prove that the connected block graphs $G_1, G_2, \ldots, G_n$ form a decomposition of G into n connected block graphs of diameter k and this decomposition satisfies condition (c) of (2).

In order to prove that the decomposition $G_1, \ldots, G_n$ satisfies the condition (b) of (2), we may assume without loss of generality that the vertex v_{k+1} belongs to G_n. Then, since $d_G(u_i, V(G_0)) = d_G(u_i, w_i) = k$ for $i = 1, \ldots, m-1$ and since there exists $u_i \in V(G_i) - V(G'_0)$ such that $d_G(u_i, V(G'_0)) = k$ for $i = m, \ldots, n$, it suffices to show that $d_G(\overline{u}_n, V(G_0)) = k$ for some vertex $\overline{u}_n \in V(G_n) - V(G_0) = V(G_n) - (V(G'_0) \cup \{v_{k+1}\})$. Suppose to the contrary that $d_G(v, V(G_0)) < k$ for each $v \in V(G_n)$. Then $d_G(v, N_G(V(G_n)) - V(G_n)) \leq k$ for each $v \in V(G_n)$. By property (c) of (2) and by the way the subgraphs $G_1, \ldots, G_{m-1}, H_m$ are defined, no two vertices of the set $N_G(V(G_n)) - V(G_n)$ $(\subset V(G_0) - V(G_n))$ belong to the same subgraph G_i ($i = 1, 2, \ldots, n-1$). Hence there exists a superset I of $N_G(V(G_n)) - V(G_n)$ such that $|I \cap V(G_i)| = 1$ for $i = 1, 2, \ldots, n$. Let z_i denote the unique vertex of I which belongs to the subgraph G_i ($i = 1, 2, \ldots, n$). We show that $I - \{z_n\}$ is a k-dominating set of G. Let $v \in V(G)$. If $v \in V(G_n)$, then $d_G(v, I - \{z_n\}) = d_G(v, N_G(V(G_n)) - V(G_n)) \leq k$. If $v \in V(G_i)$ for some $i \in \{1, 2, \ldots, n-1\}$, then $d_G(v, I - \{z_n\}) \leq d_G(v, z_i) \leq k$ since $v, z_i \in V(G_i)$ and G_i has diameter k. Hence $I - \{z_n\}$ is a k-dominating set of G of cardinality $n-1 < \gamma_n(G)$, which is impossible. We deduce, therefore, that there is a vertex $\overline{u}_n \in V(G_n) - V(G_0)$ such that $d_G(\overline{u}_n, V(G_0)) = k$. This proves the implication (i) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): The implication is obvious if the diameter of G is at most k. If the diameter of G is greater than k, then assume that we have a decomposition of G into subgraphs $G_1, G_2, \ldots, G_n$ satisfying (2). We show that $\Gamma_k(G) = n$. Let D be a minimal k-dominating set of G of cardinality $\Gamma_k(G)$. By property (b) of the decomposition $G_1, G_2, \ldots, G_n$ of G, there is a vertex u_i in G_i such that $d_G(u_i, V(G) - V(G_i)) > k$ ($i = 1, 2, \ldots, n$). Consequently, at least one vertex w_i of G_i belongs to D for each i ($i = 1, 2, \ldots, n$). However, by property (a) of (2) every vertex of G_i is within distance k from w_i. Hence $W = \{w_1, w_2, \ldots, w_n\}$ is a k-dominating set of G. In view of the minimality of D, it follows that $W = D$ and so $\Gamma_k(G) = |D| = n = \gamma_k(G)$. Hence G is well k-dominated.

(iii) $\Rightarrow$ (i): If G is well-k-dominated, then $\gamma_k(G) = \Gamma_k(G)$ and so, by Proposition 1, $\gamma_k(G) = \beta_k(G)$. ∎

Remark: If G is any connected graph, then the conditions given in (ii) of Theorem 3 are easily seen to be sufficient for G to be well-k-dominated. That the conditions are not necessary for any connected graph G, may be seen by considering the graph H_k constructed as follows. Let T be a binary tree of height k in which every leaf is at level k (and so T has order $2^{k+1}-1$). Let T_1 and T_2 be two (disjoint) copies of T. Finally, let H_k be obtained from T_1, T_2 by inserting a 1-factor between the end-vertices of T_1 and the end-vertices of T_2. (Figure 1 shows the graphs H_1 and H_2). Then it is not too difficult to see that H_k is well-2k-dominated with $\gamma_{2k}(H_k) = 2$, but H_k does not satisfy condition (ii) of Theorem 3.

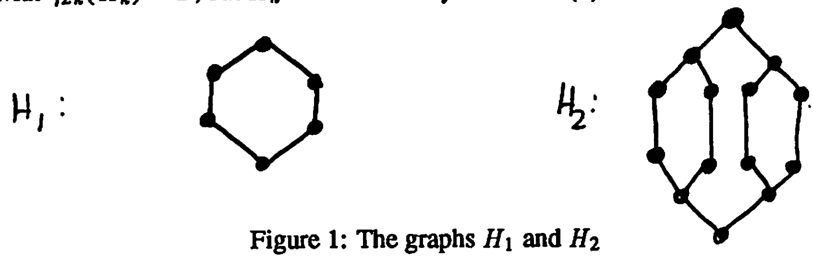

Figure 1: The graphs H_1 and H_2

Corollary 1. *If G is a connected block graph, then $\gamma_{2k}(G) = \gamma_k(G)$ if and only if G is well-2k-dominated.*

Proof: Suppose that $\gamma_{2k}(G) = \gamma_k(G)$. Then, since $\beta_{2k}(G) = \gamma_k(G)$ for any connected block graph (cf. Theorem 1), we have $\gamma_{2k}(G) = \beta_{2k}(G)$. Hence, by Theorem 3, G is well-2k-dominated.

Let G be well-2k-dominated. Then, by Theorem 3, we have $\gamma_{2k}(G) = \beta_{2k}(G)$. But $\beta_{2k}(G) = \gamma_k(G)$ for any connected block graph and so $\gamma_{2k}(G) = \gamma_k(G)$. ∎

Acknowledgements

The South African Foundation for Research Development is thanked for their financial support. Dr. Hattingh wishes to thank the Department of Mathematics and Applied Mathematics of the University of Natal, Pietermaritzburg, for their generous hospitality.

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