A Short Proof of Polya's Theorem

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Abstract. Using multisets, a short proof of Polya's theorem is given.

0. Introduction

Polya's Theorem is an important theorem of combinatorics. Among other things, it allows one to compute the number of ways of coloring a set of objects when certain configurations of these objects are considered equivalent.

A key part of the proof of Polya's Theorem is a result generally known as Burnside's Lemma. The usual proof of Burnside's Lemma involves determining the relationship between the stabilizer and the size of an orbit of an element of a set acted on by a group. Using multisets, Bogart [B], gave an elegant proof of a special version of Burnside's lemma. This allows one to prove a special case of Polya's Theorem, namely the weight one case.

We generalize the argument to give an elegant proof of Polya's Theorem. As an illustration of Polya's Theorem, we show how to compute the number of non-isomorphic graphs with a fixed number of vertices.

1. Colorings and the cycle index

A *coloring* χ is a function $\chi: D \to R$ from a finite set D called the set of objects to a finite set R called the set of colors. Let $\mathrm{X} = \mathrm{X}(D, R)$ be the set of all colorings. For a simple example, let $D = \{1, 2, 3, 4\}$ be the four squares of a 2 by 2 board labelled counterclockwise from the upper right and let $R = \{w, b\}$. There are sixteen elements in X.

Let G be a subgroup of the group of permutations on D. Each element of G has a cycle structure which can be conveniently recorded in a polynomial called the *cycle index*

$$P_G(x_1, \cdots, x_{|G|}) = \frac{1}{|G|} \sum_{g \in G} x_1^{b_1} \cdots x_{|G|}^{b_{|G|}}$$

where b_l is the number of cycles of g of length l. In the example, consider the subgroup G generated by rotations of the board by multiples of $\pi/2$. The rotation by $\pi/2$ is represented by permutation (1234), which is a single cycle of length 4. Now $G = \{(1)(2)(3)(4), (1234), (13)(24), (1432)\}$ where the elements represent, respectively, counterclockwise rotations by 0, $\pi/2$, π and $3\pi/2$. A term of the cycle index is obtained by replacing each cycle of length l by an x_l and multiplying these together. Thus $P_G(x_1, x_2, x_3, x_4) = (x_1^4 + 2x_4 + x_2^2)/4$.

2. Patterns

The group G induces an action on X, that is, for each coloring χ and group element g we obtain a coloring $g\chi = \chi \circ g^{-1}$. We wish to treat this coloring to be equivalent to the coloring χ, calling all such colorings a pattern. More formally, B is a *pattern* (or multiorbit) if $B = G\chi = \{g\chi | g \in G\}$, where $G\chi$ is considered as a multiset. In a multiset, elements are listed with repetitions allowed; the number of times an element χ appears is called the *multiplicity* mult(χ) of the element. Thus B has $|G|$ elements. This is the main advantage of using multisets since if sets had been used instead of multisets, the number of elements in B would not necessarily be $|G|$ and, in general, would vary with B.

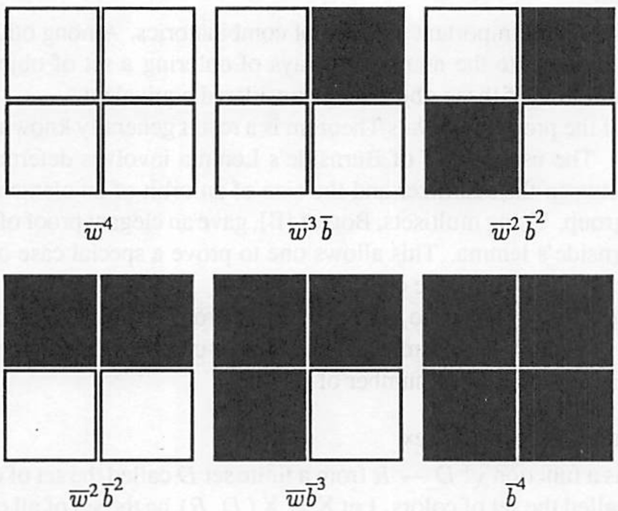

Figure 1.

The patterns for the example are listed in Figure 1. The multiplicity of the first, second and third coloring of the figure is 4, 1 and 2, respectively.

Lemma. *Two patterns are either disjoint or identical (as multisets). The multiplicity of χ in a pattern containing χ is the number of $g \in G$ with $g\chi = \chi$.*

Proof: If $\mu \in G\chi$ then for some h in G, $\mu = h\chi$. Multiplying by g or gh^{-1} shows $G\chi$ and $G\mu$ are equal as sets. Further the multiplicity of χ is the same in both multisets, since each g with $g\chi = \chi$ corresponds to $k = gh^{-1}$ with $k\mu = \chi$. This suffices to establish the lemma. ∎

For each color $r \in R$ let $\overline{r}$ denote the *weight* of the color r. Technically, weights are arbitrarily assigned elements of some algebra over the integers or of some commutative ring with unit. It is often convenient to assign integer weights, weights of 1 to some colors and 0 to the rest. Define the weight of a coloring χ to be the the product of the weights of the colors of χ,

$$\overline{\chi} = \prod_{d \in D} \overline{\chi(d)}.$$

Since the weights of all the colorings in a pattern $B = G\chi$ are the same, define $\overline{B} = \overline{\chi}$.

Let $\mathcal{B}$ denote the set of all patterns for colorings X under group G. The *pattern inventory* for $\mathcal{B}$ with given weights is, by definition,

$$\sum_{B \in \mathcal{B}} \overline{B}.$$

In the example, the pattern inventory is

$$\overline{w}^4 + \overline{w}^3\overline{b} + 2\overline{w}^2\overline{b}^2 + \overline{w}\overline{b}^3 + \overline{b}^4.$$

It turns out that this pattern inventory is, in fact,

$$P_G(\overline{w} + \overline{b}, \overline{w}^2 + \overline{b}^2, \overline{w}^3 + \overline{b}^3, \overline{w}^4 + \overline{b}^4).$$

In the example, this is the content of Polya's Theorem. Note that by considering specific terms of the pattern inventory, one can see how many patterns there are with specific numbers of each colors. Taking all weights to be 1 gives the total number of patterns.

3. Polya's Theorem

We are now ready to state and prove Polya's Theorem.

Polya's Theorem. *Let* X *be the set of all colorings* $\chi: D \to R$ *and let* G *be a subgroup of the group of permutations on* D. *Let* $\mathcal{B}$ *be the set of all patterns. Suppose weights are assigned to all colors of* R. *Then the pattern inventory is the cycle index evaluated at* $x_l = \sum_{r \in R} \overline{r}^l$, *that is,*

$$\sum_{B \in \mathcal{B}} \overline{B} = P_G(\sum_{r \in R} \overline{r}, \sum_{r \in R} \overline{r}^2, \ldots \sum_{r \in R} \overline{r}^{|G|}).$$

The proof of Polya's Theorem proceeds by showing that both sides of the equality above are equal to the sum

$$\frac{1}{|G|} \sum_{g \in G} \sum_{\substack{\chi \in X \\ g\chi = \chi}} \overline{\chi}. \qquad (*)$$

In fact, the equality with the pattern inventory is a version of Burnside's lemma, for the case when weights are not necessarily all 1. We state and prove this result first.

Burnside's Lemma. *Under the same assumptions as in Polya's Theorem,*

$$\sum_{B \in \mathcal{B}} \overline{B} = \frac{1}{|G|} \sum_{g \in G} \sum_{\substack{\chi \in X \\ g\chi = \chi}} \overline{\chi}.$$

Proof: Throughout take $\chi \in X$, $g \in G$ and $B \in \mathcal{B}$.

Interchange summation signs:

$$\frac{1}{|G|} \sum_{g \in G} \sum_{\substack{\chi \in X \\ g\chi = \chi}} \overline{\chi} = \frac{1}{|G|} \sum_{\chi} \sum_{\substack{g \\ g\chi = \chi}} \overline{\chi}$$

$$= \frac{1}{|G|} \sum_{\chi} (\overline{\chi} \sum_{\substack{g \\ g\chi = \chi}} 1)$$

$$= \frac{1}{|G|} \sum_{\chi} \overline{\chi} \operatorname{mult}(\chi) .$$

By the Lemma of the previous section, the patterns form a partition of X thus:

$$= \frac{1}{|G|} \sum_{B} \sum_{\substack{\chi \in B \\ \text{as set}}} \overline{B} \operatorname{mult}(\chi)$$

$$= \frac{1}{|G|} \sum_{B} \sum_{\substack{\chi \in B \\ \text{as multiset}}} \overline{B}$$

$$= \frac{1}{|G|} \sum_{B} \overline{B} \sum_{\substack{\chi \in B \\ \text{as multiset}}} 1$$

$$= \frac{1}{|G|} \sum_{B} \overline{B} |G| = \sum_{B} \overline{B}.$$

∎

Proof of Polya's Theorem: By Burnside's Lemma, it remains to show that sum $(*)$ is the evaluated cycle index. Write g in (disjoint) cycle form, and let $(d_1 \ldots d_l)$ be a typical cycle of g, say the k-th one. Then since $g\chi = \chi$, each of $d_1, \ldots, d_l$ must be colored the same. Thus the sum $(*)$ becomes

$$\frac{1}{|G|} \sum_{g \in G} \sum_{(\ldots, r_k \in R, \ldots)} (\cdots \overline{r_k}^{\,l} \cdots).$$

Using the distributive law gives

$$\frac{1}{|G|} \sum_{g \in G} (\cdots (\sum_{r \in R} \overline{r}^{\,l}) \cdots)$$

which is the cycle index evaluated at $x_l = \sum_{r \in R} \overline{r}^{\,l}$. ∎

4. Counting Non-Isomorphic Graphs

As a non-trivial application of Polya's Theorem we count the number of non-isomorphic graphs with n vertices and a fixed number of edges.

A graph H on n vertices is a subgraph of the complete graph K_n with n vertices. Thus H can be thought of as a coloring of the edges of K_n with the two colors "a" and "r" signifying whether to "accept" or "reject" each given edge. An isomorphism between two graphs amounts to a permutation of vertices that preserves the edge colorings. Let S_n be the group of all permutations of the n vertices of K_n and let G be the induced group of permutations of the edges. Clearly, the isomorphism classes are the patterns generated by the group G.

By Polya's Theorem, we must first compute the cycle index. Each partition of n corresponds to a cycle structure of an element of S_n. The partition $n = \sum_{l=1}^{n} b_l$ corresponds to elements of S_n with b_l cycles of length l. There are

$$n! / \prod_{l=1}^{n} (b_l! \, l^{b_l})$$

such elements in S_n. This formula is called Cauchy's formula, and can be easily seen by writing down a permutation of n numbers as a sequence and breaking this into cycles of the required types in a fixed way; the divisor comes about because without changing the final permutation, the cycles of length l can be permuted amongst themselves and each cycle of length l can be written with l different starting values.

Now each cycle of length l on vertices induces cycles on edges between these vertices. If l is even, then $l/2 - 1$ cycles of length l and one cycle of length $l/2$ are generated. If l is odd, then $(l-1)/2$ cycles of length l are generated. Also each pair of cycles of length l and k on vertices induces cycles on edges from one pair to the other. In fact, there are $\gcd(l, m)$ cycles of length $\operatorname{lcm}(l, m)$ where gcd and lcm represent the greatest common divisor and least common multiple functions, respectively.

With these observations it is possible, although tedious, to compute the cycle index. For the case of $n = 7$, for instance, we obtain Table 1, with $b_l = 0$ unless otherwise indicated.

Thus the cycle index is

$$P_G(x_1, \ldots, x_{15}) = 1/720 \; (x_1^{15} + 15 x_1^7 x_2^4 + 60 x_1^3 x_2^6 + 40 x_1^3 x_3^4 + 120 x_1 x_2 x_3^2 x_6 + 40 x_3^5 + 180 x_1 x_2 x_4^3 + 144 x_5^3 + 120 x_3 x_6^2).$$

Taking weights $\overline{a} = a$ and $\overline{r} = 1$ gives, by Polya's Theorem, the following expression for the pattern inventory:

$$a^{15} + a^{14} + 2a^{13} + 5a^{12} + 9a^{11} + 15a^{10} + 21a^9 + 24a^8 + 24a^7 + 21a^6 + 15a^5 + 9a^4 + 5a^3 + 2a^2 + a + 1.$$

number of elements	vertex cycles	edge cycles
1	$b_1 = 6$	$b_1 = 15$
15	$b_1 = 4, b_2 = 1$	$b_1 = 7, b_2 = 4$
45	$b_1 = b_2 = 2$	$b_1 = 3, b_2 = 6$
15	$b_2 = 3$	$b_1 = 3, b_2 = 6$
40	$b_1 = 3, b_3 = 1$	$b_1 = 3, b_3 = 4$
120	$b_1 = b_2 = b_3 = 1$	$b_1 = b_2 = 1, b_3 = 2, b_6 = 1$
40	$b_3 = 2$	$b_3 = 5$
90	$b_1 = 2, b_4 = 1$	$b_1 = b_2 = 1, b_4 = 3$
90	$b_2 = b_4 = 1$	$b_1 = b_2 = 1, b_4 = 3$
144	$b_1 = b_5 = 1$	$b_5 = 3$
120	$b_6 = 1$	$b_3 = 1, b_6 = 2$
720		

Table 1: Cycles for 7 vertex graphs

So there are, for instance, 24 non-isomorphic graphs with 7 edges and 6 vertices. By summing the coefficients (set $a = 1$) or by setting $x_i = 2$ in the cycle index, we see there are 156 non-isomorphic graphs with 6 vertices.

For 10 vertex graphs, there are $2^{45} = 35{,}184{,}372{,}088{,}832$ possible graphs. The $10! = 3{,}628{,}800$ permutations of vertices have 42 different cycle decompositions. There are 12,005,168 non-isomorphic graphs and the pattern inventory turns out to be: $1 + a + 2a^2 + 5a^3 + 11a^4 + 26a^5 + 66a^6 + 165a^7 + 428a^8 + 1103a^9 + 2769a^{10} + 6759a^{11} + 15772a^{12} + 34663a^{13} + 71318a^{14} + 136433a^{15} + 241577a^{16} + 395166a^{17} + 596191a^{18} + 828728a^{19} + 1061159a^{20} + 1251389a^{21} + 1358852a^{22} + 1358852a^{23} + 1251389a^{24} + 1061159a^{25} + 828728a^{26} + 596191a^{27} + 395166a^{28} + 241577a^{29} + 136433a^{30} + 71318a^{31} + 34663a^{32} + 15772a^{33} + 6759a^{34} + 2769a^{35} + 1103a^{36} + 428a^{37} + 165a^{38} + 66a^{39} + 26a^{40} + 11a^{41} + 5a^{42} + 2a^{43} + a^{44} + a^{45}$. These computations where carried out by writing a short program to compute the cycle structures. The output of the program was an expression for the cycle index, which was simplified using the symbolic manipulation package MAPLE. Then MAPLE was used to evaluate the cycle index to generate the pattern inventory. The computations took a few seconds of computer time.

Let U_n be the number of non-isomorphic graphs with n vertices. Let C_n be the number of these which are connected. Using Polya's Theorem U_n can be computed. The results for $n \leq 20$ are presented in Table 2. The components of a graph induce a partition of the vertices, each partition subset consisting of all vertices in the same component. It follows

$$C_n = U_n - \sum_{b_1 + 2b_2 + \cdots + (n-1)b_{n-1} = n} \prod_{l=1}^{n-1} \binom{C_l + b_l - 1}{b_l}$$

where the binomial coefficient counts the number of non-isomorphic graphs consisting of b_l connected components of l vertices each. The C_n can be computed inductively. The results for $n \leq 20$ are presented in Table 3.

vertices n	non-isomorphic graphs U_n
0	1
1	1
2	2
3	4
4	11
5	34
6	156
7	1044
8	12346
9	274668
10	12005168
11	1018997864
12	165091172592
13	50502031367952
14	29054155657235488
15	31426485969804308768
16	64001015704527557894928
17	245935864153532932683719776
18	1787577725145611700547878190848
19	24637809253125004524383007491432768
20	645490122795799841856164638490742749440

Table 2: Number of non-isomorphic graphs

vertices n	non-isomorphic *connected* graphs C_n
0	1
1	1
2	1
3	2
4	6
5	21
6	112
7	853
8	11117
9	261080
10	11716571
11	1006700565
12	164059830476

Table 3: Number of non-isomorphic connected graphs

vertices n	non-isomorphic *connected* graphs C_n
13	50335907869219
14	29003487462848061
15	31397381142761241960
16	63969560113225176176277
17	245871831682084026519528568
18	1787331725248899088890200576580
19	24636021429399867655322650759681644
20	645465483198722799426731128794502283004

Table 3 (continued): Number of non-isomorphic connected graphs

References

[B] Kenneth B. Bogart, *An Obvious Proof of Burnside's Lemma*, Am. Math. Monthly **98** (1991), 927–928.

[P] G. Pólya, *Kombinatorische Anzahlbestimmungen für Gruppen, Graphen und Chemische Verbindungen*, Acta Math. **68** (1937). "Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds (English translation with comments by R. C. Read)", Springer-Verlag, 1987, New York