Complementary Pairs Of Graphs Orientable To Line Digraphs

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Abstract. A graph is orientable to a line digraph (OLD, for short) if its lines can be oriented in such a way that the resulting digraph is the line digraph of some digraph. In this paper we find all graphs such that both the graph and its complement are OLD and also characterize these graphs in terms of minimal forbidden subgraphs. As shown, all of these graphs have at most nine points.

1. Introduction

Herein we consider finite graphs and digraphs without loops; multiple arcs will be allowed only in digraphs. An oriented graph is a digraph with at most one arc joining any two points. The line digraph $L(D)$ of a digraph D is obtained by taking as its point set the arc set of D, with two points $a = (u, s)$ and $b = (t, v)$ being joined by an arc whenever $s = t$. This is a standard definition of line digraphs which reflects only the head-to-tail adjacencies between arcs. However, it is clear from this definition that a line digraph has no multiple arcs, and furthermore, that it has a loop at some point a if and only if a is a loop in the original digraph. All other terminology and notation follows [7].

There are many results in the literature concerning a graph and its complement. A series of papers in this respect was initiated by Akiyama and Harary [1]. In [3] Beineke found all coderived graphs, i.e. graphs G such that both G and its complement $\overline{G}$ are line graphs. Some further generalizations of this problem are given in [4], [10], [11]. In this paper we will find all graphs such that both G and $\overline{G}$ are orientable to line digraphs. Further on, any graph from these complementary pairs of OLD graphs will be called a COLD graph.

The paper is organized as follows. Section 2 contains some basic tools for our investigation. In Section 3 we apply some results from the theory of perfect graphs to prove the sharp upper bound on the number of points of COLD graphs (Theorem 1). Finally, in Section 4 we first find all maximal graphs which are COLD (Theorem 2), and then prove that the graphs from Corollary 1 (in Section 2) do exhaust the list of graphs characterizing the class of COLD graphs in terms of forbidden induced subgraphs (Theorem 3).

2. Some Known Results

Line digraphs are characterized in the literature in several ways. The oldest and the most obvious one is due to Harary and Norman [8]. Let A and B be two disjoint sets, one of them possibly empty. The digraph $K(A, B)$ is a digraph whose point set is the union of A and B, with each point of A joined by an arc to each point of B.

Proposition 1. *A digraph is a line digraph if and only if its arcs can be partitioned into digraphs $K(A_i, B_i)$ in such a way that for all $j \neq k$ $A_j \cap A_k = \emptyset$, $B_j \cap B_k = \emptyset$, and $|A_j \cap B_k| \leq 1$.*

The next characterization (see [9]) applies to digraphs with loops and/or multiple arcs.

Proposition 2. *A digraph is a line digraph if and only if whenever (a, c), (b, c), and (b, d) are arcs, so is (a, d).*

According to Beineke [2] the next proposition holds:

Proposition 3. *An oriented graph is a line digraph of a loopless digraph if and only if it does not contain any of the three digraphs given in Fig. 1 as an induced subgraph.*

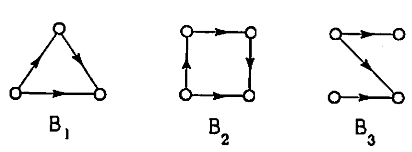

Figure 1

Corollary 1. *If G is COLD, then G does not contain any of the graphs of Fig. 2 as an induced subgraph.*

Proof: By checking that at least one graph from each complementary pair is not OLD. ∎

We shall see in Theorem 3, at the end of Section 4, that the induced subgraph property described in Corollary 1 is not only necessary but also sufficient, in the sense that it gives a complete characterization of COLD graphs.

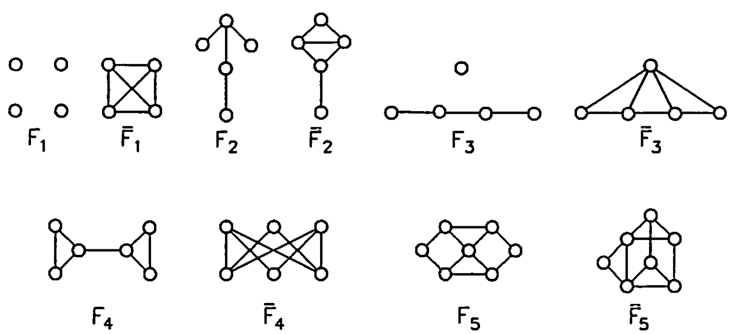

Figure 2

3. Proof Of The Upper Bound

Theorem 1 *If G is COLD, then G has at most nine points.*

Proof: Suppose to the contrary that G is an OLD graph on (at least) ten points, such that $\overline{G}$ is OLD, too. By Corollary 1, $K_4 \subset G$ or $K_4 \subset \overline{G}$ is not allowed. So, if G and $\overline{G}$ both were perfect graphs, then we would have $\chi(G) \leq 3$, $\chi(\overline{G}) \leq 3$, $10 = \chi(K_{10}) \leq \chi(G)\chi(\overline{G})$, a contradiction (here χ denotes the chromatic number of a graph).

Note further that the Strong Perfect Graph Conjecture is true for K_4-free graphs [13], implying that if G is not perfect then it contains an odd hole or an odd antihole. This (anti)hole cannot have more than five points because any cycle of length at least seven contains F_3 (of Fig. 2) as an induced subgraph. Consequently, both G and $\overline{G}$ contain C_5. Suppose now x is a point of G out of C_5. Since $P_4 \subset C_5$, x is adjacent to at least two points of C_5; otherwise $F_3 \subset G$. By the same argument on the complement, it follows that x is adjacent to at most three points of C_5. If x were adjacent to only two nonadjacent points of C_5 (or three consecutive points of C_5), then $F_2 \subset G$ (resp. $\overline{F_2} \subset G$) would hold. So, up to isomorphism x is adjacent either to x_0 and x_1 (a-type point) or to x_0, x_1, and x_3 (b-type point), where $x_0, \ldots, x_4$ are the consecutive points of C_5. Since G is supposed to have at least ten points, we can assume that G contains at least three a-type points (otherwise consider $\overline{G}$). If two of these points are adjacent to the same pair of points of C_5, then either $K_4 \subset G$ or $\overline{K}_4 \subset G$; otherwise $F_3 \subset G$. ∎

We note that an upper bound (though not sharp) on the order of COLD graphs G can be deduced from various known results of combinatorics. For example, Proposition 3 with the value of the Ramsey number $R(4,4)$ yields $|V(G)| \leq 17$.

Moreover, Proposition 1 together with a theorem of the third author's [14] on "local edge colorings" of complete graphs implies $|V(G)| \leq 16$. We thank the referee for pointing out the relation between our problem and Ramsey theory.

4. Structural Characterizations

In this section we first find all COLD graphs which are maximal, i.e. which are not proper induced subgraphs of any other COLD graph. For this purpose we prove the following sequence of lemmas. Below, $\Delta(G)$ denotes the maximum degree in G, and we put $\Delta^*(G) = \max(\Delta(G), \Delta(\overline{G}))$. The degree of a point u will be denoted by $\deg u$.

Lemma 1. *If G is COLD, then $\Delta^*(G) \leq 6$.*

Proof: If $\Delta^*(G) > 6$, there is a point in $\vec{G}$ such that its in-degree, or its out-degree, is at least 4. Thus we can separate four points which give a contradiction at once. Namely, by Corollary 1, they cannot be mutually nonadjacent (see F_1), while due to Proposition 3, no pair of them is adjacent (see B_1). ∎

For convenience, hereafter we assume the following:

- $\Delta(G) \geq \Delta(\overline{G})$, i.e. $\Delta^*(G) = \Delta(G)$;
- u is a point of G whose degree is equal to $\Delta(G)$;
- $\vec{G}$ is an orientation of G, while G, given $\vec{G}$, is its underlying graph;
- in $\vec{G}$, $V_m = \{v_1, \dots, v_m\}$ and $W_n = \{w_1, \dots, w_n\}$ are the in-neighbors and the out-neighbors of u, respectively.

Now we emphasize two facts (to be used later on repeatedly) which immediately follow from Proposition 3 and Corollary 1, respectively:

(i) points in V_m and in W_n are all mutually nonadjacent;
(ii) if m (or n) is greater than two, then any additional point of G, say x, must be adjacent to at least two points of V_m (or W_n); in particular, $m \leq 3$ and $n \leq 3$.

Case 1. $\Delta^*(G) = 6$.

We first examine the subdigraphs of $\vec{G}$, induced by $V_m \cup W_n \cup \{u\}$, i.e. the closed neighborhood of u.

Lemma 2. *If $\deg u = 6$, then the closed neighborhood of u (including orientation) is one of the digraphs shown in Fig. 3.*

Proof: We first conclude by (ii) that $m = n = 3$. Since the points in V_m and in W_n are all mutually nonadjacent, it follows that each point v_i (w_j) must be adjacent to some w_j (resp. v_i); otherwise, $\overline{K_4} \subset G$. Therefore, up to symmetry, at least the following arcs must appear in G between the neighbors of u.

(a) $w_1 v_1, w_2 v_2, w_3 v_3$ (perfect matching);
(b) $w_1 v_1, w_1 v_3, w_2 v_2, w_3 v_2$ (no perfect matching).

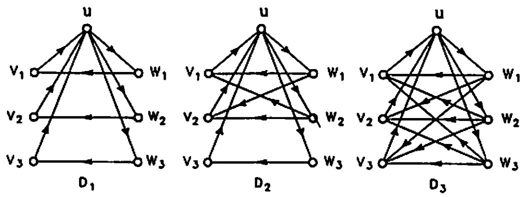

Figure 3

Since $\overline{K}_4$ already exists in (b), at least one further arc should be added to it. But then, by Proposition 2, it follows that we always have a perfect matching as in (a). (This fact could have been deduced from a particular case of the König-Hall theorem as well.) Moreover, the exclusion of B_3 implies that each connected component in the subgraph induced by $V_m \cup W_n$ is a complete bipartite digraph. ∎

In what follows, we will show that each digraph of Fig. 3 can be extended (up to orientation) to a unique digraph on nine points.

Lemma 3. *If $\Delta(G) = 6$, then G is COLD if and only if G is an induced subgraph of at least one of the three graphs $S_1 - S_3$ of Fig. 6.*

Proof: By the previous lemma, it is enough to show that S_i results as the unique maximal extension of D_i $(1 \leq i \leq 3)$. To do this, observe the point x added to D_i. By (ii), since $m = n = 3$, x is adjacent to at least two points of V_m and also to at least two points of W_n. But then, for some i $(1 \leq i \leq 3)$, x is adjacent to both v_i and w_i. By B_1, this in further implies that v_ix and xw_i are arcs. Therefore, by Proposition 2, we now get that v_ix and xw_i are arcs for each i.

If, besides x, we add to D_i any further point, say y, then the same holds for y. Since x, y, and u are mutually nonadjacent (all have degree 6), no further point can be added. So S_i is the unique maximal extension of D_i. ∎

Case 2. $\Delta^*(G) = 5$

The next lemma is analogous to Lemma 2.

Lemma 4. *If $\deg u = 5$, then the closed neighborhood of u (including orientation) is one of the digraphs $D_4 = D_1 - w_3$, $D_5 = D_2 - w_1$, $D_6 = D_2 - w_3$ and $D_7 = D_3 - w_3$ of Fig. 4, or their converse digraphs (reversing orientation on all arcs).*

Proof: We first conclude by (ii) that $m = 3, n = 2$ or, vice versa, $m = 2, n = 3$. Since the converse digraph of a line digraph is a line digraph of a converse digraph, we can restrict ourselves to the former case. With these assumptions the rest of

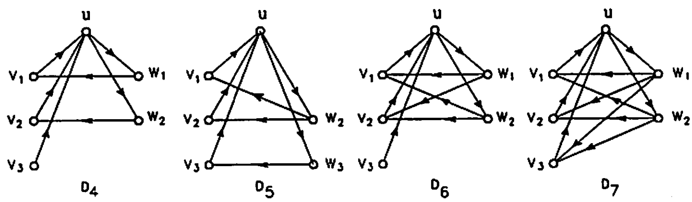

Figure 4

the proof is similar to the proof of Lemma 2, except that we now have a matching consisting of two arcs. ∎

Lemma 5. *If* $\Delta(G) = 5$, *then* G *is COLD if and only if it is an induced subgraph of at least one of the four graphs* $S_1 - S_4$ *of Fig. 6.*

Proof: By the previous lemma, it is enough to show that any extension of $D_i(4 \leq i \leq 7)$ results in a graph which is contained, as an induced subgraph, in at least one of the graphs $S_j(1 \leq j \leq 4)$. To do this, we first examine the one-point extensions of D_i. If x is the point added to D_i, then, since $m = 3$, it must be adjacent to at least two points of V_m (by (ii)). Without loss of generality, we can assume that x is adjacent to v_1 (in each D_i).

Suppose xv_1 is an arc. Then, for each D_i, $v_s x (s = 2$ or $3)$ is not an arc, since otherwise we get $B_2 (= \langle x, v_1, u, v_s \rangle$[1]) which is forbidden by Proposition 3. Then in D_4, neither xv_2 nor xv_3 is an arc, since otherwise, by Proposition 2, $w_1 v_2$ or $w_1 v_3$ should also be an arc, which is not allowed in D_4. Thus D_4 cannot be extended in this way. Applying the same argument in D_5 and D_6, it follows that xv_2 may be an arc, but not xv_3 (otherwise, by Proposition 2, some further arcs should be added to D_5 or D_6). Further, in D_5, x is not adjacent to w_2 or w_3; otherwise, in the former case we get $B_1 (= \langle x, v_1, w_2 \rangle)$, while in the latter one $F_2 (= \langle x, v_1, v_2, w_3, v_3 \rangle)$ which is forbidden by Corollary 1. Thus we get a one-point extension of D_5 (the arcs are xv_1 and xv_2). Now consider D_6. If x is adjacent to v_1, and possibly to v_2, we already have a forbidden situation due to $F_1 (= \langle x, v_3, w_1, w_2 \rangle)$. Otherwise, if x is adjacent to w_1 or w_2, then $B_1 (= \langle x, v_1, w_s \rangle$, $s = 1$ or 2) appears. So D_6 cannot be extended in this way. Finally, consider D_7. By Proposition 2, it follows that xv_2 and xv_3 both are arcs, and (as in D_6) x is nonadjacent to w_1 or w_2. So we now get a one-point extension of D_7 (the arcs are xv_1, xv_2, and xv_3).

[1] $\langle z_1, \dots z_s \rangle$ stands for the subgraph induced by $\{z_1, \dots, z_s\}$.

Assume now that v_1x is an arc. By Proposition 2, so are v_2x and v_3x in each D_i (observe x, u, v_1 and, v_2 or v_3). In D_4, x is adjacent to w_1 and w_2, since otherwise $F_2(=\langle x, v_1, v_2, v_3, w_t\rangle, t = 1$ or 2) appears. Actually, by Proposition 3, xw_1 and xw_2 are arcs. Thus we get a one-point extension of D_4 (the arcs are v_1x, v_2x, v_3x, xw_1, and xw_2). The situation is similar in D_5. Namely, xw_3 is an arc by the same argument as above, while xw_2 is implied by Proposition 2. So we arrived at another one-point extension of D_5 (the arcs are v_1x, v_2x, v_3x, xw_2, and xw_3). Consider now D_6. If x is nonadjacent to w_1 and w_2, then $F_2(=\langle u, w_1, w_2, v_3, x\rangle)$ appears. So, either xw_1 or xw_2 is an arc (the in-degree of x is at most 3), and consequently, by Proposition 2, they both are arcs. This gives a one-point extension of D_6 (the arcs are v_1x, v_2x, v_3x, xw_1, and xw_2). Finally, in D_7, x is not necessarily adjacent to w_1 or w_2, but otherwise, as above, each of these arcs implies the other. Thus, we now have two possibilities for a one-point extension of D_7 (the arcs are either v_1x, v_2x, and v_3x, or v_1x, v_2x, v_3x, w_1x, and w_2x).

To complete the proof, consider the extensions of each D_i with (possibly) more than one point added. Observe first that in D_4 and D_6 the situation is quite simple. There are unique possibilities for one-point extensions, and in both cases the points being added are nonadjacent (otherwise B_1 appears). Due to F_1, at most two points could be added. The resulting graphs are contained in S_1 or S_2. For D_5, there are two ways to get one-point extensions. If we add points of different types, they must be nonadjacent (one of them has degree 5), but then $\overline{F}_4$ appears. Otherwise, we get either a subgraph of S_2 (at most two saturated points could be added), or S_4 which is maximal. Finally, consider D_7. We now have three ways to obtain one-point extensions. If we add points of all three types, we get S_2, which is maximal as already observed (two nonsaturated points must be adjacent, due to F_1). The other possibility is to add two or more points of the same type. By F_1 it immediately follows that we can add at most two saturated points. The resulting graph is a subgraph of S_3. ∎

Case 3. $\Delta^*(G) \leq 4$.

We are now in a position to choose between the technique demonstrated above or brute force. Now, the number of points of COLD graphs, besides Theorem 1, is bounded from above by nine also by the degree conditions. With nine points only regular graphs of degree four need to be examined. It can be easily shown (see [6] as well) that within these graphs there are no COLD ones. Thus the bound is reduced to eight. The graphs with at most eight points are catalogued in system GRAPH (see [5]), and can be treated one by one. The algorithm for checking if some graph is a line digraph is given in [12]. However, the same result (with more effort) can be deduced as before. Here we only give the corresponding lemmas.

Lemma 6. *If $\Delta(G) = 4$, then G is COLD if and only if it is an induced subgraph of at least one of the graphs of Fig. 5a.*

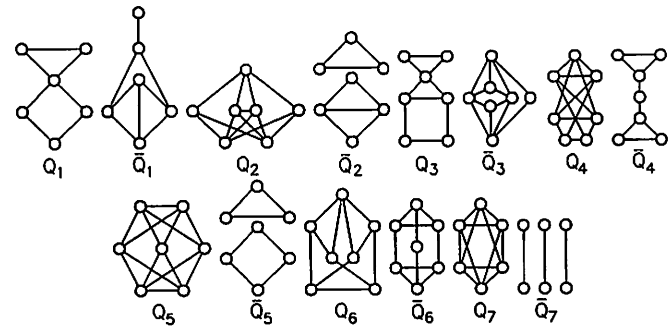

Figure 5a.

One can immediately see that Q_1, Q_2, Q_5, and Q_7 are induced subgraphs of some of the graphs of Fig. 6 (for example, of $S_4, S_1, \overline{S}_2$, and S_3 respectively), while the others are new solutions.

Lemma 7. *If $\Delta(G) = 3$, G is COLD if and only if it is an induced subgraph of at least one of the graphs of Fig. 5b.*

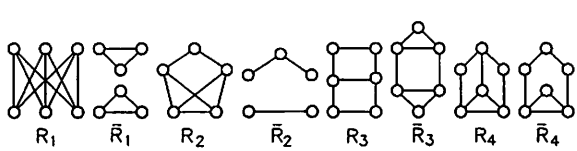

Figure 5b.

It is obvious that all these graphs are contained in some of the graphs $Q_1 - Q_7$
If $\Delta^*(G) \leq 2$, the discussion is trivial.
Collecting the above conclusions, we arrive at our main result.

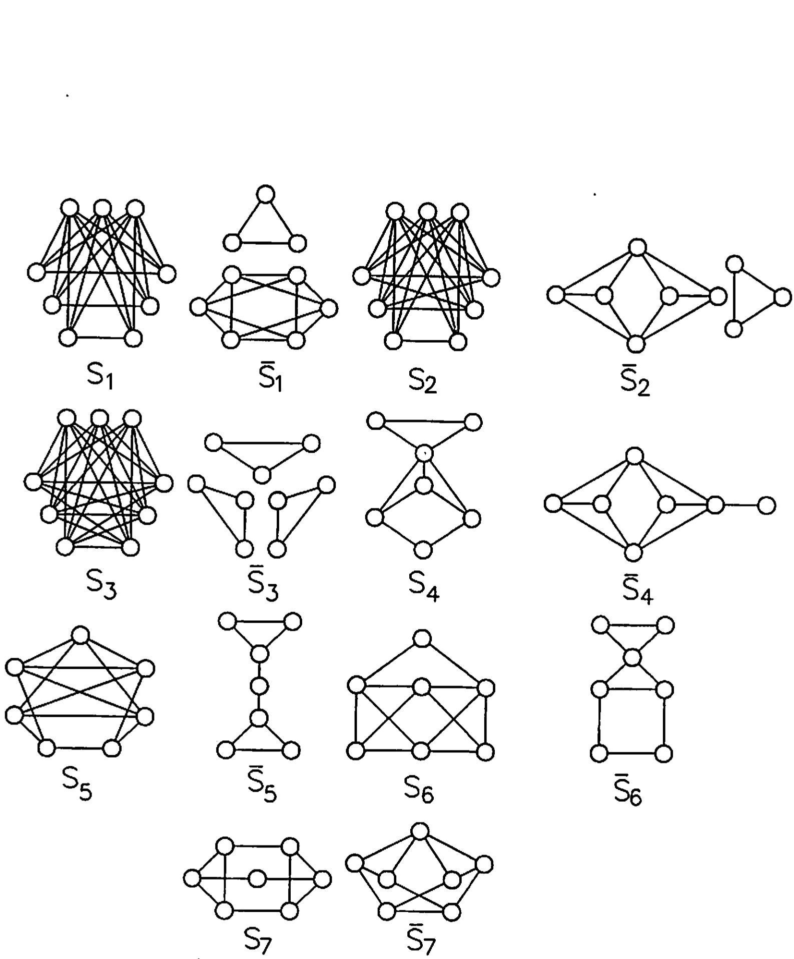

Figure 6

Theorem 2. *G is COLD if and only if it is an induced subgraph of at least one of the fourteen graphs depicted in Fig. 6.*

By Theorem 2 COLD graphs are characterized as graphs which can be embedded into at least one of fourteen maximal graphs (appearing as seven pairs of complementary graphs). The somewhat dual approach (which is more common in graph theory) is to find a list of all minimal graphs which cannot be embedded into COLD graphs, or in other words, to characterize COLD graphs in terms of forbidden induced subgraphs. Since the maximal COLD graphs have at most nine points, it follows at once that the collection of forbidden subgraphs is finite (say all graphs on ten points are forbidden although not necessarily minimal). A possible collection of all forbidden minimal graphs for COLD graphs is given by Corollary 1 (in Section 2). By a computer search it was established that the collection of all minimal forbidden graphs for COLD graphs coincides with the list of graphs of Corollary 1. So we have:

Theorem 3. *G is a COLD graph if and only if it does not contain any of the graphs of Fig. 2 as an induced subgraph.*

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