

Counting of lattices containing up to 5 reducible elements and having nullity up to 3

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ABSTRACT

In 1940, Birkhoff posed an open problem of counting all finite lattices on n elements. Recently, Bhavale counted all non-isomorphic lattices on n elements, containing up to four reducible elements, and having nullity up to three. Further, Aware and Bhavale counted all non-isomorphic lattices on n elements, containing up to five comparable reducible elements, and having nullity up to three. In this paper, we count all non-isomorphic lattices on n elements, containing five reducible elements, and having nullity three.

Keywords: chain, lattice, poset, counting

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1. Introduction

In 1940, Birkhoff [7] raised the open problem, compute for small n all non-isomorphic posets/lattices on a set of n elements. There were attempts to solve this problem by many authors. In 2002, Brinkmann and McKay [8] enumerated all non-isomorphic posets with 15 and 16 elements. The work of enumeration of all non-isomorphic (unlabeled) posets is still in progress for $n \geq 17$. In the same year, Heitzig and Reinhold [11] counted all non-isomorphic (unlabeled) lattices on up to 18 elements. For further details about counting of lattices, reader may refer [9] and [12].

In 2002, Thakare et al. [14] counted all non-isomorphic lattices on n elements, containing exactly two reducible elements. Recently, Bhavale and Aware [4] counted all non-isomorphic lattices on n elements, containing exactly three reducible elements. Bhavale and Aware [3] also counted all non-isomorphic lattices on n elements, and having nullity

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up to two. Bhavale [2] counted all non-isomorphic lattices on n elements, containing up to four reducible elements, and having nullity up to three. Aware and Bhavale [1] counted all non-isomorphic lattices on n elements, containing up to five comparable reducible elements, and having nullity up to three (see [1] also). In this paper, we count all non-isomorphic lattices on n elements, containing exactly five reducible elements, and having nullity three.

2. Preliminaries and prerequisites

Let \leq be a partial order relation on a non-empty set P , and let (P, \leq) be a poset. Elements $x, y \in P$ are said to be *comparable*, if either $x \leq y$ or $y \leq x$. A poset is called a *chain* if any two elements in it are comparable. Elements $x, y \in P$ are said to be *incomparable*, denoted by $x \parallel y$, if x, y are not comparable. An element $c \in P$ is a *lower bound* (an *upper bound*) of $a, b \in P$ if $c \leq a, c \leq b$ ($a \leq c, b \leq c$). A *meet* of $a, b \in P$, denoted by $a \wedge b$, is defined as the greatest lower bound of a and b . A *join* of $a, b \in P$, denoted by $a \vee b$, is defined as the least upper bound of a and b . A poset L is a *lattice* if $a \wedge b$ and $a \vee b$, exist in $L, \forall a, b \in L$. Lattices L_1 and L_2 are *isomorphic* (in symbol, $L_1 \cong L_2$), and the map $\phi : L_1 \rightarrow L_2$ is an *isomorphism* if and only if ϕ is one-to-one and onto, and $a \leq b$ in L_1 if and only if $\phi(a) \leq \phi(b)$ in L_2 . Algebraically, $\phi : L_1 \rightarrow L_2$ is an *isomorphism* if and only if ϕ is one-to-one and onto, and preserves both meet and join for any two elements.

An element b in P *covers* an element a in P if $a < b$, and there is no element c in P such that $a < c < b$. Denote this fact by $a \prec b$, and say that pair $\langle a, b \rangle$ is a *covering* or an *edge*. If $a \prec b$ then a is called a *lower cover* of b , and b is called an *upper cover* of a . An element of a poset P is called *doubly irreducible* if it has at most one lower cover and at most one upper cover in P . Let $Irr(P)$ denote the set of all doubly irreducible elements in the poset P . Let $Irr^*(P) = \{x \in Irr(P) : x \text{ has exactly one upper cover and exactly one lower cover in } P\}$. The set of all coverings in P is denoted by $E(P)$. The graph on a poset P with edges as covering relations is called the *cover graph* and is denoted by $C(P)$. The number of coverings in a chain is called *length* of the chain.

The *nullity* of a graph G is given by $m - n + c$, where m is the number of edges in G , n is the number of vertices in G , and c is the number of connected components of G . Bhavale and Waphare [5] defined *nullity of a poset* P , denoted by $\eta(P)$, to be the nullity of its cover graph $C(P)$. For $a < b$, the interval $[a, b] = \{x \in P : a \leq x \leq b\}$, and $[a, b) = \{x \in P : a \leq x < b\}$; similarly, (a, b) and $(a, b]$ can also be defined. For integer $n \geq 3$, *crown* is a partially ordered set $\{x_1, y_1, x_2, y_2, x_3, y_3, \dots, x_n, y_n\}$ in which $x_i \leq y_i, y_i \geq x_{i+1}$, for $i = 1, 2, \dots, n - 1$, and $x_1 \leq y_n$ are the only comparability relations.

An element x in a lattice L is *join-reducible*(*meet-reducible*) in L if there exist $y, z \in L$ both distinct from x , such that $y \vee z = x$ ($y \wedge z = x$). An element x in a lattice L is *reducible* if it is either join-reducible or meet-reducible. x is *join-irreducible*(*meet-irreducible*) if it is not join-reducible(meet-reducible); x is *doubly irreducible* if it is both join-irreducible and meet-irreducible. The set of all doubly irreducible elements in L is denoted by $Irr(L)$, and its complement in L is denoted by $Red(L)$. An *ear* of a loopless connected graph G is a subgraph of G which is a maximal path in which all internal vertices are of degree

2 in G . *Trivial ear* is an ear containing no internal vertices. A *non-trivial ear* is an ear which is not an edge. A vertex of a graph is called *pendant* if its degree is one.

Definition 2.1. [15] A finite lattice of order n is called *dismantlable* if there exists a chain $L_1 \subset L_2 \subset \dots \subset L_n (= L)$ of sublattices of L such that $|L_i| = i$, for all i .

Theorem 2.2. [13] *A finite lattice is dismantlable if and only if it contains no crown.*

The concepts of ‘1-sum’ and ‘2-sum’ are introduced by Bhavale and Waphare [6]. Let P_1 and P_2 be disjoint posets. Let $a \in P_1$. Define a partial order on $P = P_1 \cup P_2$ with respect to a as follows. For $x, y \in P$, we say that $x \leq y$ in P if $x, y \in P_1$ and $x \leq y$ in P_1 or $x, y \in P_2$ and $x \leq y$ in P_2 or $x \in P_1, y \in P_2$ and $x \leq a$ in P_1 . It is easy to see that P is a poset containing P_1 and P_2 as subposets. The procedure for obtaining P in this way is called an *up 1-sum of P_1 with P_2 with respect to a* , denoted by $P_1]_a P_2$. A diagram of P is obtained by placing a diagram of P_1 and a diagram of P_2 side by side in such a way that the minimal elements of P_2 are at higher positions than a and then by adding the coverings $\langle a, x \rangle$ for all $x \in S$, the set of all minimal elements of P_2 . This clearly gives $|E(P)| = |E(P_1)| + |E(P_2)| + |S|$.

Dually, one can define a *down 1-sum of two posets*. If P is a down 1-sum of P_1 with P_2 with respect to a in P_1 then we write $P = P_1]_a^P P_2$. We call the element a an *adjunct element* of the 1-sum. We say that P is a *1-sum of posets P_1 and P_2 with respect to an element $a \in P_1$* if P is either an up 1-sum or a down 1-sum of P_1 and P_2 with respect to a . A 1-sum $P_1]_a P_2$ or $P_1]_a^P P_2$ is called *trivial 1-sum* if P_2 is a chain and a is respectively a maximal or a minimal element of P_1 ; otherwise, we say that the 1-sum is *non-trivial*.

The concept of *adjunct operation of lattices*, is introduced by Thakare et al. [14]. Bhavale and Waphare [6] extended the concept of adjunct of lattices to 2-sum of posets.

A *2-sum of the posets P_1 and P_2 with respect to a pair (a, b) with $a < b$ but $a \not\leq b$ in P_1* , is the poset $P = P_1 \cup P_2$ with a partial order defined on P , which is the union of the partial orders in $P_1]_a P_2$ and $P_1]_b P_2$. We denote the 2-sum of the posets P_1 and P_2 with respect to a pair (a, b) by $P_1]_a^b P_2$. The pair (a, b) is called an *adjunct pair* of the 2-sum.

A lattice L is called an *adjunct of lattices L_1, L_2, \dots, L_k* , if it is of the form $L = L_1]_{a_1}^{b_1} L_2 \cdots]_{a_{k-1}}^{b_{k-1}} L_k$. Following result is due to Thakare et al. [14].

Theorem 2.3. [14] *A finite lattice is dismantlable if and only if it is an adjunct of chains.*

Corollary 2.4. [4] *A dismantlable lattice with n elements has nullity k if and only if it is an adjunct of $k + 1$ chains.*

According to Bhavale and Waphare [5], if a dismantlable lattice contains r reducible elements and has nullity $k \geq 1$ then $2 \leq r \leq 2k$.

If M and N are two disjoint posets then the *direct sum*, denoted by $M \oplus N$, is defined by taking the following order relation on $M \cup N$: $x \leq y$ if and only if $x, y \in M$ and $x \leq y$ in M , or $x, y \in N$ and $x \leq y$ in N , or $x \in M, y \in N$. In general, $M \oplus N \neq N \oplus M$. Also, if M and N are lattices then $|E(M \oplus N)| = |E(M)| + |E(N)| + 1$.

Thakare et al. [14] defined a *block* as a finite lattice in which the largest element is join-reducible and the least element is meet-reducible. Let L be a finite lattice which is not a chain. Then L contains a unique maximal sublattice which is a block, called the *maximal block*. The lattice L has the form $C_1 \oplus \mathbf{B}$ or $\mathbf{B} \oplus C_2$ or $L = C_1 \oplus \mathbf{B} \oplus C_2$, where C_1, C_2 are the chains and \mathbf{B} is the maximal block. It follows that $\eta(\mathbf{B}) = \eta(L)$.

Bhavale and Waphare [5] introduced the concepts namely, retractible element, basic retract, basic block, and basic block associated to a poset.

Definition 2.5. [5] Let P be a poset. Let $x \in Irr(P)$. Then x is called a *retractible* element of P if it satisfies either of the following conditions.

1. There are no $y, z \in Red(P)$ such that $y \prec x \prec z$.
2. There are $y, z \in Red(P)$ such that $y \prec x \prec z$ and there is no other directed path from y to z in P .

Definition 2.6. [5] A poset P is a *basic retract* if no element of $Irr^*(P)$ is retractible in the poset P .

Definition 2.7. [5] Let P be a poset. Consider a (Hasse) diagram of P . If $Irr^*(P) = \emptyset$ then we say that P is a basic retract associated to itself; otherwise, go on removing elements of $Irr^*(P)$ one by one as long as $\eta(P)$ does not alter. Ultimately we get a subposet P' of P such that no element of $Irr^*(P')$ is retractible in P' . The resultant subposet P' of P is called a *basic retract associated to P* .

Definition 2.8. [5] A poset P is a *basic block* if it is one element or $Irr(P) = \emptyset$ or removal of any doubly irreducible element reduces nullity by one.

Definition 2.9. [5] B is a *basic block associated to a poset P* if B is obtained from the basic retract associated to P by successive removal of all the pendant vertices.

Theorem 2.10. [5] Let B be a basic retract associated to a poset P . Then $Red(B) = Red(P)$ and $\eta(B) = \eta(P)$.

For the other definitions, notation, and terminology see [10] and [16].

3. Lattices in which reducible elements are comparable

Bhavale and Waphare [5] introduced the concept of RC-lattices as the class of lattices in which the reducible elements are all lying on a chain. Using Theorem 2.2, they have proved that RC-lattices are dismantlable. Interestingly, the lattices on up to three reducible elements are RC-lattices.

Let $\mathcal{L}(n; r, k)$ denote the class of all non-isomorphic RC-lattices on n elements such that each lattice in it has nullity k and contains r reducible elements. Note that there is only one lattice (i.e., a chain) having nullity zero. Therefore $\mathcal{L}(n; 0, 0)$ consists of the chain on n elements.

Theorem 3.1. [14] For an integer $n \geq 4$,

$$|\mathcal{L}(n; 2, 1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & \text{if } n = 2m + 1; \\ \frac{m(m-1)(4m-5)}{6} & \text{if } n = 2m. \end{cases}$$

Recently, Bhavale and Aware [3] counted all non-isomorphic lattices having nullity up to two. Further, Aware and Bhavale [1] counted all non-isomorphic lattices containing up to five comparable reducible elements, and having nullity up to three (see also [1] and [4]).

Bhavale [2] defined the class $\mathcal{L}'(n)$ as the subclass of $\mathcal{L}(n; 2, 1)$, containing the lattices in which 1 is the reducible element.

Lemma 3.2. [2] For $n \geq 4$, $|\mathcal{L}'(n)| = \sum_{i=0}^{n-4} \left\lfloor \frac{n-i-2}{2} \right\rfloor$.

In Section 4, we count the class of all non-isomorphic dismantlable lattices on n elements such that each lattice in it has nullity three, and at least two of the five reducible elements in each lattice in it are incomparable. For that purpose we need the following remark due to Bhavale [2].

Remark 3.3. [2] For any $j \geq 3$, let S_j be the class of all non-isomorphic posets Y on j elements such that $Y = C]_x C'$, where C, C' are chains. Then $Y \in S_j$ if and only if $Y \oplus \{1\} \in \mathcal{L}'(j+1)$. Therefore $|S_j| = |\mathcal{L}'(j+1)|$. If $s_j = |S_j|$ for all j , then $s_j = |\mathcal{L}'(j+1)| = \sum_{i=0}^{j-3} \left\lfloor \frac{j-i-1}{2} \right\rfloor$.

4. Lattices in which reducible elements are incomparable

It is known that the lattices of nullity up to two are RC-lattices (see [3]). But a lattice of nullity at least three may not be RC-lattice. We define the class of RI-lattices as the class of lattices such that each lattice in it contains at least two incomparable reducible elements. Hereafter, we will limit ourselves to dismantlable lattices only. By Theorem 2.2, it follows that the lattices of nullity up to three, containing at most 7 reducible elements are dismantlable, since cube (2^3) is the smallest lattice of nullity 3, containing 8 reducible elements, and which also contains the crown on 6 reducible elements.

Let $\mathcal{L}_r^k(n)$ denote the class of all non-isomorphic dismantlable RI-lattices on n elements such that each lattice in it contains r reducible elements and has nullity k . Let $\mathcal{B}_r^k(m)$ denote the class of all non-isomorphic dismantlable maximal blocks on m elements such that each maximal block in it is an RI-lattice of nullity k and contains r reducible elements.

Theorem 4.1. [2] For $n \geq 8$,

$$|\mathcal{B}_4^3(n)| = \begin{cases} \sum_{n=i+j+2, i>j} 4s_i s_j & \text{if } n \text{ is odd;} \\ \sum_{n=i+j+2, i>j} 4s_i s_j + s_{\frac{n-2}{2}} (2s_{\frac{n-2}{2}} + 1) & \text{if } n \text{ is even,} \end{cases}$$

$$\text{where } s_i = \sum_{k=0}^{i-3} \left\lfloor \frac{i-k-1}{2} \right\rfloor.$$

The following result follows from Theorem 4.1.

Theorem 4.2. [2] For $n \geq 8$, $|\mathcal{L}_4^3(n)| = \sum_{i=0}^n |\mathcal{B}_4^3(n-i)|$.

Now in the following we prove that, there are eight non-isomorphic basic blocks containing five reducible elements and having nullity three such that at least two reducible elements are incomparable.

Proposition 4.3. If B is the basic block associated to a lattice $L \in \mathcal{L}_5^3(n)$ then $B \in \{B_1, B_2, \dots, B_8\}$ (see Figure I).

Proof. As B is the basic block associated to a lattice $L \in \mathcal{L}_5^3(n)$, not all 5 reducible elements of B are comparable. Let $0, 1, a, b, c$ be the reducible elements of B . Now at least two of them are incomparable. Without loss of generality, suppose $a||b$. Then we have the following four cases.

1. If $c||a$ and $c||b$ then B contains a diamond, $M_3 = \{0, a, b, c, 1\}$ as a sublattice. Without loss of generality, suppose a is join reducible. Then there exist $a', a'' \in Irr(B)$ such that $a'||a''$ with $a' \vee a'' = a$ and $a' \wedge a'' = 0$. Consider a sublattice $L' = M_3 \cup \{a''\}$ of B . By Corollary 2.4, $\eta(L'_0^a\{a'\}) = 3$, since $\eta(M_3) = 2 = \eta(L')$. Again without loss of generality, suppose b is join reducible. Then there exist $b', b'' \in Irr(B)$ such that $b'||b''$ with $b' \vee b'' = b$ and $b' \wedge b'' = 0$. Consider a sublattice $L'' = M_3 \cup \{a'', b''\}$ of B . Again by Corollary 2.4, $\eta(L''_0^a\{a'\}_0^b\{b'\}) = 4$, since $\eta(L'') = \eta(M_3)$. This is not possible, since $\eta(B) = 3$.

2. Without loss of generality, suppose $c||a$ and c is comparable to b with $b < c$. Then B contains a pentagon, $N_5 = \{0, a, b, c, 1\}$ as a sublattice. Now we have the following three subcases.

a. Suppose a is a join reducible element only. Then there exist $a', a'' \in Irr(B)$ such that $a'||a''$ with $a' \vee a'' = a$ and $a' \wedge a'' = 0$. Consider a sublattice $L' = N_5 \cup \{a''\}$ of B . By Corollary 2.4, $\eta(L'_0^a\{a'\}) = 2$, since $\eta(N_5) = 1 = \eta(L')$.

If b and c are both join reducible elements then by similar arguments there exist $b', b'', c', c'' \in Irr(B)$ such that $b'||b'', c'||c''$ and $\eta(L''_0^a\{a'\}_0^b\{b'\}_0^c\{c'\}) = 4$, where $L'' = L' \cup \{b'', c''\}$ is a sublattice of B . Note that $\eta(L'') = \eta(L')$. Thus B contains a sublattice having nullity greater than that of B . This is not possible. Similar arguments can be applied if b and

c are both meet reducible elements as well as if b is a join reducible element and c is a meet reducible element. Further in both the cases, we get sublattices of B having nullity greater than that of B . Finally, if b is a meet reducible element and c is a join reducible element then we left with only one possibility and in that case B is isomorphic to the basic block B_2 (see Figure 1).

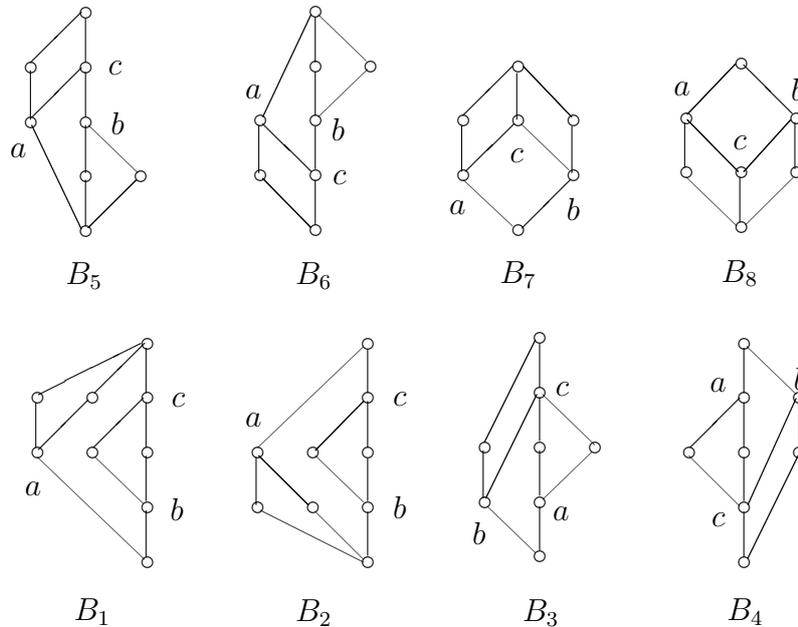


Fig. 1. Basic blocks of nullity 3 containing 5 reducible elements

b. Suppose a is a meet reducible element only. By duality (and also by similar arguments as applied above), in this case, we find B is isomorphic to the basic block B_1 (see Figure 1).

c. Suppose a is a meet reducible as well as join reducible element. Then by similar arguments as applied above, it can be shown that B contains a sublattice having nullity 4. So this is also not possible.

3. If c is comparable to both a and b with $c < a$ and $c < b$. Clearly $a \wedge b = c$. Further, c can not be join reducible element, since otherwise B contains a sublattice having nullity greater than 3. Now we have the following three subcases.

a. a and b are both join reducible elements. Then (c, a) and (c, b) can not both be adjunct pairs in an adjunct representation of B , since otherwise B contains a sublattice having nullity greater than 3. Therefore at the most one of (c, a) and (c, b) may be an adjunct pair in an adjunct representation of B .

Suppose without loss of generality, (c, a) is an adjunct pair in an adjunct representation of B . But then there exists $x \in Irr(B)$ such that $x \wedge c = 0$ and $x \vee c = b$, since b is a join reducible element. In this case B is isomorphic to the basic block B_4 (see Figure 1).

If none of (c, a) and (c, b) is an adjunct pair in an adjunct representation of B then there exist $x, y \in Irr(B)$ such that $x \wedge c = 0, x \vee c = a, y \wedge c = 0$ and $y \vee c = b$, since a and b are join reducible elements. In this case B is isomorphic to the basic block B_8 (see Figure 1).

b. a is join(meet) reducible and b is meet(join) reducible element then B is isomorphic to the basic block B_6 (see Figure 1). c. a and b are both meet reducible elements. But this is not possible, since otherwise B contains a sublattice having nullity > 3 .

4. If c is comparable to both a and b with $a < c$ and $b < c$. The proof is similar to the earlier case, and by duality, we get the basic blocks B_3, B_7 and B_5 up to isomorphism. \square

5. Counting of RI-lattices containing 5 reducible elements and having nullity 3

For each i , $1 \leq i \leq 8$, let $\mathbb{B}_i = \{\mathbf{B} \in \mathcal{B}_5^3(n) \mid B_i \text{ is the basic block associated to } \mathbf{B}\}$. Then $\mathcal{B}_5^3(n) = \mathbb{B}_1 \dot{\cup} \mathbb{B}_2 \dot{\cup} \cdots \dot{\cup} \mathbb{B}_8$.

Proposition 5.1. For $n \geq 9$, $|\mathbb{B}_1| = \sum_{i=4}^{n-5} \left(|\mathcal{L}(i; 2, 1)| \times \sum_{k=0}^{n-i-5} \left\lfloor \frac{n-i-k-3}{2} \right\rfloor \right)$, where

$$|\mathcal{L}(i; 2, 1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & \text{if } i = 2m+1; \\ \frac{m(m-1)(4m-5)}{6} & \text{if } i = 2m. \end{cases}$$

Proof. Let $\mathbf{B} \in \mathbb{B}_1$. As B_1 (see Figure 1) is the basic block associated to \mathbf{B} , $\mathbf{B} \setminus \{0, 1\}$ is the disjoint union of a sublattice $M \in \mathcal{L}_2^1(i)$ and a subposet $Y \in S_j$ of \mathbf{B} , where $i \geq 4$ and $j \geq 3$ with $|\mathbf{B}| = n = i + j + 2 \geq 9$. Also, observe that $\mathbf{B} = (\{0\} \oplus M \oplus \{1\})_0^1 Y$ (or $\mathbf{B} = (\{0\} \oplus Y \oplus \{1\})_0^1 M$). Further, if $\mathbf{B}' = (\{0\} \oplus M' \oplus \{1\})_0^1 Y' \in \mathbb{B}_1$ then $\mathbf{B} \cong \mathbf{B}'$ if and only if $M \cong M'$ and $Y \cong Y'$. Now for fixed i , there are $|\mathcal{L}(i; 2, 1)| \times |S_{n-i-2}|$ maximal blocks in \mathbb{B}_1 up to isomorphism. Further, $4 \leq i = n - j - 2 \leq n - 5$ and there are $\sum_{i=4}^{n-5} (|\mathcal{L}(i; 2, 1)| \times |S_{n-i-2}|)$ maximal blocks in \mathbb{B}_1 up to isomorphism. Therefore

$|\mathbb{B}_1| = \sum_{i=4}^{n-5} (|\mathcal{L}(i; 2, 1)| \times |S_{n-i-2}|)$. But by Remark 3.3, $|S_j| = \sum_{k=0}^{j-3} \left\lfloor \frac{j-k-1}{2} \right\rfloor$. Hence the proof follows from Theorem 3.1. \square

Corollary 5.2. For $n \geq 9$, $|\mathbb{B}_2| = \sum_{i=4}^{n-5} \left(|\mathcal{L}(i; 2, 1)| \times \sum_{k=0}^{n-i-5} \left\lfloor \frac{n-i-k-3}{2} \right\rfloor \right)$, where

$$|\mathcal{L}(i; 2, 1)| = \begin{cases} \frac{m(m-1)(4m+1)}{6} & \text{if } i = 2m+1; \\ \frac{m(m-1)(4m-5)}{6} & \text{if } i = 2m. \end{cases}$$

Proof. Clearly $|\mathbb{B}_2| = |\mathbb{B}_1|$, since $\mathbf{B} \in \mathbb{B}_2$ if and only if the dual of \mathbf{B} , $\mathbf{B}^* \in \mathbb{B}_1$. Thus the proof follows from Proposition 5.1. \square

For $n \geq 6$, Thakare et al. [14] denoted the class of all non-isomorphic maximal blocks (containing n elements and having nullity two) of type \mathbf{B} , where $\mathbf{B} = C_1]_{a_1}^{b_1} C_2]_{a_2}^{b_2} C_3$ (where $0 = a_1 < a_2 < b_1 = b_2 = 1$ on a maximal chain C_1 and C_2, C_3 are chains) by $\mathcal{B}_{1,2}^n$, and obtained the following result.

Proposition 5.3. [14] For $n \geq 6$, $|\mathcal{B}_{1,2}^n| = \sum_{r=1}^{\lfloor \frac{n-4}{2} \rfloor} \sum_{l=1}^{n-2r-3} (n-l-2r-2)$.

Proposition 5.4. For $n \geq 8$,

$$|\mathbb{B}_3| = \sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} \sum_{r=1}^{\lfloor \frac{n-i-j-4}{2} \rfloor} \sum_{l=1}^{n-i-j-2r-3} l(n-i-j-l-2r-2).$$

Proof. Let $\mathbf{B} \in \mathbb{B}_3$. Clearly B_3 (see Figure 1) is the basic block associated to \mathbf{B} . As the nullity of B_3 (and hence of \mathbf{B}) is 3, by Corollary 2.4, $\mathbf{B} = C_1]_0^c C_2]_a^c C_3]_b^1 C_4$, where C_1 is a maximal chain containing a and c , and C_2, C_3, C_4 are chains with $b \in C_2$. Note that $a, b, c \in \text{Red}(B_3)$ and by Theorem 2.10, $\text{Red}(\mathbf{B}) = \text{Red}(B_3)$. Let $\mathbf{B}' = \mathbf{B} \cap [0, c]$ and $C'_1 = C_1 \cap (c, 1]$ with $|C'_1| = i \geq 1$ and $|\mathbf{B}'| = k \geq 6$. Suppose $|C_4| = j \geq 1$. Then $\mathbf{B} = (\mathbf{B}' \oplus C'_1)]_b^1 C_4$ with $\mathbf{B}' \in \mathcal{B}_{1,2}^k$ and $|\mathbf{B}| = n = i + j + k \geq 8$. Further, if $\mathbf{D} = (\mathbf{D}' \oplus C'')]_b^1 C''' \in \mathbb{B}_3$, where $\mathbf{D}' \in \mathcal{B}_{1,2}^k$ and C', C'' are chains with $|C'| = i$ and $|C''| = j$, then $\mathbf{B} \cong \mathbf{D}$ if and only if $\mathbf{B}' \cong \mathbf{D}'$, $C'_1 \cong C'$ and $C_4 \cong C''$. By Proposition 5.3,

$|\mathcal{B}_{1,2}^k| = \sum_{r=1}^{\lfloor \frac{k-4}{2} \rfloor} \sum_{l=1}^{k-2r-3} (k-l-2r-2)$, where $l = |C_2|$, $r = |C_3|$, and for fixed l and r , $k-l-2r-2$ is the number of possible positions of a in the block $\mathbf{B}' \in \mathcal{B}_{1,2}^k$ (and hence in the block $\mathbf{B} \in \mathbb{B}_3$).

Now for fixed i and j , there are $|\mathcal{B}_{1,2}^{n-i-j}|$ maximal blocks in \mathbb{B}_3 up to isomorphism. Therefore for fixed j , we have $1 \leq i = n - j - k \leq n - j - 6$ and there are $\sum_{i=1}^{n-j-6} |\mathcal{B}_{1,2}^{n-i-j}|$ maximal blocks in \mathbb{B}_3 up to isomorphism. Further, $1 \leq j = n - i - k \leq n - 1 - 6 = n - 7$ and there are $\sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} |\mathcal{B}_{1,2}^{n-i-j}|$ maximal blocks in \mathbb{B}_3 up to isomorphism.

Now b takes l number of positions in the block $\mathbf{B}' \in \mathcal{B}_{1,2}^k$ and hence in the block $\mathbf{B} \in \mathbb{B}_3$. Therefore there are $\sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} (|\mathcal{B}_{1,2}^{n-i-j}| \times l)$ maximal blocks in \mathbb{B}_3 up to isomorphism. Thus the proof follows from Proposition 5.3. □

Corollary 5.5. For $n \geq 8$,

$$|\mathbb{B}_4| = \sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} \sum_{r=1}^{\lfloor \frac{n-i-j-4}{2} \rfloor} \sum_{l=1}^{n-i-j-2r-3} l(n-i-j-l-2r-2).$$

Proof. Clearly $|\mathbb{B}_4| = |\mathbb{B}_3|$, since $\mathbf{B} \in \mathbb{B}_4$ if and only if the dual of \mathbf{B} , $\mathbf{B}^* \in \mathbb{B}_3$. Thus the proof follows by Proposition 5.4. \square

Corollary 5.6. For $n \geq 8$,

$$|\mathbb{B}_6| = \sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} \sum_{r=1}^{\lfloor \frac{n-i-j-4}{2} \rfloor} \sum_{l=1}^{n-i-j-2r-3} l(n-i-j-l-2r-2).$$

Proof. Let $\mathbf{B} \in \mathbb{B}_6$. Clearly B_6 (see Figure 1) is the basic block associated to \mathbf{B} . As the nullity of B_6 (and hence of \mathbf{B}) is 3, by Corollary 2.4, $\mathbf{B} = C_1]_c^1 C_2]_b^1 C_3]_0^a C_4$, where C_1 is a maximal chain containing b and c , and C_2, C_3, C_4 are chains with $a \in C_2$. Note that $a, b, c \in \text{Red}(B_6)$ and by Theorem 2.10, $\text{Red}(\mathbf{B}) = \text{Red}(B_6)$. Let $\mathbf{B}' = \mathbf{B} \cap [c, 1]$ and $C'_1 = C_1 \cap [0, c]$ with $|C'_1| = i \geq 1$ and $|\mathbf{B}'| = k \geq 6$. Suppose $|C_4| = j \geq 1$. Then $\mathbf{B} = (C'_1 \oplus \mathbf{B}')]_0^a C_4$ with $\mathbf{B}' \in \mathcal{B}_{1,2}^k$ and $|\mathbf{B}| = n = i + j + k \geq 8$.

Now by applying the similar arguments as applied in the proof of Proposition 5.4, we have $|\mathbb{B}_6| = \sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} (|\mathcal{B}_{1,2}^{n-i-j}| \times l)$. Hence the proof follows from Proposition 5.3. \square

Corollary 5.7. For $n \geq 8$,

$$|\mathbb{B}_5| = \sum_{j=1}^{n-7} \sum_{i=1}^{n-j-6} \sum_{r=1}^{\lfloor \frac{n-i-j-4}{2} \rfloor} \sum_{l=1}^{n-i-j-2r-3} l(n-i-j-l-2r-2).$$

Proof. Clearly $|\mathbb{B}_5| = |\mathbb{B}_6|$, since $\mathbf{B} \in \mathbb{B}_5$ if and only if the dual of \mathbf{B} , $\mathbf{B}^* \in \mathbb{B}_6$. Thus the proof follows by Corollary 5.6. \square

Proposition 5.8. For $n \geq 7$,

$$|\mathbb{B}_7| = \sum_{l=1}^{n-7} \sum_{r=1}^{n-l-6} \sum_{q=1}^{\lfloor \frac{n-l-r-4}{2} \rfloor} \sum_{p=q+1}^{n-q-l-r-3} (p \times q) + \sum_{r=1}^{\lfloor \frac{n-6}{2} \rfloor} \sum_{l=r+1}^{n-r-5} \sum_{p=1}^{\lfloor \frac{n-l-r-3}{2} \rfloor} p^2 + \sum_{p=1}^{\lfloor \frac{n-5}{2} \rfloor} \sum_{l=1}^{\lfloor \frac{n-2p-3}{2} \rfloor} \frac{p(p+1)}{2}.$$

Proof. Let $\mathbf{B} \in \mathbb{B}_7$. Clearly B_7 (see Figure 1) is the basic block associated to \mathbf{B} . As the nullity of B_7 (and hence of \mathbf{B}) is 3, by Corollary 2.4, $\mathbf{B} = C_1]_0^c C_2]_a^1 C_3]_b^1 C_4$, where C_1 is a maximal chain containing a and c , and C_2, C_3, C_4 are chains with $b \in C_2$. Note that $a, b, c \in \text{Red}(B_7)$ and by Theorem 2.10, $\text{Red}(\mathbf{B}) = \text{Red}(B_7)$. Suppose $C_1 : x_0 \prec x_1 \prec x_2 \prec \dots \prec x_{p+t}$, $C_2 : y_1 \prec y_2 \prec \dots \prec y_q$, $C_3 : z_1 \prec z_2 \prec \dots \prec z_l$, and $C_4 : w_1 \prec w_2 \prec \dots \prec w_r$ are disjoint chains with $x_0 = 0, x_{p+1} = c, a \in C_1 \cap [x_1, x_p]$ and $x_{p+t} = 1$. Then $p + q + l + r + t + 1 = n \geq 8$ with $t \geq 2$. Note that, a has p choices and b has q choices in \mathbf{B} . Further, as C_1 is maximal, $p \geq q$. Now we have the following two cases.

1. Suppose $p > q$. Let $S_1 = \{(p, q, l, r, t) \mid p \geq q + 1, t \geq 2, p, q, l, r, t \in \mathbb{N}, 1 + p + q + l + r + t = n \geq 8\}$. Suppose $\mathbf{B}' \in \mathbb{B}_7$ and $(p', q', l', r', t') \in S_1$ corresponds to \mathbf{B}' . Then

$\mathbf{B} \cong \mathbf{B}'$ if and only if $(p, q, l, r, t) = (p', q', l', r', t')$, and $h(a), h(b)$ are the same in both \mathbf{B} and \mathbf{B}' , where $h(a)$ is a height of a .

Now for fixed $p, q, l, r, 2 \leq t = n - p - q - l - r - 1$, and there are $p \times q$ maximal blocks in \mathbb{B}_7 up to isomorphism. Therefore for fixed $q, l, r, q + 1 \leq p = n - q - l - r - t - 1 \leq n - q - l - r - 3$ since $t \geq 2$, and there are $\sum_{p=q+1}^{n-q-l-r-3} (p \times q)$ maximal blocks in \mathbb{B}_7 up to isomorphism. Again for fixed $l, r, 1 \leq q = n - p - l - r - t - 1 \leq n - (q + 1) - l - r - 3$ since $p \geq q + 1$ and $t \geq 2$. That is, $1 \leq q \leq \lfloor \frac{n-l-r-4}{2} \rfloor$. Therefore for fixed l, r , there are $\sum_{q=1}^{\lfloor \frac{n-l-r-4}{2} \rfloor} \left(\sum_{p=q+1}^{n-q-l-r-3} (p \times q) \right)$ maximal blocks in \mathbb{B}_7 up to isomorphism. Further, for fixed $l, 1 \leq r = n - p - q - l - t - 1 \leq n - (q + 1) - q - l - 3 = n - 2q - l - 4 \leq n - l - 6$ since $p \geq q + 1, t \geq 2$ and $q \geq 1$. Therefore for fixed l , there are $\sum_{r=1}^{n-l-6} \left(\sum_{q=1}^{\lfloor \frac{n-l-r-4}{2} \rfloor} \sum_{p=q+1}^{n-q-l-r-3} (p \times q) \right)$ maximal blocks in \mathbb{B}_7 up to isomorphism. Also $1 \leq l = n - p - q - r - t - 1 \leq n - 7$ since $p \geq q + 1, t \geq 2, q \geq 1$ and $r \geq 1$. Thus in this case, there are $\sum_{l=1}^{n-7} \left(\sum_{r=1}^{n-l-6} \sum_{q=1}^{\lfloor \frac{n-l-r-4}{2} \rfloor} \sum_{p=q+1}^{n-q-l-r-3} (p \times q) \right)$ maximal blocks in \mathbb{B}_7 up to isomorphism.

2. Suppose $p = q$. In this case, without loss, suppose $l \geq r$. But then again we have to consider the following two subcases.

a. Suppose $l > r$. Let $S_2 = \{(p, l, r, t) \mid l \geq r + 1, t \geq 2, p, l, r, t \in \mathbb{N}, 1 + 2p + l + r + t = n \geq 8\}$. Suppose $\mathbf{B}' \in \mathbb{B}_7$ and $(p', l', r', t') \in S_2$ corresponds to \mathbf{B}' . Then $\mathbf{B} \cong \mathbf{B}'$ if and only if $(p, l, r, t) = (p', l', r', t')$, and $h(a), h(b)$ are the same in both \mathbf{B} and \mathbf{B}' .

Now for fixed $p, l, r, 2 \leq t = n - 2p - l - r - 1$, and there are $p \times p = p^2$ maximal blocks in \mathbb{B}_7 up to isomorphism. Again for fixed $l, r, 1 \leq p = \frac{n-l-r-t-1}{2} \leq \frac{n-l-r-3}{2}$ since $t \geq 2$. That is, $1 \leq p \leq \lfloor \frac{n-l-r-3}{2} \rfloor$. Therefore for fixed l, r , there are $\sum_{p=1}^{\lfloor \frac{n-l-r-3}{2} \rfloor} p^2$ maximal blocks in \mathbb{B}_7 up to isomorphism. Further, for fixed $r, r + 1 \leq l = n - 2p - r - t - 1 \leq n - r - 5$ since $t \geq 2, p \geq 1$, and there are $\sum_{l=r+1}^{n-r-5} \left(\sum_{p=1}^{\lfloor \frac{n-l-r-3}{2} \rfloor} p^2 \right)$ maximal blocks in \mathbb{B}_7 up to isomorphism. Also $1 \leq r = n - 2p - l - t - 1 \leq n - (r + 1) - 5$ since $t \geq 2, p \geq 1$. That is, $1 \leq r \leq \frac{n-6}{2}$. Thus in this subcase, $1 \leq r \leq \lfloor \frac{n-6}{2} \rfloor$, and there are $\sum_{r=1}^{\lfloor \frac{n-6}{2} \rfloor} \left(\sum_{l=r+1}^{n-r-5} \sum_{p=1}^{\lfloor \frac{n-l-r-3}{2} \rfloor} p^2 \right)$ maximal blocks in \mathbb{B}_7 up to isomorphism.

b. Suppose $l = r$. Let $S_3 = \{(p, l, t) \mid t \geq 2, p, l, r, t \in \mathbb{N}, 1 + 2p + 2l + t = n \geq 7\}$. Suppose $\mathbf{B}' \in \mathbb{B}_7$ and $(p', l', t') \in S_3$ corresponds to \mathbf{B}' . Then $\mathbf{B} \cong \mathbf{B}'$ if and only if $(p, l, t) = (p', l', t')$, and $h(a), h(b)$ are the same in both \mathbf{B} and \mathbf{B}' along with $h(a) \leq h(b)$.

Now for fixed $p, l, 2 \leq t = n - 2p - 2l - 1$, and there are $p + (p - 1) + \dots + 3 + 2 + 1 = \frac{p(p+1)}{2}$ maximal blocks in \mathbb{B}_7 up to isomorphism. Note that, if $h(a) = i$ ($1 \leq i \leq p$) then there are $p - i + 1$ choices for b in a maximal block $\mathbf{B} \in \mathbb{B}_7$. Again for fixed $p, 1 \leq l = \frac{n-2p-t-1}{2} \leq \frac{n-2p-3}{2}$ since $t \geq 2$. That is, $1 \leq l \leq \lfloor \frac{n-2p-3}{2} \rfloor$. Therefore for fixed p , there are $\sum_{l=1}^{\lfloor \frac{n-2p-3}{2} \rfloor} \frac{p(p+1)}{2}$ maximal

blocks in \mathbb{B}_7 up to isomorphism. Also $1 \leq p = \frac{n-2l-t-1}{2} \leq \frac{n-5}{2}$ since $t \geq 2$ and $l \geq 1$. That is, $1 \leq p \leq \lfloor \frac{n-5}{2} \rfloor$. Thus in this subcase, there are $\sum_{p=1}^{\lfloor \frac{n-5}{2} \rfloor} \left(\sum_{l=1}^{\lfloor \frac{n-2p-3}{2} \rfloor} \frac{p(p+1)}{2} \right)$ maximal blocks in \mathbb{B}_7 up to isomorphism. □

Corollary 5.9. For $n \geq 7$,

$$|\mathbb{B}_8| = \sum_{l=1}^{n-7} \sum_{r=1}^{n-l-6} \sum_{q=1}^{\lfloor \frac{n-l-r-4}{2} \rfloor} \sum_{p=q+1}^{n-q-l-r-3} (p \times q) + \sum_{r=1}^{\lfloor \frac{n-6}{2} \rfloor} \sum_{l=r+1}^{n-r-5} \sum_{p=1}^{\lfloor \frac{n-l-r-3}{2} \rfloor} p^2 + \sum_{p=1}^{\lfloor \frac{n-5}{2} \rfloor} \sum_{l=1}^{\lfloor \frac{n-2p-3}{2} \rfloor} \frac{p(p+1)}{2}.$$

Proof. Clearly $|\mathbb{B}_8| = |\mathbb{B}_7|$, since $\mathbf{B} \in \mathbb{B}_8$ if and only if the dual of \mathbf{B} , $\mathbf{B}^* \in \mathbb{B}_7$. Thus the proof follows from Proposition 5.8. □

Using Proposition 5.1, Corollary 5.2, Proposition 5.4, Corollary 5.5, Corollary 5.6, Corollary 5.7, Proposition 5.8 and Corollary 5.9, we obtain the following result.

Theorem 5.10. For $n \geq 7$, $|\mathcal{B}_5^3(n)| = \sum_{i=1}^8 |\mathbb{B}_i|$.

Proof. The proof follows from the fact that $\{\mathbb{B}_i : 1 \leq i \leq 8\}$ forms a partition of the class $\mathcal{B}_5^3(n)$. □

The following result follows from Theorem 5.10.

Theorem 5.11. For $n \geq 7$, $|\mathcal{L}_5^3(n)| = \sum_{i=0}^n |\mathcal{B}_5^3(n-i)|$.

Table 1. Values of $|\mathcal{L}_5^3(n)|$ for $7 \leq n \leq 24$

n	7	8	9	10	11	12	13	14	15
$ \mathcal{L}_5^3(n) $	2	14	72	258	760	1914	4320	8918	17178
n	16	17	18	19	20	21	22	23	24
$ \mathcal{L}_5^3(n) $	31220	54096	89964	144480	225036	341280	505428	732930	1042866

6. Conclusion

Aware and Bhavale [1] counted all non-isomorphic lattices containing up to five comparable reducible elements, and having nullity up to three. Bhavale [2] obtained the number of all non-isomorphic lattices on n elements, having nullity three, and containing four reducible elements such that at least two of them are incomparable (see Theorem 4.2). In Theorem 5.11 we obtain the number of all non-isomorphic lattices on n elements, having nullity three, and containing five reducible elements such that at least two of them are

incomparable. Thus, this counting achieve the counting of all non-isomorphic lattices on n elements, containing up to five reducible elements, and having nullity up to three.

We now raise the following three problems. Find up to isomorphism all RC-lattices containing $r \geq 6$ reducible elements. Find up to isomorphism all dismantlable lattices containing $r \geq 6$ reducible elements. Find up to isomorphism all lattices containing $r \geq 4$ reducible elements.

References

- [1] B. P. Aware and A. N. Bhavale. Counting of lattices containing up to five comparable reducible elements, and having nullity up to three. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 127:179–197, 2025.
- [2] A. N. Bhavale. Counting of lattices containing up to 4 reducible elements and having nullity up to 3. *Journal of the Indian Mathematical Society (Communicated)*:179–197, 2025.
- [3] A. N. Bhavale and B. P. Aware. Counting of lattices having nullity up to two. *Journal of Combinatorial Mathematics and Combinatorial Computing*, 126:215–223, 2024. <https://doi.org/10.61091/jcmcc126-14>.
- [4] A. N. Bhavale and B. P. Aware. Counting of lattices on up to three reducible elements. *Journal of the Indian Mathematical Society*, 92(3):512–529, 2024. <https://doi.org/10.18311/jims/2025/36377>.
- [5] A. N. Bhavale and B. N. Waphare. Basic retracts and counting of lattices. *Australasian Journal of Combinatorics*, 78(1):73–99, 2020.
- [6] A. N. Bhavale and B. N. Waphare. Posets dismantlable by doubly irreducibles. *Journal of the Indian Mathematical Society*, 88(1-2):46–59, 2021.
- [7] G. Birkhoff. *Lattice Theory*, volume 25. Amer. Math. Soc. Colloq. Pub., New Delhi, 1979.
- [8] G. Brinkmann and B. D. McKay. Posets on up to 16 points. *Order*, 19(1-2):147–179, 2002. <https://doi.org/10.1023/A:1016543307592>.
- [9] V. Gebhardt and S. Tawn. Constructing unlabelled lattices. *Journal of Algebra*, 545(1-2):213–236, 2020. <https://doi.org/10.1016/j.jalgebra.2018.10.017>.
- [10] G. Grätzer. *General Lattice Theory*. Birkhäuser Verlag, Basel, second edition, 1998.
- [11] J. Heitzig and J. Reinhold. Counting finite lattices. *Algebra Universalis*, 48(1):43–53, 2002. <https://doi.org/10.1007/PL00013837>.
- [12] P. Jipsen and N. Lawless. Generating all finite modular lattices of a given size. *Algebra Universalis*, 74(1-2):253–264, 2015. <https://doi.org/10.1007/s00012-015-0348-x>.
- [13] D. Kelly and I. Rival. Crowns, fences and dismantlable lattices. *Canadian Journal of Mathematics*, 26(5):1257–1271, 1974. <https://doi.org/10.4153/CJM-1974-120-2>.
- [14] M. M. P. N. K. Thakare and B. N. Waphare. A structure theorem for dismantlable lattices and enumeration. *Periodica Mathematica Hungarica*, 45(1-2):147–160, 2002. <https://doi.org/10.1023/A:1022314517291>.
- [15] I. Rival. Lattices with doubly irreducible elements. *Canadian Mathematical Bulletin*, 17:91–95, 1974. <https://doi.org/10.4153/CMB-1974-016-3>.

- [16] D. B. West. *Introduction to Graph Theory*. Prentice Hall of India, New Delhi, 2002.

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