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Decomposition of graphs into trees of order five

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ABSTRACT

A $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition of a graph is a partition of its edge set into α copies of P_5, β copies of S_5, and γ copies of Y_5, where P_5, S_5, and Y_5 denote the three non-isomorphic trees of order five. In this paper, we study the existence of a $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition of the complete bipartite graph $K_{m,n}$ for $m \geq 4$ and $n \geq 2$, and of the complete graph K_n for $n \geq 8$. In fact, we establish necessary and sufficient conditions for the existence of such decompositions in $K_{m,n}$ and K_n.

Keywords: paths, stars, Y-trees, complete bipartite graphs, complete graphs, decomposition

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1. Introduction

All graphs considered in this paper are finite, simple, and undirected. Let P_n, S_n, C_n, K_n, and $K_{m,n}$ denote a path, a star, a cycle, a complete graph on n vertices, and a complete bipartite graph with partite sets of cardinalities m and n, respectively. A Y-tree on five vertices, denoted by Y_5, is the tree obtained from P_4 by attaching a pendant vertex to a vertex of degree 2. A *decomposition* of a graph G is a partition of its edge set into subgraphs $H_1, H_2, \ldots, H_r$, where $r \in \mathbb{N}$ and $H_i \subseteq G$ for $1 \leq i \leq r$. If $H_i \cong H$ for all $1 \leq i \leq r$, then G is said to be *H-decomposable.* If G can be decomposed into $\alpha_i \geq 0$ copies of H_i for $1 \leq i \leq k$, then we say that G admits an $(H_1, H_2, \ldots, H_k)$-multi-decomposition, or a $\{H_1^{\alpha_1}, H_2^{\alpha_2}, \ldots, H_k^{\alpha_k}\}$-decomposition. A k-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_k)$ is said to be *admissible* if it satisfies the necessary condition for the existence of a $\{H_1^{\alpha_1}, H_2^{\alpha_2}, \ldots, H_k^{\alpha_k}\}$-

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decomposition of G, namely

$$\sum_{i=1}^{k} \alpha_i |E(H_i)| = |E(G)|.$$

This condition is simply the edge-count, equivalently edge-divisibility, requirement, ensuring that the total number of edges contributed by the α_i copies of each H_i, for $i = 1, 2, \dots, k$, equals $|E(G)|$.

Graph decomposition is a well-established topic in graph theory. The concept of multi-decomposition was introduced by Abuieda and Devan [1]. Among the various structures used in decomposition, trees are particularly useful because of their simplicity and hierarchical nature [9, 14]. The decomposition of graphs into trees has a long history. Huang and Rosa [6] studied decompositions of complete graphs into various trees, including P_5 and Y_5. Cain [2] investigated decompositions of complete graphs into stars. Yamamoto et al. [19] established necessary and sufficient conditions for the decomposition of $K_{m,n}$ into stars, while Parker [12] obtained necessary and sufficient conditions for the decomposition of $K_{m,n}$ into paths. Joseph and Issacraj [13] established necessary and sufficient conditions for the decomposition of $K_{m,n}$ into Y_5-trees, where the Y_5-tree is referred to as a fork. Shyu [17] obtained several results on $\{P_k^\alpha, S_k^\beta\}$-decomposition of $K_{m,n}$ and K_n. In addition, Shyu [18] determined necessary and sufficient conditions for the $\{P_5^\alpha, S_5^\beta, C_5^\gamma\}$-decomposition of $K_{m,n}$ and K_n. More recently, Ilayaraja and Muthusamy [7] established necessary and sufficient conditions for the existence of a $\{P_4^\alpha, S_5^\beta\}$-decomposition of K_n.

Chaadhanaa and Hemalatha [3] established necessary and sufficient conditions for the existence of a $\{P_5^\alpha, Y_5^\beta\}$-decomposition of $K_{m,n}$ and K_n. Lin and Jou [11] studied $\{C_k^\alpha, P_k^\beta, S_k^\gamma\}$-decomposition of balanced complete bipartite multigraphs. Sethuraman and Murugan [16] proposed the conjecture that K_{4m+1}, $m \geq 1$, admits a $\{H_1^\alpha, H_2^\beta\}$-decomposition, where H_1 and H_2 are arbitrary trees of size m. Lee and Chen [10] considered multi-decomposition of complete graphs into Hamiltonian paths and stars with three edges. Saranya et al. [15] studied $\{S_{n-4}^\alpha, P_5^\beta\}$-decomposition, for $n \geq 5$, and $\{S_{n-2}^\alpha, C_5^\beta\}$-decomposition, for $n \geq 4$, of n-dimensional hypercube graphs. Multi-decompositions of product graphs into trees were studied in [4, 5, 8]. In this paper, we focus on trees with four edges. There are three non-isomorphic trees of this size: the path P_5, the star S_5, and the Y-tree Y_5. We determine necessary and sufficient conditions for the $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{m,n}$ and K_n.

Notation

- $G = (\alpha_1, \alpha_2, \dots, \alpha_k)$ indicates that G has a $\{H_1^{\alpha_1}, H_2^{\alpha_2}, \dots, H_k^{\alpha_k}\}$-decomposition.
- rG denotes r disjoint copies of the graph G.
- We write $G = H_1 \oplus H_2$ if G can be decomposed into H_1 and H_2.
- The vertices of K_n are denoted by v_i, $1 \leq i \leq n$.
- The vertex partite sets of $K_{m,n}$ are denoted by $\{v_{1i} : 1 \leq i \leq m\}$ and $\{v_{2j} : 1 \leq j \leq n\}$.

- The path P_5 with vertices v_i, $1 \leq i \leq 5$, having v_1 and v_5 as pendant vertices, is denoted by $(v_1, v_2, v_3, v_4, v_5)$.
- The star S_5 with vertices v_i, $1 \leq i \leq 5$, is denoted by $(v_1; v_2, v_3, v_4, v_5)$, where v_1 is the center of S_5.
- The tree Y_5 with vertices v_i, $1 \leq i \leq 5$, is denoted by $(v_1, v_2, \underline{v_3}, v_4; \underline{v_5})$, where v_i, $1 \leq i \leq 4$, form a path of length three and the underlined vertices indicate the edge v_3v_5. The vertex v_3, which has degree three, is called the center of Y_5.
- Let $\mathbb{N}_0$ denote the set of all non-negative integers. Then we define

$$\mathbb{N}_0^k = \{(a_1, a_2, \dots, a_k) \mid a_i \in \mathbb{N}_0,\ 1 \leq i \leq k\}.$$

Remark 1.1. Since each tree of order five contains four edges, the necessary condition given in the definition of admissible tuples becomes

$$4\sum_{i=1}^{k} \alpha_i = mn,$$

where $1 \leq k \leq 3$. Hence $mn \equiv 0 \pmod 4$. Since $K_{m,n} \cong K_{n,m}$, any decomposition result established for $K_{m,n}$ also holds for $K_{n,m}$. Therefore, without loss of generality, in our treatment of complete bipartite graphs, we consider only the following arithmetic possibilities:
(i) $m \equiv 0 \pmod 4$ and $n \geq 2$;
(ii) $m \equiv 2 \pmod 4$ and $n \equiv 2 \pmod 4$, with $(m, n) \neq (2, 2)$ since $K_{2,2} \cong C_4$.

2. $\{S_5^\alpha, Y_5^\beta\}$-decomposition of $K_{m,n}$

In this section, we obtain necessary and sufficient conditions for the $\{S_5^\alpha, Y_5^\beta\}$-decomposition of $K_{m,n}$, where $m \geq 4$ and $n \geq 2$. We first recall some results required for the subsequent discussion.

Theorem 2.1 ([19]). *Let k, m, and n be positive integers such that $m \leq n$. There exists an S_{k+1}-decomposition of $K_{m,n}$ if and only if one of the following conditions holds:*
(i) *$k \leq m$ and $mn \equiv 0 \pmod k$;*
(ii) *$m < k \leq n$ and $n \equiv 0 \pmod k$.*

Theorem 2.2 ([13]). *The complete bipartite graph $K_{m,n}$ is Y_5-decomposable if and only if*

$$mn \equiv 0 \pmod 4,$$

except for $K_{2,4i+2}$, where $i = 1, 2, \dots$.

If $X_1, \dots, X_n$ are sets of ordered pairs of non-negative integers, then

$$X_1 + \cdots + X_n = \{(p_1 + \cdots + p_n,\ q_1 + \cdots + q_n) : (p_i, q_i) \in X_i,\ 1 \leq i \leq n\}.$$

Lemma 2.3. *Suppose that $n \geq 2$ and $m \geq 3$. Let*

$$\{(p,q) \in \mathbb{N}_0^2 : p+q=n,\ q \neq 1\} \subseteq X_1$$

and

$$\{(a,b) \in \mathbb{N}_0^2 : a+b=m,\ b \neq 1\} \subseteq X_2.$$

Define

$$G_1 := \{(s,t) \in \mathbb{N}_0^2 : s+t=n+m,\ t \neq 1\}.$$

Then $G_1 \subseteq X_1 + X_2$.

Proof. It is easy to see that $(s,t) = (m+n-1, 1) \notin X_1 + X_2$. To prove the claim, it is enough to show that

$$\forall (s,t) \in G_1,\ \exists\, p_1, q_1, a_1, b_1 \in \mathbb{N}_0 : \begin{cases} (s,t) = (p_1 + a_1,\ q_1 + b_1), \\ p_1 + q_1 = n, \quad a_1 + b_1 = m, \\ q_1 \neq 1 \quad \text{and} \quad b_1 \neq 1. \end{cases} \tag{1}$$

Note that $p_1, q_1 \leq n$ and $a_1, b_1 \leq m$. The elements p_1, q_1, a_1, b_1 satisfying the conditions given in (1) can be obtained in the following two cases.

Case 1. $t = m+1$.

Then choose

$$(p_1, q_1) = (n-2, 2) \quad \text{and} \quad (a_1, b_1) = (1, m-1).$$

Case 2. $t \neq m+1$.

In this case, let

$$b_1 = \min(t,m), \qquad a_1 = m - b_1, \qquad q_1 = t - b_1, \qquad p_1 = n - q_1.$$

It is clear that $0 \leq a_1, b_1 \leq m$, $a_1 + b_1 = m$, $p_1 \leq n$, $q_1 \geq 0$, and $p_1 + q_1 = n$.

Subclaim 1. $q_1 \leq n$.

If $b_1 = \min(t,m) = t$, then $q_1 = t - b_1 = 0 \leq n$. If $b_1 = m$, then $q_1 \leq n$ because

$$s + t = n + m \implies t \leq m + n \implies t - m \leq n.$$

Subclaim 2. $p_1 \geq 0$.

Since $q_1 \leq n$ by Subclaim 1, it follows that $p_1 \geq 0$. □

Corollary 2.4. *Suppose that $n, m \geq 2$,*

$$\{(p,q) \in \mathbb{N}_0^2 : p+q=n,\ q \neq 1\} \subseteq X_3$$

and

$$\{(a,b) \in \mathbb{N}_0^2 : a+b=m\} \subseteq X_4.$$

Define

$$G_2 := \{(s,t) \in \mathbb{N}_0^2 : s+t=n+m\}.$$

Then $G_2 \subseteq X_3 + X_4$.

Proof. The elements p_1, q_1, a_1, b_1 constructed in Lemma 2.3 remain valid for any $(s,t) \in G_2$ when $m, n \geq 2$, since no restriction on b is required in this case. Hence, $G_2 \subseteq X_3 + X_4$. □

Lemma 2.5. *There is no $\{S_5^{n-1}, Y_5^1\}$-decomposition of $K_{4,n}$, $n \geq 3$, whenever $\alpha + \beta = n$.*

Proof. Suppose that there exists a $\{S_5^{n-1}, Y_5^1\}$-decomposition of $G := K_{4,n}$. Let $G^1 = K_{4,n} - Y_5$. The center of Y_5 is either v_{2j}, $1 \leq j \leq n$, or v_{1i}, $1 \leq i \leq 4$.

Case 1. Y_5 has center at v_{2j}.

Then the degree sequence of the vertices v_{1i} in G^1 is $\{n, n-1, n-1, n-2\}$. At least two of n, $n-1$, and $n-2$ are not divisible by 4. Let v_{1k_1}, $1 \leq k_1 \leq 4$, be a vertex whose degree is not a multiple of 4. Then at least one edge, say e_1, incident with v_{1k_1} cannot belong to an S_5 with center v_{1k_1}. Hence e_1 must lie in an S_5 whose center is v_{2j_1}, $1 \leq j_1 \leq n$. Note that v_{2j_1} cannot be a vertex of Y_5. Let S_5^1 be the star containing e_1. Then the degree sequence of the vertices v_{1i} in $G^2 := G^1 - S_5^1$ is $\{n-1, n-2, n-2, n-3\}$. Again, at least two of $n-1$, $n-2$, and $n-3$ are not divisible by 4. Let v_{1k_2}, $1 \leq k_2 \leq 4$, be a vertex whose degree is not a multiple of 4. Then at least one edge, say e_2, incident with v_{1k_2} cannot belong to an S_5 with center v_{1k_2}. Hence e_2 must lie in an S_5 whose center is v_{2j_2}, $1 \leq j_2 \leq n$, where $v_{2j_2} \neq v_{2j_1}$ and v_{2j_2} is not a vertex of Y_5. Let S_5^2 be the star containing e_2. Then the degree sequence of the vertices v_{1i} in $G^3 := G^2 - S_5^2$ is $\{n-2, n-3, n-3, n-4\}$. Continuing this process, we obtain $G^{n-1} := G^{n-2} - S_5^{n-2}$ with degree sequence $\{2, 1, 1, 0\}$ for the vertices v_{1i}. This degree sequence cannot yield an S_5, which is a contradiction.

Case 2. Y_5 has center at v_{1i}.

The proof follows by the same argument as in Case 1. In this case, the degree sequence of v_{1i} in G is $\{n, n, n-1, n-3\}$, and the degree sequence of v_{1i} in G^{n-2} is $\{3, 3, 2, 0\}$. Clearly, G^{n-2} cannot be decomposed into $2S_5$, which gives a contradiction. Hence, $K_{4,n}$ does not have a $\{S_5^{n-1}, Y_5^1\}$-decomposition whenever $\alpha + \beta = n$. □

Lemma 2.6. *$K_{4,n}$, $n \geq 3$, admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = n$, except when $(\alpha, \beta) = (n-1, 1)$.*

Proof. If $\alpha = 0$ or $\beta = 0$, then Theorems 2.1 and 2.2 give the required decompositions. For $1 \leq \alpha \leq n-2$, we write

$$K_{4,n} = \alpha S_5 \oplus K_{4,n-\alpha},$$

where the center of each S_5 is v_{2j}, $1 \leq j \leq n$. By Theorem 2.2, $K_{4,n-\alpha} = (n-\alpha)Y_5$. Therefore, $K_{4,n}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = n$ and $\beta \neq 1$. □

Remark 2.7. It is easy to see that $K_{4,2}$ does not have a $\{S_5^1, Y_5^1\}$-decomposition whenever $\alpha + \beta = 2$. If $\alpha = 0$ or $\beta = 0$, then Theorems 2.1 and 2.2 give the required decompositions.

Lemma 2.8. *There is no $\{S_5^{3p-1}, Y_5^1\}$-decomposition of $K_{4p,3}$ whenever $\alpha + \beta = 3p$.*

Proof. Since the vertices v_{1i}, $1 \leq i \leq 4p$, have degree 3 in $K_{4p,3}$, none of them can be the center of an S_5. Hence, the center of every S_5 must be a vertex v_{2j}, $1 \leq j \leq 3$. Each vertex v_{2j} can be the center of at most p stars. To obtain a decomposition of $K_{4p,3}$ into $(3p-1)S_5$, two of the vertices v_{2j} would have to be centers of p stars each, using all their incident edges, while the remaining vertex v_{2j} would be the center of only $p-1$ stars. Consequently, no edges would remain to form the required Y_5. Therefore, $K_{4p,3}$ admits no $\{S_5^{3p-1}, Y_5^1\}$-decomposition whenever $\alpha + \beta = 3p$. □

Lemma 2.9. *$K_{4p,3}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = 3p$, except when $(\alpha, \beta) = (3p-1, 1)$.*

Proof. When $n = 3$, we write

$$K_{4p,3} = pK_{4,3}.$$

By Lemma 2.6, $K_{4,3}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = 3$, except when $\beta = 1$. Hence, by Lemma 2.3, $K_{4p,3}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha+\beta = 3p$ and $\beta \neq 1$. □

Lemma 2.10. *$K_{8,i}$, $i \in \{5, 6\}$, admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = 2i$.*

Proof. We write

$$K_{8,i} = 2K_{4,i}.$$

By Lemma 2.6, $K_{4,i}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = i$, except when $\beta = 1$. Hence, by Lemma 2.3, $K_{8,i}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha+\beta = 2i$, except when $\beta = 1$. For $\beta = 1$, when $i = 5$, the edge set for the admissible pair $(9, 1)$ is given by

$$\begin{array}{ll}
(v_{25}; v_{11}, v_{13}, v_{17}, v_{18}), & (v_{24}; v_{11}, v_{14}, v_{15}, v_{16}), \\
(v_{12}; v_{22}, v_{23}, v_{24}, v_{25}), & (v_{13}; v_{21}, v_{22}, v_{23}, v_{24}), \\
(v_{14}; v_{21}, v_{22}, v_{23}, v_{25}), & (v_{15}; v_{21}, v_{22}, v_{23}, v_{25}), \\
(v_{16}; v_{21}, v_{22}, v_{23}, v_{25}), & (v_{17}; v_{21}, v_{22}, v_{23}, v_{24}), \\
(v_{18}; v_{21}, v_{22}, v_{23}, v_{24}), & (v_{12}, v_{21}, \underline{v_{11}}, v_{23}; \underline{v_{22}}).
\end{array}$$

When $i = 6$, we write

$$K_{8,6} = 2S_5 \oplus K_{8,5},$$

which gives $(11, 1) = (2, 0)+(9, 1)$. Thus, $K_{8,i}$, $i \in \{5, 6\}$, admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = 2i$. □

Lemma 2.11. *$K_{4p,n}$, $p \geq 2$, $n \geq 5$, admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = pn$.*

Proof. We write

$$K_{4p,n} = K_{8,i} \oplus 2(K_{4,n-i}) \oplus (p-2)K_{4,n},$$

where $i = 5$ if n is odd and $i = 6$ if n is even. By Lemma 2.10, $K_{8,i}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = 2i$. Further, by Lemma 2.6 and Remark 2.7, both $K_{4,n-i}$ and $K_{4,n}$ admit a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha+\beta = n-i$ and $\alpha+\beta = n$, respectively. Hence, by Lemma 2.3 and Corollary 2.4, $K_{4p,n}$, $p \geq 2$, $n \geq 5$, admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = pn$. □

Lemma 2.12. *There is no $\{S_5^\alpha, Y_5^\beta\}$-decomposition of $K_{2p,2}$, $p > 1$, whenever $\alpha + \beta = p$ and one of the following holds:*
(i) *β is odd;*
(ii) *$\beta = 0$ and p is odd.*

Proof. If $\alpha = 0$, or if $\beta = 0$ and p is odd, then Theorems 2.2 and 2.1, respectively, imply that no such decompositions exist. Now let $\beta \in \{1, \ldots, p-1\}$ be odd. Suppose that $K_{2p,2}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition. Then the vertices v_{2j}, $j \in \{1, 2\}$, must serve as centers of the trees in the decomposition. After removing β copies of Y_5, each vertex v_{2j} has odd degree in $K_{2p,2} - \beta Y_5$. Since v_{2j} must be the center of every S_5, it follows that $K_{2p,2} - \beta Y_5$ cannot be decomposed into $(p-\beta)S_5$, which is a contradiction. □

Remark 2.13. $K_{6,2}$ admits an (S_5^1, Y_5^2)-decomposition whenever $\alpha+\beta = 3$ and does not have an (S_5^α, Y_5^β)-decomposition for other possible admissible pairs by Lemma 2.12. For $(1, 2)$, the edge sets are given by

$$(v_{21}; v_{11}, v_{12}, v_{13}, v_{14}),$$

$$(v_{21}, v_{15}, \underline{v_{22}}, v_{14}; \underline{v_{13}})$$

and

$$(v_{21}, v_{16}, \underline{v_{22}}, v_{11}; \underline{v_{12}}).$$

Lemma 2.14. *$K_{2p,2}$, $p > 1$, admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = p$, β is even, and $\beta \neq 0$ if p is odd.*

Proof. When p is odd, let $p = 2p_1 + 1$ for some $p_1 \in \mathbb{N}$. We write

$$K_{2p,2} = (p_1 - 1)K_{4,2} \oplus K_{6,2}.$$

Take $K_{6,2} = (1, 2)$. Any even value of β such that $2 \leq \beta \leq p - 1$ can be obtained by choosing appropriate admissible pairs between $(0, 2)$ and $(2, 0)$ of $K_{4,2}$, which exist by Remark 2.7.

When p is even, let $p = 2p_1$ for $p_1 \in \mathbb{N}$. We write

$$K_{2p,2} = p_1 K_{4,2}.$$

The proof follows by an argument similar to that for p odd. □

Lemma 2.15. *$K_{6,6}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = 9$.*

Proof. If $\alpha = 0$ or $\beta = 0$, then Theorems 2.2 and 2.1, respectively, give the required decompositions. We write

$$K_{6,6} = K_{6,4} \oplus K_{6,2}.$$

Since $K_{6,4} \cong K_{4,6}$, by Lemma 2.6, $K_{6,4}$ has a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition for the admissible pairs

$$(\alpha, \beta) \in \{(0,6), (1,5), (2,4), (3,3), (4,2), (6,0)\}.$$

Take $K_{6,2} = (1, 2)$. Then, for each admissible pair (α, β) of $K_{6,4}$, we obtain a corresponding admissible pair for $K_{6,6}$ by addition, namely $(\alpha + 1, \beta + 2)$. This yields

$$(\alpha, \beta) \in \{(1,8), (2,7), (3,6), (4,5), (5,4), (7,2)\}.$$

For the remaining admissible pairs, we provide explicit constructions. For $(\alpha, \beta) = (6, 3)$, the edge sets are given by

$$(v_{22}; v_{11}, v_{12}, v_{13}, v_{14}), \quad (v_{24}; v_{11}, v_{12}, v_{13}, v_{14}),$$

$$(v_{25}; v_{11}, v_{12}, v_{13}, v_{14}), \quad (v_{26}; v_{11}, v_{12}, v_{13}, v_{14}),$$

$$(v_{16}; v_{22}, v_{23}, v_{24}, v_{25}), \quad (v_{21}; v_{11}, v_{12}, v_{14}, v_{16}),$$

$$(v_{21}, v_{13}, \underline{v_{23}}, v_{12}; \underline{v_{11}}), \quad (v_{14}, v_{23}, \underline{v_{15}}, v_{22}; \underline{v_{21}}),$$

and

$$(v_{16}, v_{26}, \underline{v_{15}}, v_{24}; \underline{v_{25}}).$$

For $(\alpha, \beta) = (8, 1)$, the edge sets are given by

$$(v_{21}; v_{11}, v_{12}, v_{13}, v_{14}), \quad (v_{12}; v_{23}, v_{24}, v_{25}, v_{26}),$$

$$(v_{22}; v_{11}, v_{12}, v_{13}, v_{14}), \quad (v_{23}; v_{11}, v_{13}, v_{14}, v_{16}),$$

$$(v_{24}; v_{11}, v_{13}, v_{14}, v_{16}), \quad (v_{26}; v_{11}, v_{13}, v_{14}, v_{15}),$$

$$(v_{25}; v_{11}, v_{13}, v_{14}, v_{16}), \quad (v_{15}; v_{22}, v_{23}, v_{24}, v_{25}),$$

and

$$(v_{15}, v_{21}, \underline{v_{16}}, v_{26}; \underline{v_{22}}).$$

Thus, $K_{6,6}$ admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = 9$. □

Lemma 2.16. *$K_{2p,2q}$, where $p, q > 1$ are odd, admits a $\{S_5^{\alpha}, Y_5^{\beta}\}$-decomposition whenever $\alpha + \beta = pq$.*

Proof. Let $p = 2p_1 + 1$ and $q = 2q_1 + 1$ for some $p_1, q_1 \in \mathbb{N}$. We write

$$K_{2p,2q} = (p_1 - 1)(q_1 - 1)K_{4,4} \oplus (p_1 + q_1 - 2)K_{6,4} \oplus K_{6,6}.$$

Since $K_{6,4} \cong K_{4,6}$, by Lemma 2.6, $K_{6,4}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha+\beta = 6$, except when $\beta = 1$. Similarly, by Lemma 2.6, $K_{4,4}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha+\beta = 4$, except when $\beta = 1$. Further, by Lemma 2.15, $K_{6,6}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = 9$. Hence, by Lemma 2.3 and Corollary 2.4, $K_{2p,2q}$, for $p, q > 1$ odd, admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition whenever $\alpha + \beta = pq$. □

Theorem 2.17. *For $m \geq 4$ and $n \geq 2$, $K_{m,n}$ admits a $\{S_5^\alpha, Y_5^\beta\}$-decomposition if and only if $\alpha + \beta = \frac{mn}{4}$, $mn \equiv 0 \pmod 4$, and one of the following holds. Let $p, q \in \mathbb{N}$.*

1. *$m = 4$, $n \geq 4$, and $\beta \neq 1$;*
2. *$m = 4p$, $n = 3$, and $\beta \neq 1$;*
3. *$m = 4p$, $p \geq 2$, and $n \geq 5$;*
4. *$m = 2p$, $p > 1$, $n = 2$, and*
   (i) *β is even;*
   (ii) *$\beta \neq 0$ if p is odd;*
5. *$m = 2p$, $n = 2q$, and $p, q > 1$ are odd.*

Proof. Assume that $K_{m,n}$ has an (S_5^α, Y_5^β)-decomposition. Then, from the edge-divisibility condition for the existence of an (S_5^α, Y_5^β)-decomposition of $K_{m,n}$, we have

$$\alpha + \beta = \frac{mn}{4}.$$

Thus, any admissible pair must satisfy this arithmetic condition, and in particular $mn \equiv 0 \pmod 4$. Moreover, Lemmas 2.5, 2.8, and 2.12 restrict the admissible pairs to the five cases listed above. Hence, the stated conditions are necessary.

Conversely, let (α, β) be an admissible pair satisfying one of the above cases. Then the required decompositions of $K_{m,n}$ follow from Lemmas 2.9, 2.6, 2.11, 2.14, and 2.16, respectively, corresponding to Cases *(1)–(5)*. Hence, the sufficiency follows. □

3. $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{m,n}$

In this section, we obtain necessary and sufficient conditions for the $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{m,n}$, where $m \geq 4$ and $n \geq 2$.

In the work of T. W. Shyu [17], Theorems 2.5–2.8 were established to address the $\{P_k^\alpha, S_k^\beta\}$-decomposition of $K_{m,n}$ for $k \geq 4$. Since, in the present study, we focus exclusively on the case $k = 5$, we restate and combine the relevant results into a single theorem accordingly, which is given in Theorem 3.1.

Theorem 3.1 ([17])**.** *For $m \geq 4$ and $n \geq 2$, $K_{m,n}$ admits a $\{P_5^\alpha, S_5^\beta\}$-decomposition if one of the following holds. Let $p, q \in \mathbb{N}$.*

1. *$m = 4$, $n \geq 3$, and $\alpha \neq 1$;*
2. *$m = 4p$, $n = 3, 4$, and $\alpha \neq 1$;*
3. *$m = 4p$, $p \geq 2$, and $n \geq 5$;*
4. *$m = 2p$, p is even, $n = 2$, and β is even;*
5. *$m = 2p$, $n = 2q$, $p \geq 3$, and $q \geq 7$ are odd.*

Lemma 3.2 ([18])**.** *For $\alpha, \beta > 0$, $K_{2p,2}$, $p \in \mathbb{N}$, admits a $\{P_5^\alpha, S_5^\beta\}$-decomposition if and only if $p \geq 2$ whenever $\alpha + \beta = p$ and β is even.*

Lemma 3.3 ([8])**.** *$K_{6,6}$ admits a $\{P_5^\alpha, S_5^\beta\}$-decomposition whenever $\alpha + \beta = 9$.*

Remark 3.4. We write

$$K_{6,10} = K_{6,6} \oplus K_{6,4} \quad \text{and} \quad K_{10,10} = K_{6,10} \oplus K_{4,10}.$$

By Lemma 3.3, $K_{6,6}$ admits a $\{P_5^\alpha, S_5^\beta\}$-decomposition whenever $\alpha + \beta = 9$. By Theorem 3.1, $K_{6,4}$ and $K_{4,10}$ admit a $\{P_5^\alpha, S_5^\beta\}$-decomposition whenever $\alpha + \beta = 6$ and $\alpha + \beta = 10$, respectively, except when $\alpha = 1$. Hence, by Corollary 2.4, we have a $\{P_5^\alpha, S_5^\beta\}$-decomposition of $K_{6,10}$ and $K_{10,10}$ whenever $\alpha + \beta = 15$ and $\alpha + \beta = 25$, respectively.

Observation 3.5. *Theorem 3.1, Lemmas 3.2 and 3.3, together with Remark 3.4, give necessary and sufficient conditions for the $\{P_5^\alpha, S_5^\beta\}$-decomposition of $K_{m,n}$, where $m \geq 4$ and $n \geq 2$.*

Theorem 3.6 ([3])**.** *There exists a $\{P_5^\alpha, Y_5^\beta\}$-decomposition of $K_{m,n}$ if and only if one of the following holds. Let $p, q \in \mathbb{N}$.*

1. *$m = 2p$, p is even, $n = 2$, and α is even;*
2. *$m = 2p$, $p \geq 3$ is odd, $n = 2$, and α is odd;*
3. *$m = 4p$ and $n \geq 3$;*
4. *$m = 2p$ and $n = 2q$, where $p, q \geq 3$ are odd.*

Remark 3.7. For any admissible triplet (α, β, γ) with at most two of the three entries in the triplet equal to 0, the proof follows directly from the results on $\{H_1^\alpha, H_2^\beta\}$-decomposition, where $\{H_1, H_2\} \in \{P_5, S_5, Y_5\}$. These results are given in Observation 3.5, Theorem 3.6, and Theorem 2.17.

If $Y_1, \ldots, Y_n$ are sets of ordered triplets of non-negative integers, then

$$Y_1 + \cdots + Y_n = \{(a_1 + \cdots + a_n,\, b_1 + \cdots + b_n,\, c_1 + \cdots + c_n) : (a_i, b_i, c_i) \in Y_i \text{ for } 1 \leq i \leq n\}.$$

Lemma 3.8. *Suppose that $n, m \geq 3$,*

$$\{(p, q, r) \in \mathbb{N}_0^3 : p + q + r = n,\ (p, r) \notin \{(1,0), (1,1), (0,1)\}\} \subseteq Y_1$$

and

$$\{(a, b, c) \in \mathbb{N}_0^3 : a + b + c = m,\ (a, c) \notin \{(1,0), (1,1), (0,1)\}\} \subseteq Y_2.$$

Define

$$G_3 := \{(s,t,u) \in \mathbb{N}_0^3 : s+t+u = n+m,\ (s,u) \notin \{(1,0),(1,1),(0,1)\}\}.$$

Then $G_3 \subseteq Y_1 + Y_2$.

Proof. It is easy to see that $(s,t,u) \notin Y_1 + Y_2$ whenever

$$(s,u) \in \{(1,0),(1,1),(0,1)\}.$$

To prove the claim, it suffices to show that

$$\forall (s,t,u) \in G_3,\ \exists\, p_1,q_1,r_1,a_1,b_1,c_1 \in \mathbb{N}_0 : \begin{cases} (s,t,u) = (p_1+a_1,\ q_1+b_1,\ r_1+c_1), \\ p_1+q_1+r_1 = n, \quad a_1+b_1+c_1 = m, \\ (p_1,r_1) \notin \{(1,0),(1,1),(0,1)\}, \\ (a_1,c_1) \notin \{(1,0),(1,1),(0,1)\}. \end{cases} \tag{2}$$

Note that $p_1, q_1, r_1 \le n$ and $a_1, b_1, c_1 \le m$. The elements $p_1, q_1, r_1, a_1, b_1, c_1$ satisfying the conditions given in (2), for $m, n \ge 3$, can be obtained in the following seven cases.

Case 1. $u = 0$, $s = n+1$, and $t = m-1$.
Choose

$$(p_1,q_1,r_1) = (n-1,1,0) \quad \text{and} \quad (a_1,b_1,c_1) = (2,m-2,0).$$

Case 2. $u = 1$, $s = n+1$, and $t = m-2$.
Choose

$$(p_1,q_1,r_1) = (n-1,0,1) \quad \text{and} \quad (a_1,b_1,c_1) = (2,m-2,0).$$

Case 3. $u = 1$ and $s \le n$.
Choose

$$(p_1,q_1,r_1) = (s,n-1-s,1) \quad \text{and} \quad (a_1,b_1,c_1) = (0,m,0).$$

Since $u = 1$, we have $s \ne 0, 1$.

Case 4. $s = 0$, $u = m+1$, and $t = n-1$.
Choose

$$(p_1,q_1,r_1) = (0,n-2,2) \quad \text{and} \quad (a_1,b_1,c_1) = (0,1,m-1).$$

Case 5. $s = 1$, $u = m+1$, and $t = n-2$.
Choose

$$(p_1,q_1,r_1) = (1,n-3,2) \quad \text{and} \quad (a_1,b_1,c_1) = (0,1,m-1).$$

Case 6. $s = 1$ and $u \le m$.
Choose

$$(p_1,q_1,r_1) = (0,n,0) \quad \text{and} \quad (a_1,b_1,c_1) = (1,m-1-u,u).$$

Case 7. Any (s,t,u) that does not fall under Cases 1–6.
Let

$$c_1 = \min(u,m), \quad p_1 = \min(s,n), \quad a_1 = s - p_1,$$
$$b_1 = m - (a_1 + c_1), \quad q_1 = t - b_1, \quad r_1 = u - c_1.$$

It is clear that $a_1 + b_1 + c_1 = m$, $a_1, r_1 \geq 0$, $b_1 \leq m$, $0 \leq c_1 \leq m$, and $0 \leq p_1 \leq n$.

Subclaim 1. $a_1 \leq m$.

If $p_1 = \min(s, n) = s$, then $a_1 = s - p_1 = 0 \leq m$. If $p_1 = n$, then $a_1 \leq m$ because

$$s + t + u = m + n \implies s \leq m + n \iff s - n \leq m.$$

Subclaim 2. $b_1 \geq 0$.

It is enough to show that

$$m - s \geq \min(u, m) - \min(s, n). \tag{3}$$

If $\min(u, m) = m$, then $\min(s, n) = s$ because

$$s + t + u = n + m \implies s + t \leq n \implies s \leq n.$$

The inequality becomes equality in this case. When $\min(u, m) = u$ and $\min(s, n) = n$, since

$$s + t + u = n + m \implies s + u \leq n + m \iff u - n \leq m - s,$$

(3) holds. Also, for $\min(u, m) = u$ and $\min(s, n) = s$,

$$m \geq u \iff m - s \geq u - s.$$

Therefore, (3) holds.

Subclaim 3. $0 \leq q_1 \leq n$.

To prove $q_1 \geq 0$, it is enough to show that

$$t + s - m \geq \min(s, n) - \min(u, m). \tag{4}$$

As in Subclaim 2, if $\min(u, m) = m$, then $\min(s, n) = s$. Therefore,

$$t \geq 0 \iff t + s - m \geq s - m,$$

and (4) holds. When $\min(u, m) = u$ and $\min(s, n) = n$, the inequality becomes equality because $s + t + u = n + m$. Also, for $\min(u, m) = u$ and $\min(s, n) = s$, since $n \geq s$, we have

$$m + n \geq m + s \implies s + t + u \geq m + s,$$

because $s + t + u = n + m$. Therefore, (4) holds.

Now, to prove $q_1 \leq n$, we use

$$s + t + u = n + m \implies t - m + s + u = n$$

which implies

$$t - m + s + \min(u, m) \leq n$$

and hence

$$t - m + s + \min(u, m) - \min(s, n) \leq n.$$

Subclaim 4. $r_1 \leq n$.

If $c_1 = \min(u, m) = u$, then $r_1 = u - c_1 = 0 \le n$. If $c_1 = m$, then $r_1 \le n$ because

$$s + t + u = m + n \implies u \le m + n \iff u - m \le n.$$

Subclaim 5. $p_1 + q_1 + r_1 = n$.
Indeed,

$$p_1 + q_1 + r_1 = p_1 + t - b_1 + u - c_1 = p_1 + t - m + a_1 + c_1 + u - c_1$$

$$= p_1 + t - m + s - p_1 + u = t - m + s + u = n.$$

□

Corollary 3.9. *Suppose that* $n, m \ge 3$,

$$\{(p, q, r) \in \mathbb{N}_0^3 : p + q + r = n,\ (p, r) \notin \{(1, 0), (1, 1), (0, 1)\}\} \subseteq Y_3$$

and

$$\{(a, b, c) \in \mathbb{N}_0^3 : a + b + c = m,\ (a, c) \notin \{(1, 0), (0, 1)\}\} \subseteq Y_4.$$

Define

$$G_4 := \{(s, t, u) \in \mathbb{N}_0^3 : s + t + u = n + m,\ (s, u) \notin \{(1, 0), (0, 1)\}\}.$$

Then $G_4 \subseteq Y_3 + Y_4$.

Proof. It is easy to see that if $(s, u) \in \{(1, 0), (0, 1)\}$, then $(s, t, u) \notin Y_3 + Y_4$. When

$$(s, t, u) = (1, m + n - 2, 1),$$

take

$$(p_1, q_1, r_1) = (0, n, 0) \quad \text{and} \quad (a_1, b_1, c_1) = (1, m - 2, 1).$$

For the remaining $(s, t, u) \in G_4$, the elements $p_1, q_1, r_1, a_1, b_1, c_1$ given in Lemma 3.8 suffice. □

Corollary 3.10. *Suppose that*

$$\{(p, q, r) \in \mathbb{N}_0^3 : p + q + r = n\} \subseteq Y_5$$

and

$$\{(a, b, c) \in \mathbb{N}_0^3 : a + b + c = m\} \subseteq Y_6.$$

Then

$$\{(s, t, u) \in \mathbb{N}_0^3 : s + t + u = n + m\} \subseteq Y_5 + Y_6.$$

Proof. For any (s, t, u) satisfying $s + t + u = n + m$, the elements $p_1, q_1, r_1, a_1, b_1, c_1$ constructed in Case 7 of Lemma 3.8 suffice. □

Lemma 3.11. *There is no $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition of $K_{2p,2}$, $p > 1$, whenever $\alpha + \beta + \gamma = p$, $\alpha, \beta, \gamma > 0$, and γ is odd.*

Proof. The vertices v_{2j}, $j \in \{1, 2\}$, must serve as centers of the Y_5 in a $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition of $K_{2p,2}$. If γ is odd, then each v_{2j} has odd degree in $K_{2p,2} - \gamma Y_5$. However, to decompose $K_{2p,2} - \gamma Y_5$ into αP_5 and βS_5, where $\alpha + \beta + \gamma = p$, each v_{2j} must have even degree, since it must be the center of an S_5 and cannot serve as a pendant vertex of a P_5. This is a contradiction, and hence the result follows. □

Remark 3.12.

- There is no admissible triplet (α, β, γ) in $K_{4,2}$ with $\alpha, \beta, \gamma > 0$.
- The triplet $(1, 1, 1)$ is the only admissible triplet in $K_{6,2}$ with $\alpha, \beta, \gamma > 0$, and $K_{6,2}$ does not admit a $\{P_5^1, S_5^1, Y_5^1\}$-decomposition by Lemma 3.11.
- In $K_{8,2}$, the edge set for the admissible triplet $(1, 1, 2)$ is given by

$$(v_{18}, v_{21}, v_{17}, v_{22}, v_{12}),$$

$$(v_{21}; v_{11}, v_{12}, v_{13}, v_{14}),$$

$$(v_{21}, v_{15}, \underline{v_{22}}, v_{14}; \underline{v_{13}})$$

and

$$(v_{21}, v_{16}, \underline{v_{22}}, v_{18}; \underline{v_{11}}).$$

For all other admissible triplets with $\alpha, \beta, \gamma > 0$, $K_{8,2}$ does not admit a $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition by Lemma 3.11.

Lemma 3.13. *$K_{2p,2}$, $p > 1$, admits a $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition whenever $\alpha+\beta+\gamma = p$ and one of the following holds:*
(i) *$\gamma = 0$ and β is even;*
(ii) *$\beta = 0$ and γ is even;*
(iii) *$\alpha = 0$, γ is even, and $\gamma \neq 0$ if p is odd;*
(iv) *$\alpha, \beta, \gamma > 0$ and γ is even.*

Proof. For *(i)*, *(ii)*, and *(iii)*, Remark 3.7 gives the required proof.

(iv) Case 1. p is odd.

Let $p = 2p_1 + 1$ for some $p_1 \in \mathbb{N}$. Then

$$K_{2p,2} = K_{2(2p_1+1),2} = K_{6,2} \oplus (p_1 - 1)K_{4,2}.$$

Subcase 1. β is even.

In this case, take $K_{6,2} = (1, 0, 2)$. Also, in the subgraph $(p_1 - 1)K_{4,2}$, for $\frac{\gamma-2}{2}K_{4,2}$, take $K_{4,2} = (0, 0, 2)$; for $\frac{\beta}{2}K_{4,2}$, take $K_{4,2} = (0, 2, 0)$; and for $\frac{\alpha-1}{2}K_{4,2}$, take $K_{4,2} = (2, 0, 0)$, so that

$$\frac{\gamma - 2}{2} + \frac{\beta}{2} + \frac{\alpha - 1}{2} = p_1 - 1,$$

because $\alpha + \beta + \gamma = p = 2p_1 + 1$.

Subcase 2. β is odd.

Take $K_{6,2} = (0,1,2)$. In the subgraph $(p_1-1)K_{4,2}$, for $\frac{\gamma-2}{2}K_{4,2}$, take $K_{4,2} = (0,0,2)$; for $\frac{\beta-1}{2}K_{4,2}$, take $K_{4,2} = (0,2,0)$; and for $\frac{\alpha}{2}K_{4,2}$, take $K_{4,2} = (2,0,0)$, so that

$$\frac{\gamma-2}{2} + \frac{\beta-1}{2} + \frac{\alpha}{2} = p_1 - 1.$$

Case 2. p is even.

Let $p = 2p_1$ for some $p_1 \in \mathbb{N}$. Then

$$K_{2p,2} = K_{4p_1,2} = K_{8,2} \oplus (p_1-2)K_{4,2}$$

for $p_1 \geq 2$. When $p_1 = 1$, we have $K_{4p_1,2} = K_{4,2}$.

Subcase 1. β is even.

Take $K_{8,2} = (0,2,2)$. In the subgraph $(p_1-2)K_{4,2}$, for $\frac{\gamma-2}{2}K_{4,2}$, take $K_{4,2} = (0,0,2)$; for $\frac{\beta-2}{2}K_{4,2}$, take $K_{4,2} = (0,2,0)$; and for $\frac{\alpha}{2}K_{4,2}$, take $K_{4,2} = (2,0,0)$, so that

$$\frac{\gamma-2}{2} + \frac{\beta-2}{2} + \frac{\alpha}{2} = p_1 - 2.$$

Subcase 2. β is odd.

Take $K_{8,2} = (1,1,2)$. In the subgraph $(p_1-2)K_{4,2}$, for $\frac{\gamma-2}{2}K_{4,2}$, take $K_{4,2} = (0,0,2)$; for $\frac{\beta-1}{2}K_{4,2}$, take $K_{4,2} = (0,2,0)$; and for $\frac{\alpha-1}{2}K_{4,2}$, take $K_{4,2} = (2,0,0)$, so that

$$\frac{\gamma-2}{2} + \frac{\beta-1}{2} + \frac{\alpha-1}{2} = p_1 - 2.$$

□

Lemma 3.14. *There is no* $\{P_5^1, S_5^{3p-2}, Y_5^1\}$*-decomposition of* $K_{4p,3}$ *whenever* $\alpha+\beta+\gamma = 3p$.

Proof. The vertices v_{2j}, $1 \leq j \leq 3$, must serve as centers of the S_5 in a $\{P_5^1, S_5^{3p-2}, Y_5^1\}$-decomposition of $K_{4p,3}$. Each vertex v_{2j} can be the center of at most p stars. Hence, at least one of the vertices v_{2j}, say v_{21}, must be the center of p copies of S_5. Removing these p stars, we obtain

$$K_{4p,3} - pS_5 \cong K_{4p,2},$$

where each removed S_5 has center at v_{21}. Thus, the problem reduces to determining whether $K_{4p,2}$ admits a $\{P_5^1, S_5^{2p-2}, Y_5^1\}$-decomposition. By Lemma 3.11, no such decomposition exists, and hence the result follows. □

Remark 3.15. The only admissible triplet (α,β,γ) in $K_{4,3}$ with $\alpha,\beta,\gamma > 0$ is $(1,1,1)$, and $K_{4,3}$ does not admit a $\{P_5^1, S_5^1, Y_5^1\}$-decomposition by Lemma 3.14.

Lemma 3.16. $K_{4p,3}$ *admits a* $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$*-decomposition whenever* $\alpha+\beta+\gamma = 3p$ *and*

$$(\alpha,\gamma) \notin \{(0,1),(1,1),(1,0)\}.$$

Proof. We write

$$K_{4p,3} = pK_{4,3}.$$

By Remarks 3.7 and 3.15, $K_{4,3}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha+\beta+\gamma = 3$ and

$$(\alpha, \gamma) \notin \{(0,1), (1,1), (1,0)\}.$$

Hence, by Lemma 3.8, $K_{4p,3}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha+\beta+\gamma = 3p$ and

$$(\alpha, \gamma) \notin \{(0,1), (1,1), (1,0)\}.$$

□

Lemma 3.17. *$K_{4,n}$, $n \geq 4$, admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha+\beta+\gamma = n$,*

$$(\alpha, \gamma) \notin \{(0,1), (1,0)\},$$

and, in addition,

$$(\alpha, \beta, \gamma) \neq (1, 2, 1)$$

when $n = 4$.

Proof. If one or two of α, β, γ are zero, the result follows from Remark 3.7. Hence, assume that $\alpha, \beta, \gamma > 0$. We write

$$K_{4,i} = S_5 \oplus K_{4,i-1}.$$

This recursion starts from the base case $K_{4,3}$, which admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 3$ and

$$(\alpha, \gamma) \notin \{(0,1), (1,1), (1,0)\}$$

by Remarks 3.7 and 3.15. Given an admissible triplet (α, β, γ) in $K_{4,i}$, consider the triplet $(\alpha, \beta - 1, \gamma)$ in $K_{4,i-1}$. By this recursive construction, we obtain a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition for all $\alpha, \beta, \gamma > 0$, except when

$$(\alpha, \beta, \gamma) = (1, n-2, 1).$$

For $i = 4$, a $\{P_5^1, S_5^2, Y_5^1\}$-decomposition does not exist, since

$$K_{4,4} - 2S_5 \cong K_{4,2},$$

and $K_{4,2}$ cannot be decomposed into P_5 and Y_5 by Theorem 3.6. For $i = 5$, the edge set for the admissible triplet $(1, 3, 1)$ is given by

$$(v_{11}, v_{21}, v_{12}, v_{22}, v_{13}),$$

$$(v_{11}; v_{22}, v_{23}, v_{24}, v_{25}), \quad (v_{13}; v_{21}, v_{23}, v_{24}, v_{25}),$$

$$(v_{14}; v_{21}, v_{22}, v_{24}, v_{25})$$

and

$$(v_{14}, v_{23}, \underline{v_{12}}, v_{25}; \underline{v_{24}}).$$

For $n \geq 6$, we obtain

$$(1, n-2, 1) = (0, 1, 0) + (1, n-3, 1),$$

with the base case $(1, 3, 1)$ arising from $K_{4,5}$. □

Lemma 3.18. *$K_{4p,n}$, $p \geq 2$ and $n \geq 5$, admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = pn$.*

Proof. Let $n = 4q + r$, where $q \in \mathbb{N}$ and $r \in \{0, 1, 2, 3\}$.

Case 1. $r = 0$.

If $q = 1$, then $n = 4$. Hence, assume $q \geq 2$. We write

$$K_{4p,4q} = pK_{4,8} \oplus p(q-2)K_{4,4}.$$

Case 2. $r \in \{1, 2, 3\}$.

We write

$$K_{4p,4q+r} = p(q-1)K_{4,4} \oplus pK_{4,i},$$

where $i = 5$ if $r = 1$, $i = 6$ if $r = 2$, and $i = 7$ if $r = 3$.

By Lemma 3.17, $K_{4,4}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 4$ and

$$(\alpha, \gamma) \notin \{(0, 1), (1, 1), (1, 0)\}.$$

Further, for $i \in \{5, 6, 7, 8\}$, $K_{4,i}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha+\beta+\gamma = i$ and

$$(\alpha, \gamma) \notin \{(0, 1), (1, 0)\}.$$

Hence, by Corollary 3.9, we obtain a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{4p,n}$ whenever $\alpha + \beta + \gamma = pn$ and

$$(\alpha, \gamma) \notin \{(0, 1), (1, 0)\}.$$

Finally, for $(\alpha, \gamma) \in \{(0, 1), (1, 0)\}$, the required decomposition exists by Remark 3.7. □

Lemma 3.19. *$K_{6,6}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 9$.*

Proof. We write

$$K_{6,6} = K_{4,6} \oplus K_{2,6}.$$

Except for the admissible triplet $(1, 7, 1)$, this partition yields a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition for all admissible triplets, as listed in Table 1. For $(1, 7, 1)$, the edge set is given by

$$(v_{24}, v_{13}, v_{25}, v_{14}, v_{26}),$$

$$(v_{21}; v_{11}, v_{12}, v_{13}, v_{14}), \quad (v_{22}; v_{11}, v_{12}, v_{13}, v_{14}),$$

$$(v_{23}; v_{11}, v_{12}, v_{13}, v_{14}), \quad (v_{24}; v_{11}, v_{12}, v_{14}, v_{15}),$$
$$(v_{25}; v_{11}, v_{12}, v_{15}, v_{16}), \quad (v_{26}; v_{11}, v_{12}, v_{13}, v_{15}),$$
$$(v_{16}; v_{21}, v_{22}, v_{24}, v_{26})$$

and

$$(v_{16}, v_{23}, \underline{v_{15}}, v_{22}; \underline{v_{21}}).$$

□

Lemma 3.20. *$K_{2p,2q}$, where $p, q > 1$ are odd, admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = pq$.*

Proof. Let $p = 2p_1 + 1$ and $q = 2q_1 + 1$ for some $p_1, q_1 \in \mathbb{N}$. We write

$$K_{2p,2q} = (p_1 - 1)(q_1 - 1)K_{4,4} \oplus (p_1 + q_1 - 2)K_{6,4} \oplus K_{6,6}.$$

Since $K_{6,4} \cong K_{4,6}$, by Lemma 3.17, $K_{6,4}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 6$ and

$$(\alpha, \gamma) \notin \{(0, 1), (1, 0)\}.$$

Similarly, by Lemma 3.17, $K_{4,4}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha+\beta+\gamma = 4$ and

$$(\alpha, \gamma) \notin \{(0, 1), (1, 1), (1, 0)\}.$$

Further, by Lemma 3.19, $K_{6,6}$ admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha+\beta+\gamma = 9$. Hence, by Lemma 3.8 and Corollary 3.9, we obtain a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{2p,2q}$ whenever $\alpha + \beta + \gamma = pq$ and

$$(\alpha, \gamma) \notin \{(0, 1), (1, 0)\}.$$

Finally, for $(\alpha, \gamma) \in \{(0, 1), (1, 0)\}$, the required decomposition exists by Remark 3.7. □

Theorem 3.21. *For $m \geq 4$ and $n \geq 2$, $K_{m,n}$ has a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition if and only if $\alpha + \beta + \gamma = \frac{mn}{4}$, $mn \equiv 0 \pmod 4$, and one of the following holds. Let $p, q \in \mathbb{N}$.*

1. *$m = 2p$, $p \geq 2$, $n = 2$, and*
(i) *$\gamma = 0$ and β is even;*
(ii) *$\beta = 0$ and γ is even;*
(iii) *$\alpha = 0$, γ is even, and $\gamma \neq 0$ if p is odd;*
(iv) *$\alpha, \beta, \gamma > 0$ and γ is even.*
2. *$m = 4p$, $n = 3$, and*
$$(\alpha, \gamma) \notin \{(0, 1), (1, 1), (1, 0)\}.$$
3. *$m = 4$, $n \geq 4$, and*
(i) *$(\alpha, \gamma) \notin \{(0, 1), (1, 0)\}$;*
(ii) *$(\alpha, \beta, \gamma) \neq (1, 2, 1)$ when $n = 4$.*
4. *$m = 4p$, $p \geq 2$, and $n \geq 5$.*
5. *$m = 2p$, $n = 2q$, and $p, q > 1$ are odd.*

Proof. Assume that $K_{m,n}$ has a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition. Then, from the edge-divisibility condition for the existence of a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{m,n}$, we have

$$\alpha + \beta + \gamma = \frac{mn}{4}.$$

Thus, any admissible triplet must satisfy this arithmetic condition, and in particular $mn \equiv 0 \pmod 4$. Moreover, Lemmas 3.11, 3.13, 3.14, and 3.17 restrict the admissible triplets to the five cases listed above. Hence, the stated conditions are necessary.

Conversely, let (α, β, γ) be an admissible triplet satisfying one of the above cases. Then the required decompositions of $K_{m,n}$ follow from Lemmas 3.13, 3.16, 3.17, 3.18, and 3.20, corresponding respectively to Cases *(1)–(5)*. Hence, the sufficiency follows. □

4. $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of K_n

In this section, we obtain necessary and sufficient conditions for the $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of K_n, where $n \geq 8$. We recall some results required for our main theorem.

Lemma 4.1 ([8]). *There exists a $\{P_5^\alpha, S_5^\beta\}$-decomposition of K_n for $n = 8, 9$ whenever $\alpha + \beta = 7$ and $\alpha + \beta = 9$, respectively.*

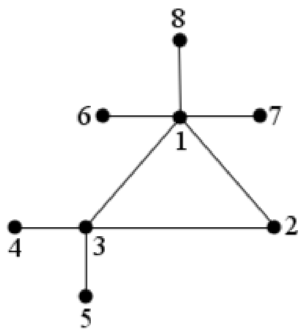
Theorem 4.2 ([3]). *For non-negative integers α, β, and $n \geq 8$,*

$$K_n = \alpha P_5 \oplus \beta Y_5$$

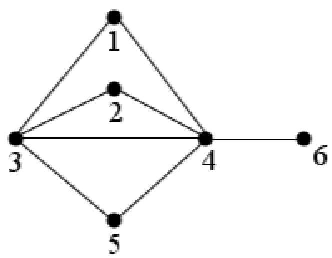
if and only if

$$4(\alpha + \beta) = \binom{n}{2}.$$

Note 4.3. Figure 1 illustrates two graphs with 8 edges that can be decomposed into different combinations of P_5, S_5, and Y_5, namely $2S_5$, $2Y_5$, $1S_5 \oplus 1Y_5$, and $1P_5 \oplus 1Y_5$.



(a) Graph G_1



(b) Graph G_2

Fig. 1.

	Corresponding edge sets			
Graph	$2S_5$	$2Y_5$	$1S_5 \oplus 1Y_5$	$1P_5 \oplus 1Y_5$
G_1	$\{3;1,2,4,5\}$, $\{1;2,6,7,8\}$	$\{8,1,\underline{3},5;\underline{4}\}$, $\{3,2,\underline{1},7;\underline{6}\}$	$\{1;3,6,7,8\}$, $\{1,2,\underline{3},5;\underline{4}\}$	$\{8,1,2,3,4\}$, $\{5,3,\underline{1},6;\underline{7}\}$
G_2	$\{3;1,2,4,5\}$, $\{4;1,2,5,6\}$	$\{6,4,\underline{3},1;\underline{2}\}$, $\{3,5,\underline{4},1;\underline{2}\}$	$\{4;1,2,3,6\}$, $\{4,5,\underline{3},1;\underline{2}\}$	$\{2,4,5,3,1\}$, $\{2,3,\underline{4},1;\underline{6}\}$

Definition 4.4 ([4]). A connected graph G with 8 edges is called a $PY5$ graph if

$$G = 2P_5 = 2Y_5 = P_5 \oplus Y_5.$$

Note 4.5. Here we recall two $PY5$ graphs from [4] that can also be decomposed into $1P_5 \oplus 1S_5$.

(a) Graph T_1 (b) Graph T_2

Fig. 2. $PY5$ graphs

	Corresponding edge sets			
Graph	$2P_5$	$2Y_5$	$1P_5 \oplus 1Y_5$	$1P_5 \oplus 1S_5$
T_1	$\{5,6,1,2,3\}$, $\{6,2,5,4,3\}$	$\{4,5,\underline{6},1;\underline{2}\}$, $\{4,3,\underline{2},1;\underline{5}\}$	$\{1,2,5,4,3\}$, $\{3,2,\underline{6},5;\underline{1}\}$	$\{1,6,5,4,3\}$, $\{2;1,3,5,6\}$
T_2	$\{6,1,5,2,4\}$, $\{1,2,3,4,5\}$	$\{1,5,\underline{4},2;\underline{3}\}$, $\{6,1,\underline{2},5;\underline{3}\}$	$\{6,1,5,4,2\}$, $\{4,3,\underline{2},5;\underline{1}\}$	$\{6,1,5,4,3\}$, $\{2;1,3,4,5\}$

Lemma 4.6. *For $\alpha \geq 0$ and $\beta, \gamma > 0$, K_8 admits a $\{P_5^{\alpha}, S_5^{\beta}, Y_5^{\gamma}\}$-decomposition whenever $\alpha + \beta + \gamma = 7$.*

Proof. Since $|E(K_8)| = 28$ and each tree of order five has four edges, any such decomposition must consist of seven trees. The construction proceeds by partitioning the edge set into three edge-disjoint subgraphs H_1, H_2, and H_3, together with a tree of order five, where $H_i \in \{G_1, G_2, T_1, T_2\}$, such that the subgraphs G_1 and G_2 are the graphs given in Note 4.3, and T_1 and T_2 are the graphs given in Note 4.5. Depending on the degree structure of G_1, G_2, T_1, and T_2, each can be further decomposed into two trees of order five of type P_5, S_5, or Y_5. By combining these decompositions appropriately, we obtain the required admissible triplets (α, β, γ).

First write

$$K_8 = 1G_1 \oplus 2G_2 \oplus 1S_5,$$

where the edge sets of the subgraphs in the given order are

$$\{v_3v_6, v_3v_7, v_3v_8, v_3v_5, v_4v_3, v_4v_5, v_4v_2, v_4v_1\},$$

$$\{v_2v_1, v_2v_3, v_2v_5, v_2v_8, v_1v_3, v_1v_5, v_1v_8, v_1v_6\},$$

$$\{v_8v_4, v_8v_5, v_8v_6, v_8v_7, v_6v_2, v_6v_4, v_6v_5, v_6v_7\},$$

and

$$\{v_7; v_1, v_2, v_4, v_5\}.$$

This decomposition yields admissible triplets (α, β, γ) for $\alpha = 0, 1$, as listed in Table 2.

Next write

$$K_8 = 1T_2 \oplus 2G_2 \oplus 1P_5,$$

where the corresponding edge sets are

$$\{v_6v_1, v_6v_5, v_6v_2, v_6v_7, v_8v_7, v_7v_2, v_2v_5, v_5v_1\},$$

$$\{v_2v_1, v_2v_3, v_2v_4, v_2v_8, v_1v_3, v_1v_4, v_1v_7, v_1v_8\},$$

$$\{v_4v_3, v_4v_5, v_4v_6, v_4v_8, v_3v_5, v_3v_6, v_3v_7, v_3v_8\},$$

and

$$\{v_4, v_7, v_5, v_8, v_6\}.$$

This construction gives the required decompositions for $\alpha = 2, 3, 4$, as listed in Table 2. Finally, for $(\alpha, \beta, \gamma) = (5, 1, 1)$, the edge sets are

$$\{v_8, v_6, v_1, v_2, v_3\}, \quad \{v_5, v_2, v_8, v_1, v_3\},$$

$$\{v_2, v_6, v_4, v_8, v_5\}, \quad \{v_1, v_5, v_6, v_7, v_8\},$$

$$\{v_7, v_3, v_5, v_4, v_1\}, \quad \{v_7; v_1, v_2, v_4, v_5\},$$

and

$$\{v_2, v_4, \underline{v_3}, v_8; \underline{v_6}\}.$$

A direct verification shows that the listed subgraphs are pairwise edge-disjoint, and their union equals $E(K_8)$. Thus, K_8 admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition for $\alpha \geq 0$ and $\beta, \gamma > 0$ whenever $\alpha + \beta + \gamma = 7$. □

Lemma 4.7. *For $\alpha \geq 0$ and $\beta, \gamma > 0$, K_9 admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 9$.*

Proof. Since $|E(K_9)|= 36$ and each tree of order five has four edges, any such decomposition must consist of nine trees. The construction proceeds similarly by partitioning the edge set into four edge-disjoint subgraphs H_1, H_2, H_3, and H_4, together with a tree of order five, where $H_i \in \{G_1, G_2, T_1, T_2\}$. Each of these subgraphs can be further decomposed into two trees of order five of type P_5, S_5, or Y_5, depending on its degree structure. By combining these decompositions appropriately, we obtain the required admissible triplets (α, β, γ).

First write

$$K_9 = 1G_1 \oplus 3G_2 \oplus 1S_5,$$

where the edge sets of the subgraphs in the given order are

$$\{v_4v_1, v_4v_2, v_4v_3, v_4v_5, v_3v_5, v_3v_6, v_3v_7, v_3v_9\},$$

$$\{v_2v_1, v_2v_3, v_2v_5, v_2v_9, v_1v_3, v_1v_5, v_1v_9, v_1v_6\},$$

$$\{v_6v_2, v_6v_4, v_6v_7, v_6v_5, v_7v_1, v_7v_2, v_7v_4, v_7v_5\},$$

$$\{v_8v_5, v_8v_6, v_8v_7, v_8v_9, v_9v_4, v_9v_5, v_9v_6, v_9v_7\},$$

and

$$\{v_8; v_1, v_2, v_3, v_4\}.$$

This decomposition yields admissible triplets (α, β, γ) for $\alpha = 0, 1$, as listed in Table 3.

Next write

$$K_9 = 1T_1 \oplus 3G_2 \oplus 1P_5,$$

where the corresponding edge sets are

$$\{v_6v_1, v_6v_2, v_6v_3, v_6v_7, v_3v_7, v_7v_1, v_1v_5, v_5v_2\},$$

$$\{v_2v_1, v_2v_3, v_2v_4, v_2v_9, v_1v_3, v_1v_4, v_1v_8, v_1v_9\},$$

$$\{v_3v_4, v_3v_5, v_3v_8, v_3v_9, v_4v_5, v_4v_7, v_4v_8, v_4v_9\},$$

$$\{v_9v_5, v_9v_6, v_9v_7, v_9v_8, v_8v_2, v_8v_5, v_8v_6, v_8v_7\},$$

and

$$\{v_4, v_6, v_5, v_7, v_2\}.$$

This construction yields the required decompositions for $\alpha = 2, 3, 4, 5$, except $(5, 3, 1)$, as listed in Table 3.

For $(\alpha, \beta, \gamma) = (5, 3, 1)$, the edge sets are

$$\{v_3, v_2, v_1, v_6, v_7\}, \quad \{v_5, v_2, v_9, v_1, v_3\},$$

$$\{v_1, v_7, v_2, v_6, v_5\}, \quad \{v_1, v_5, v_7, v_4, v_6\},$$

$$\{v_6, v_9, v_7, v_8, v_5\},$$

$$\{v_8; v_1, v_2, v_3, v_4\}, \quad \{v_3; v_5, v_6, v_7, v_9\},$$

$$\{v_4; v_1, v_2, v_3, v_5\}$$

and

$$\{v_6, v_8, \underline{v_9}, v_5; \underline{v_4}\}.$$

Finally, by rearranging the edges of the three copies of G_2 in the decomposition

$$K_9 = 1G_1 \oplus 3G_2 \oplus 1S_5,$$

we obtain six copies of P_5:

$$\{v_3, v_2, v_1, v_6, v_7\}, \quad \{v_5, v_2, v_9, v_1, v_3\},$$

$$\{v_9, v_6, v_5, v_1, v_7\}, \quad \{v_4, v_6, v_2, v_7, v_5\},$$

$$\{v_5, v_8, v_9, v_4, v_7\}$$

and

$$\{v_6, v_8, v_7, v_9, v_5\}.$$

This gives the required decompositions for $\alpha = 6, 7$, as listed in Table 3.

A direct verification shows that the listed subgraphs are pairwise edge-disjoint, and their union equals $E(K_9)$. Thus, K_9 admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition for $\alpha \geq 0$ and $\beta, \gamma > 0$ whenever $\alpha + \beta + \gamma = 9$. □

Lemma 4.8. *K_n admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $n = 4p$ or $4p+1$, where p is even and*

$$\alpha + \beta + \gamma = \frac{n(n-1)}{8}.$$

Proof. We prove this by induction on p.

Case 1. $n = 4p$.

When $p = 2$, K_8 admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 7$ by Lemmas 4.1, 4.6, and Theorem 4.2. Assume that, for some even $p \geq 4$, $K_{4(p-2)}$ admits the required decomposition. Since

$$V(K_{4p}) = \{v_i : 1 \leq i \leq 4p\},$$

let

$$V_1 = \{v_i : 1 \leq i \leq 8\} \quad \text{and} \quad V_2 = \{v_i : 9 \leq i \leq 4p\}.$$

Then the subgraph induced by V_1 is K_8. Also, the subgraph induced by V_2 is $K_{4(p-2)}$, which admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever

$$\alpha + \beta + \gamma = \frac{(p-2)(4p-9)}{2}$$

by the induction hypothesis. The edges between the vertices in V_1 and V_2 give rise to $K_{8,4(p-2)}$, which admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever

$$\alpha + \beta + \gamma = 8(p-2)$$

by Case *4* of Theorem 3.21. Therefore, for $p \geq 4$, we write

$$K_{4p} = K_8 \oplus K_{4(p-2)} \oplus K_{8,4(p-2)}.$$

Now, by Corollary 3.10, K_{4p} admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever

$$\alpha + \beta + \gamma = \frac{p(4p-1)}{2}.$$

Case 2. $n = 4p + 1$.

When $p = 2$, K_9 admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever $\alpha + \beta + \gamma = 9$ by Lemmas 4.1, 4.7, and Theorem 4.2. For $p \geq 4$, we write

$$K_{4p+1} = K_8 \oplus K_{4(p-2)+1} \oplus K_{8,4(p-2)+1}.$$

With an argument similar to that of Case 1, the proof follows. □

Theorem 4.9. *Let $n \geq 8$. Then K_n admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition if and only if*

$$\alpha + \beta + \gamma = \frac{n(n-1)}{8}$$

and

$$n \equiv 0 \text{ or } 1 \pmod 8.$$

Proof. Suppose that K_n admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition. From the definition of admissible triplets, we have

$$4(\alpha + \beta + \gamma) = \binom{n}{2}.$$

Therefore,

$$n(n-1) \equiv 0 \pmod 8,$$

which implies that

$$n \equiv 0, 1 \pmod 8.$$

Conversely, suppose that $n \geq 8$ and $n \equiv 0$ or $1 \pmod 8$. Then $n = 4p$ or $4p + 1$ with p even. By Lemma 4.8, K_n admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition whenever

$$\alpha + \beta + \gamma = \frac{n(n-1)}{8}.$$

This completes the proof. □

5. Conclusion

This paper determines the necessary and sufficient conditions for the $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition of $K_{m,n}$ and K_n. The main result for $K_{m,n}$ is stated in Theorem 3.21. For $m \geq 4$ and $n \geq 2$, a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition exists if and only if

$$\alpha + \beta + \gamma = \frac{mn}{4} \quad \text{and} \quad mn \equiv 0 \pmod 4,$$

subject to the additional parity condition in the case $m = 2p$, $p \geq 2$, $n = 2$, and to the excluded admissible triplets in the cases $m = 4p$, $n = 3$, and $m = 4$, $n \geq 4$. For complete graphs, Theorem 4.9 shows that K_n admits a $\{P_5^\alpha, S_5^\beta, Y_5^\gamma\}$-decomposition if and only if

$$n \equiv 0 \text{ or } 1 \pmod 8,$$

with $n \geq 8$.

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Appendix

The admissible triplets (α, β, γ) are obtained by summing the contributions of the constituent subgraphs in each decomposition. The corresponding values are listed below.

Table 1. Admissible triplet construction for $K_{6,6}$

Decomposition	(α, β, γ)	Component-wise sum
$K_{6,6} = K_{6,4} \oplus K_{6,2}$	$(1,1,7)$	$(0,1,5)+(1,0,2)$
	$(1,2,6)$	$(0,2,4)+(1,0,2)$
	$(1,3,5)$	$(0,3,3)+(1,0,2)$
	$(1,4,4)$	$(0,4,2)+(1,0,2)$
	$(1,5,3)$	$(1,4,1)+(0,1,2)$
	$(1,6,2)$	$(0,6,0)+(1,0,2)$
	$(2,1,6)$	$(1,1,4)+(1,0,2)$
	$(2,2,5)$	$(1,2,3)+(1,0,2)$
	$(2,3,4)$	$(1,3,2)+(1,0,2)$
	$(2,4,3)$	$(1,4,1)+(1,0,2)$
	$(2,5,2)$	$(1,3,2)+(1,2,0)$
	$(2,6,1)$	$(1,4,1)+(1,2,0)$
	$(3,1,5)$	$(2,1,3)+(1,0,2)$
	$(3,2,4)$	$(2,2,2)+(1,0,2)$
	$(3,3,3)$	$(2,3,1)+(1,0,2)$
	$(3,4,2)$	$(2,4,0)+(1,0,2)$
	$(3,5,1)$	$(2,3,1)+(1,2,0)$
	$(4,1,4)$	$(1,1,4)+(3,0,0)$
	$(4,2,3)$	$(1,2,3)+(3,0,0)$
	$(4,3,2)$	$(1,3,2)+(3,0,0)$
	$(4,4,1)$	$(1,4,1)+(3,0,0)$
	$(5,1,3)$	$(2,1,3)+(3,0,0)$
	$(5,2,2)$	$(2,2,2)+(3,0,0)$
	$(5,3,1)$	$(2,3,1)+(3,0,0)$
	$(6,1,2)$	$(3,1,2)+(3,0,0)$
	$(6,2,1)$	$(3,2,1)+(3,0,0)$
	$(7,1,1)$	$(4,1,1)+(3,0,0)$

Table 2. Admissible triplet construction for K_8

Decomposition	(α, β, γ)	Component-wise sum
$K_9 = 1G_1 \oplus 3G_2 \oplus 1S_5$	$(0,1,6)$	$(0,0,2)+(0,0,2)+(0,0,2)+(0,1,0)$
	$(0,2,5)$	$(0,1,1)+(0,0,2)+(0,0,2)+(0,1,0)$
	$(0,3,4)$	$(0,2,0)+(0,0,2)+(0,0,2)+(0,1,0)$
	$(0,4,3)$	$(0,2,0)+(0,1,1)+(0,0,2)+(0,1,0)$
	$(0,5,2)$	$(0,2,0)+(0,2,0)+(0,0,2)+(0,1,0)$
	$(0,6,1)$	$(0,1,1)+(0,2,0)+(0,2,0)+(0,1,0)$
	$(1,1,5)$	$(0,0,2)+(0,0,2)+(1,0,1)+(0,1,0)$
	$(1,2,4)$	$(0,0,2)+(1,0,1)+(0,1,1)+(0,1,0)$
	$(1,3,3)$	$(1,0,1)+(0,2,0)+(0,0,2)+(0,1,0)$
	$(1,4,2)$	$(1,0,1)+(0,1,1)+(0,2,0)+(0,1,0)$
	$(1,5,1)$	$(1,0,1)+(0,2,0)+(0,2,0)+(0,1,0)$
$K_8 = 1T_2 \oplus 2G_2 \oplus 1P_5$	$(2,1,4)$	$(1,1,0)+(0,0,2)+(0,0,2)+(1,0,0)$
	$(2,2,3)$	$(1,1,0)+(0,1,1)+(0,0,2)+(1,0,0)$
	$(2,3,2)$	$(1,1,0)+(0,2,0)+(0,0,2)+(1,0,0)$
	$(2,4,1)$	$(1,1,0)+(0,1,1)+(0,2,0)+(1,0,0)$
	$(3,1,3)$	$(2,0,0)+(0,1,1)+(0,0,2)+(1,0,0)$
	$(3,2,2)$	$(2,0,0)+(0,2,0)+(0,0,2)+(1,0,0)$
	$(3,3,1)$	$(2,0,0)+(0,2,0)+(0,1,1)+(1,0,0)$
	$(4,1,2)$	$(2,0,0)+(1,0,1)+(0,1,1)+(1,0,0)$
	$(4,2,1)$	$(2,0,0)+(1,0,1)+(0,2,0)+(1,0,0)$

Table 3. Admissible triplet construction for K_9

Decomposition	(α, β, γ)	Component-wise sum
$K_9 = 1G_1 \oplus 3G_2 \oplus 1S_5$	$(0,1,6)$	$(0,0,2)+(0,0,2)+(0,0,2)+(0,1,0)$
	$(0,2,5)$	$(0,1,1)+(0,0,2)+(0,0,2)+(0,1,0)$
	$(0,3,4)$	$(0,2,0)+(0,0,2)+(0,0,2)+(0,1,0)$
	$(0,4,3)$	$(0,2,0)+(0,1,1)+(0,0,2)+(0,1,0)$
	$(0,5,2)$	$(0,2,0)+(0,2,0)+(0,0,2)+(0,1,0)$
	$(0,6,1)$	$(0,1,1)+(0,2,0)+(0,2,0)+(0,1,0)$
	$(1,1,5)$	$(0,0,2)+(0,0,2)+(1,0,1)+(0,1,0)$
	$(1,2,4)$	$(0,0,2)+(1,0,1)+(0,1,1)+(0,1,0)$
	$(1,3,3)$	$(1,0,1)+(0,2,0)+(0,0,2)+(0,1,0)$
	$(1,4,2)$	$(1,0,1)+(0,1,1)+(0,2,0)+(0,1,0)$
	$(1,5,1)$	$(1,0,1)+(0,2,0)+(0,2,0)+(0,1,0)$
$K_8 = 1T_2 \oplus 2G_2 \oplus 1P_5$	$(2,1,4)$	$(1,1,0)+(0,0,2)+(0,0,2)+(1,0,0)$
	$(2,2,3)$	$(1,1,0)+(0,1,1)+(0,0,2)+(1,0,0)$
	$(2,3,2)$	$(1,1,0)+(0,2,0)+(0,0,2)+(1,0,0)$
	$(2,4,1)$	$(1,1,0)+(0,1,1)+(0,2,0)+(1,0,0)$
	$(3,1,3)$	$(2,0,0)+(0,1,1)+(0,0,2)+(1,0,0)$
	$(3,2,2)$	$(2,0,0)+(0,2,0)+(0,0,2)+(1,0,0)$
	$(3,3,1)$	$(2,0,0)+(0,2,0)+(0,1,1)+(1,0,0)$
	$(4,1,2)$	$(2,0,0)+(1,0,1)+(0,1,1)+(1,0,0)$
	$(4,2,1)$	$(2,0,0)+(1,0,1)+(0,2,0)+(1,0,0)$
$K_9 = 1G_1 \oplus 6P_5 \oplus 1S_5$	$(6,1,2)$	$(0,0,2)+6(1,0,0)+(0,1,0)$
	$(6,2,1)$	$(0,1,1)+6(1,0,0)+(0,1,0)$
	$(7,1,1)$	$(1,0,1)+6(1,0,0)+(0,1,0)$