

Decompositions of $K_n(\lambda)$ into P_4 and S_4

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ABSTRACT

Let P_{k+1} denote a path of length k , let S_m denote a star with m edges, and let $K_n(\lambda)$ denote the complete multigraph on n vertices in which every edge is taken λ times. In this paper, we prove that the necessary conditions are also sufficient for a $\{P_4, S_4\}$ -decomposition of $K_n(\lambda)$.

Keywords: decomposition, complete multigraph, path, star

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1. Introduction

All graphs considered here are finite and undirected, with no loops. For standard graph-theoretic terminology, the reader is referred to [1]. A simple graph in which each pair of distinct vertices is joined by an edge is called a *complete graph*. A complete graph on n vertices is denoted by K_n . If more than one edge joining two vertices is allowed, the resulting object is called a *multigraph*. Let $K_n(\lambda)$ denote the *complete multigraph* on n vertices in which every edge is taken λ times. A *complete bipartite graph* is a simple bipartite graph with bipartition (X, Y) in which each vertex of X is joined to each vertex of Y ; if $|X| = a$ and $|Y| = b$, such a graph is denoted by $K_{a,b}$. If $a = b$, the complete bipartite graph is referred to as *balanced*. In $K_{a,b}(\lambda)$, we label the vertices in the partite set X as $1, 2, \dots, a$ and those in Y as $a+1, a+2, \dots, a+b$. A *path* is an open trail with no repeated vertex. A path with k edges is denoted by P_{k+1} . The complete bipartite graph $K_{1,m}$ is called a *star* and is denoted by S_m . For $m \geq 3$, the vertex of degree m in S_m is called the *center*, and any vertex of degree 1 in S_m is called an end vertex.

Let G be a graph, and let G_1 be a subgraph of G . Then $G \setminus G_1$ is obtained from G

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by deleting the edges of G_1 . Let G_1 and G_2 be subgraphs of G . The union $G_1 \cup G_2$ of G_1 and G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. We say that G_1 and G_2 are *edge-disjoint* if they have no edge in common. If G_1 and G_2 are edge-disjoint, we denote their union by $G_1 + G_2$. A *decomposition* of a graph G is a collection of edge-disjoint subgraphs G_1, G_2, \dots, G_n of G such that every edge of G is in exactly one G_i . In this case, G is said to be *decomposed* or *decomposable* into G_1, G_2, \dots, G_n . If G has a decomposition into p_1 copies of G_1, \dots, p_n copies of G_n , then we say that G has a $\{p_1 G_1, \dots, p_n G_n\}$ -decomposition. If such a decomposition exists for all values of p_1, \dots, p_n satisfying the trivial necessary conditions, then we say that G has a $\{G_1, \dots, G_n\}_{\{p_1, \dots, p_n\}}$ -decomposition or that G is fully $\{G_1, \dots, G_n\}$ -decomposable. We say that G is decomposed into P_4 and S_4 if each $G_i \simeq P_4$ or S_4 . We say that G is decomposed into H , or that G has an H -decomposition, if each $G_i \simeq H$.

In [10], Priyadharsini and Muthusamy gave necessary and sufficient conditions for the existence of a $\{pG_1, qG_2\}$ -decomposition of $K_n(\lambda)$, when

$$(G_1, G_2) \in \{(P_n, S_{1,n-1}), (C_n, S_{1,n-1}), (P_n, C_n)\}.$$

In [9], Priyadharsini gave necessary and sufficient conditions for the existence of a $\{pP_n, qS_{1,n-1}\}$ -decomposition of $K_{n+1}(\lambda)$. In [11], Shyu gave the necessary conditions for a $\{pP_{k+1}, qS_k\}$ -decomposition of K_n and proved that K_n is fully $\{P_{k+1}, S_k\}$ -decomposable when $n \geq 4k$, such that either k is even and $p \geq \frac{k}{2}$, or k is odd and $p \geq k$; Shyu also settled the case $k = 3$ completely, i.e., for all $n \geq 6$. In [12], Shyu proved that K_n is fully $\{P_{k+1}, S_k\}$ -decomposable when $n \geq 4k$. In [7], Lee and Chen showed the existence of a $\{pP_{k+1}, qS_k\}$ -decomposition of $K_n(\lambda)$ and $K_{b,b}(\lambda)$. In [8], Lee and Chen gave the necessary conditions for a $\{pF, qS_3\}$ -decomposition of K_n and proved that K_n is fully $\{F, S_3\}$ -decomposable when $F \in \{P_n, C_n\}$. In [13], Shyu gave the necessary conditions for a $\{pC_k, qP_{k+1}, rS_k\}$ -decomposition of K_n and proved that K_n is fully $\{C_4, P_5, S_4\}$ -decomposable when n is odd. In [5], Ilayaraja and Muthusamy gave the necessary conditions for a $\{pP_{k+1}, qS_m\}$ -decomposition of K_n and proved that K_n is fully $\{P_4, S_4\}$ -decomposable. In [2], the authors gave the necessary conditions for a $\{pP_{k+1}, qS_m\}$ -decomposition of $K_n(\lambda)$ and proved that $K_n(\lambda)$ is fully $\{P_7, S_4\}$ -decomposable. In [3, 4], the authors proved that $K_n(\lambda)$ is fully $\{P_7, S_3\}$ -decomposable and that $K_n(\lambda)$ is fully $\{P_5, S_3\}$ -decomposable. In this paper, we prove that $K_n(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

2. Preliminaries

For convenience, we denote $V(K_n(\lambda)) = \{x_1, x_2, \dots, x_n\}$. The notation $S(x_1; x_2, \dots, x_m)$ denotes an m -star with x_1 as the center vertex and x_2, \dots, x_m as end vertices, and $(x_1 x_2 \dots x_{k+1})$ denotes a path with vertices x_1, x_2, \dots, x_{k+1} and edges $x_1 x_2, x_2 x_3, \dots, x_k x_{k+1}$.

We recall here some results on P_{k+1} - and S_m -decompositions that are useful for our proofs.

Theorem 2.1. [14] *A necessary and sufficient condition for the existence of a P_{k+1} -decomposition of $K_n(\lambda)$ is that $\lambda \binom{n}{2} \equiv 0 \pmod{k}$ and $n \geq k + 1$.*

Theorem 2.2. [15] *A necessary and sufficient condition for the existence of an S_m -decomposition of $K_n(\lambda)$ is that:*

- (i) $\lambda \binom{n}{2} \equiv 0 \pmod{m}$;
- (ii) $n \geq 2m$ for $\lambda = 1$;
- (iii) $n \geq m + 1$ for even λ ;
- (iv) $n \geq m + 1 + \frac{m}{\lambda}$ for odd $\lambda \geq 3$.

Theorem 2.3. [16] *Let k be a positive integer, and let a and b be positive even integers such that $a \geq b$. Then $K_{a,b}(\lambda)$ has a P_{k+1} -decomposition if and only if $a \geq \lceil \frac{k+1}{2} \rceil$, $b \geq \lceil \frac{k}{2} \rceil$, and $\lambda ab \equiv 0 \pmod{k}$.*

Theorem 2.4. [6] *For positive integers a and b with $a \geq b$, the complete bipartite multi-graph $K_{a,b}(\lambda)$ is S_m -decomposable if and only if $a \geq m$ and*

$$\begin{cases} \lambda a \equiv 0 \pmod{m}, & \text{if } b < m, \\ \lambda ab \equiv 0 \pmod{m}, & \text{if } b \geq m. \end{cases}$$

In [2], the authors discussed the necessary conditions for a $\{pP_{k+1}, qS_m\}$ -decomposition of $K_n(\lambda)$ when $\lambda \geq 1$, which are as follows.

Theorem 2.5. [2] *Let λ, n, k , and m be positive integers. Let p and q be non-negative integers. The necessary conditions for a $\{pP_{k+1}, qS_m\}$ -decomposition of $K_n(\lambda)$ are $pk + qm = \lambda \binom{n}{2}$ and $n \geq \max\{k + 1, m + 1\}$.*

In this paper, we prove that the above necessary conditions are sufficient for a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_n(\lambda)$ in Theorem 3.4.

3. Main Result

In this section, we discuss a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_n(\lambda)$ when $\lambda \geq 1$. Since $K_n(\lambda)$ cannot be decomposed into P_4 and S_4 when $n \leq 4$, we consider the decompositions for $n \geq 5$.

In the following lemma, we discuss $\{P_4, S_4\}_{\{p,q\}}$ -decompositions of $K_{4,6}$, which will be used later to decompose $K_n(\lambda)$ into $\{pP_4, qS_4\}$.

Lemma 3.1. *If p and q are non-negative integers such that $3p + 4q = 24$, then $K_{4,6}$ is fully $\{P_4, S_4\}$ -decomposable.*

Proof. We have $(p, q) \in \{(8, 0), (4, 3), (0, 6)\}$. By Theorem 2.3, $K_{4,6}$ is $\{8P_4, 0S_4\}$ -decomposable. The graph $K_{4,6}$ can be decomposed into four copies of P_4 :

$$(x_8x_3x_5x_4), \quad (x_4x_6x_3x_{10}), \quad (x_1x_9x_2x_{10}), \quad (x_1x_7x_3x_9),$$

and three copies of S_4 :

$$S(x_1; x_5, x_6, x_8, x_{10}), \quad S(x_2; x_5, x_6, x_7, x_8), \quad S(x_4; x_7, x_8, x_9, x_{10}).$$

By Theorem 2.4, $K_{4,6}$ is $\{0P_4, 6S_4\}$ -decomposable. Therefore, $K_{4,6}$ is fully $\{P_4, S_4\}$ -decomposable. \square

Lemma 3.2. *If p and q are non-negative integers such that $3p + 4q = 24$, then $K_{2,12}$ is fully $\{P_4, S_4\}$ -decomposable.*

Proof. We have $(p, q) \in \{(8, 0), (4, 3), (0, 6)\}$. By Theorem 2.3, $K_{2,12}$ is $\{8P_4, 0S_4\}$ -decomposable. The graph $K_{2,12}$ can be decomposed into four copies of P_4 :

$$(x_3x_2x_7x_1), \quad (x_4x_2x_8x_1), \quad (x_5x_2x_9x_1), \quad (x_6x_2x_{10}x_1),$$

and three copies of S_4 :

$$S(x_1; x_3, x_4, x_5, x_6), \quad S(x_1; x_{11}, x_{12}, x_{13}, x_{14}), \quad S(x_2; x_{11}, x_{12}, x_{13}, x_{14}).$$

By Theorem 2.4, $K_{2,12}$ is $\{0P_4, 6S_4\}$ -decomposable. Therefore, $K_{2,12}$ is fully $\{P_4, S_4\}$ -decomposable. \square

Lemma 3.3. *The graph K_6 cannot be decomposed into one copy of P_4 and three copies of S_4 .*

Proof. Let $V(K_6) = \{x_1, x_2, \dots, x_6\}$, and let P_4 be the path $(x_1x_2x_3x_4)$. Then x_2 and x_3 cannot be center vertices of any copy of S_4 . There are two cases.

Case 1. Suppose that we fix x_1 as a center vertex. Then x_4 cannot be a center vertex. Hence either x_5 or x_6 can be a center vertex of a copy of S_4 , but not both.

Case 2. Suppose that we fix x_5 as a center vertex of a copy of S_4 . Then x_6 is the only possible remaining center vertex of a copy of S_4 .

Hence, in both cases, we can have only two copies of S_4 . Therefore, K_6 cannot be decomposed into one copy of P_4 and three copies of S_4 . \square

We now prove our main result.

Theorem 3.4. *For any non-negative integers p and q and any integer $n \geq 5$, there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_n(\lambda)$ if and only if $3p + 4q = \lambda \binom{n}{2}$, except possibly when $(\lambda, n, p, q) = (1, 6, 1, 3)$.*

Proof. The necessary conditions are obvious. First, we prove the result for odd n with $5 \leq n \leq 27$ and for even n with $6 \leq n \leq 30$. Then we generalize it for any odd $n > 27$ and any even $n > 30$ by applying mathematical induction. Since we discuss the $\{pP_4, qS_4\}$ -decomposition of $K_n(\lambda)$ for all possible choices of p and q , we have the following cases.

Case 1. $n = 5$.

If $\lambda = 1$, then $(p, q) \in \{(2, 1)\}$. The graph K_5 can be decomposed into two copies of P_4 :

$$(x_1x_3x_4x_2), \quad (x_3x_2x_1x_4),$$

and one copy of S_4 :

$$S(x_5; x_1, x_2, x_3, x_4).$$

Therefore, K_5 is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda = 2$, then $(p, q) \in \{(0, 5), (4, 2)\}$. By Theorem 2.2, $K_5(2)$ is $\{0P_4, 5S_4\}$ -decomposable. We write

$$K_5(2) = K_5 + K_5 = \{(2, 1)\} + \{(2, 1)\} = \{(4, 2)\}.$$

Therefore, $K_5(2)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda = 3$, then $(p, q) \in \{(2, 6), (6, 3), (10, 0)\}$. By Theorem 2.1, $K_5(3)$ is $\{10P_4, 0S_4\}$ -decomposable. We write

$$K_5(3) = K_5(2) + K_5 = \{(0, 5), (4, 2)\} + \{(2, 1)\} = \{(2, 6), (6, 3)\}.$$

Therefore, $K_5(3)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda = 4$, then $(p, q) \in \{(0, 10), (4, 7), (8, 4), (12, 1)\}$. By Theorem 2.2, $K_5(4)$ is $\{0P_4, 10S_4\}$ -decomposable. We write

$$K_5(4) = K_5(3) + K_5 = \{(2, 6), (6, 3), (10, 0)\} + \{(2, 1)\} = \{(4, 7), (8, 4), (12, 1)\}.$$

Therefore, $K_5(4)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda = 5$, then $(p, q) \in \{(2, 11), (6, 8), (10, 5), (14, 2)\}$. We write

$$\begin{aligned} K_5(5) &= K_5(3) + K_5(2) \\ &= \{(2, 6), (6, 3), (10, 0)\} + \{(0, 5), (4, 2)\} \\ &= \{(2, 11), (6, 8), (10, 5), (14, 2)\}. \end{aligned}$$

Therefore, $K_5(5)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda = 6$, then

$$(p, q) \in \{(0, 15), (4, 12), (8, 9), (12, 6), (16, 3), (20, 0)\}.$$

By Theorem 2.2, $K_5(6)$ is $\{0P_4, 15S_4\}$ -decomposable. We write

$$\begin{aligned} K_5(6) &= K_5(3) + K_5(3) \\ &= \{(2, 6), (6, 3), (10, 0)\} + \{(2, 6), (6, 3), (10, 0)\} \\ &= \{(4, 12), (8, 9), (12, 6), (16, 3), (20, 0)\}. \end{aligned}$$

Therefore, $K_5(6)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda \geq 7$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{6}$, then $K_5(\lambda)$ can be decomposed into $\frac{\lambda}{6}$ copies of $K_5(6)$. Therefore, $K_5(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda \equiv 1 \pmod{6}$, then we write

$$K_5(\lambda) = K_5(\lambda - 1) + K_5.$$

The graph $K_5(\lambda - 1)$ can be decomposed into $\frac{\lambda-1}{6}$ copies of $K_5(6)$. Therefore, $K_5(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda \equiv 2 \pmod{6}$, then we write

$$K_5(\lambda) = K_5(\lambda - 2) + K_5(2).$$

The graph $K_5(\lambda - 2)$ can be decomposed into $\frac{\lambda-2}{6}$ copies of $K_5(6)$. Therefore, $K_5(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda \equiv 3 \pmod{6}$, then we write

$$K_5(\lambda) = K_5(\lambda - 3) + K_5(3).$$

The graph $K_5(\lambda - 3)$ can be decomposed into $\frac{\lambda-3}{6}$ copies of $K_5(6)$. Therefore, $K_5(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda \equiv 4 \pmod{6}$, then we write

$$K_5(\lambda) = K_5(\lambda - 4) + K_5(4).$$

The graph $K_5(\lambda - 4)$ can be decomposed into $\frac{\lambda-4}{6}$ copies of $K_5(6)$. Therefore, $K_5(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

If $\lambda \equiv 5 \pmod{6}$, then we write

$$K_5(\lambda) = K_5(\lambda - 5) + K_5(5).$$

The graph $K_5(\lambda - 5)$ can be decomposed into $\frac{\lambda-5}{6}$ copies of $K_5(6)$. Therefore, $K_5(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 2. $n = 6$.

If $\lambda = 1$, then $(p, q) \in \{(5, 0)\}$, except when $(p, q) = (1, 3)$. By Theorem 2.1, K_6 is $\{5P_4, 0S_4\}$ -decomposable.

If $\lambda = 2$, then $(p, q) \in \{(2, 6), (6, 3), (10, 0)\}$. The graph $K_6(2)$ can be decomposed into two copies of P_4 :

$$(x_4x_3x_5x_6), \quad (x_1x_6x_3x_2),$$

and six copies of S_4 :

$$\begin{aligned} S(x_1; x_2, x_3, x_4, x_5), & \quad S(x_1; x_2, x_3, x_4, x_5), \\ S(x_2; x_3, x_4, x_5, x_6), & \quad S(x_4; x_2, x_3, x_5, x_6), \\ S(x_5; x_2, x_3, x_4, x_6), & \quad S(x_6; x_1, x_2, x_3, x_4). \end{aligned}$$

Also, $K_6(2)$ can be decomposed into six copies of P_4 :

$$\begin{aligned} (x_6x_5x_4x_3), & \quad (x_6x_5x_4x_3), & \quad (x_1x_5x_3x_6), \\ (x_2x_1x_6x_4), & \quad (x_1x_4x_2x_3), & \quad (x_1x_3x_5x_2), \end{aligned}$$

and three copies of S_4 :

$$S(x_1; x_2, x_3, x_4, x_5), \quad S(x_2; x_3, x_4, x_5, x_6), \quad S(x_6; x_1, x_2, x_3, x_4).$$

By Theorem 2.1, $K_6(2)$ is $\{10P_4, 0S_4\}$ -decomposable.

If $\lambda = 3$, then $(p, q) \in \{(3, 9), (7, 6), (11, 3), (15, 0)\}$. The graph $K_6(3)$ can be decomposed into three copies of P_4 :

$$(x_1x_6x_2x_5), \quad (x_4x_5x_6x_3), \quad (x_5x_6x_4x_2),$$

and nine copies of S_4 :

$$\begin{aligned} &S(x_1; x_2, x_3, x_4, x_5), \quad S(x_1; x_2, x_3, x_4, x_5), \\ &S(x_2; x_1, x_3, x_4, x_5), \quad S(x_3; x_1, x_2, x_4, x_5), \\ &S(x_3; x_2, x_4, x_5, x_6), \quad S(x_4; x_1, x_2, x_3, x_5), \\ &S(x_5; x_1, x_2, x_3, x_4), \quad S(x_6; x_1, x_2, x_3, x_4), \\ &S(x_6; x_1, x_2, x_4, x_5). \end{aligned}$$

By taking $K_6(3) = K_6(2) + K_6$, we obtain all the other possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 15), (4, 12), (8, 9), \dots, (20, 0)\},$$

where the values of p increase by 4 and the values of q decrease by 3. By Theorem 2.2, $K_6(4)$ is $\{0P_4, 15S_4\}$ -decomposable. By taking

$$K_6(4) = K_6(2) + K_6(2),$$

we obtain all the above possible decompositions.

If $\lambda = 5$, then

$$(p, q) \in \{(1, 18), (5, 15), (9, 12), \dots, (25, 0)\}.$$

The graph $K_6(5)$ can be decomposed into one copy of P_4 :

$$(x_6x_5x_3x_4),$$

and eighteen copies of S_4 :

$$\begin{aligned} &S(x_1; x_3, x_4, x_5, x_6), \quad S(x_1; x_2, x_3, x_4, x_5), \\ &S(x_1; x_2, x_3, x_4, x_5), \quad S(x_2; x_1, x_3, x_4, x_5), \\ &S(x_2; x_1, x_3, x_4, x_6), \quad S(x_2; x_1, x_3, x_4, x_5), \\ &S(x_3; x_1, x_2, x_4, x_6), \quad S(x_3; x_1, x_2, x_4, x_6), \\ &S(x_4; x_1, x_2, x_3, x_6), \quad S(x_4; x_1, x_2, x_3, x_5), \\ &S(x_5; x_1, x_2, x_3, x_4), \quad S(x_5; x_1, x_3, x_4, x_6), \\ &S(x_5; x_2, x_3, x_4, x_6), \quad S(x_5; x_2, x_3, x_4, x_6), \\ &S(x_6; x_1, x_2, x_3, x_4), \quad S(x_6; x_1, x_2, x_3, x_4), \\ &S(x_6; x_1, x_2, x_3, x_4), \quad S(x_6; x_1, x_2, x_4, x_5). \end{aligned}$$

By taking

$$K_6(5) = K_6(3) + K_6(2),$$

we obtain all the other possible decompositions.

If $\lambda \geq 6$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_6(\lambda) = \frac{\lambda}{4}K_6(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_6(\lambda) = K_6(\lambda - 5) + K_6(5) = \frac{\lambda - 5}{4}K_6(4) + K_6(5).$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_6(\lambda) = K_6(\lambda - 2) + K_6(2) = \frac{\lambda - 2}{4}K_6(4) + K_6(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_6(\lambda) = K_6(\lambda - 3) + K_6(3) = \frac{\lambda - 3}{4}K_6(4) + K_6(3).$$

Therefore, $K_6(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 3. $n = 7$.

If $\lambda = 1$, then $(p, q) \in \{(3, 3), (7, 0)\}$. The graph K_7 can be decomposed into three copies of P_4 :

$$(x_6x_5x_7x_4), \quad (x_2x_5x_3x_4), \quad (x_3x_2x_4x_5),$$

and three copies of S_4 :

$$S(x_1; x_2, x_3, x_4, x_5), \quad S(x_6; x_1, x_2, x_3, x_4), \quad S(x_7; x_1, x_2, x_3, x_6).$$

By Theorem 2.1, K_7 is $\{7P_4, 0S_4\}$ -decomposable.

If $\lambda = 2$, then $(p, q) \in \{(2, 9), (6, 6), (10, 3), (14, 0)\}$. The graph $K_7(2)$ can be decomposed into two copies of P_4 :

$$(x_3x_2x_4x_7), \quad (x_2x_3x_4x_6),$$

and nine copies of S_4 :

$$\begin{aligned} &S(x_1; x_2, x_3, x_4, x_5), \quad S(x_1; x_2, x_3, x_4, x_5), \\ &S(x_4; x_2, x_3, x_6, x_7), \quad S(x_5; x_2, x_3, x_4, x_6), \\ &S(x_5; x_2, x_3, x_4, x_6), \quad S(x_6; x_1, x_2, x_3, x_7), \\ &S(x_6; x_1, x_2, x_3, x_7), \quad S(x_7; x_1, x_2, x_3, x_5), \\ &S(x_7; x_1, x_2, x_3, x_5). \end{aligned}$$

By taking

$$K_7(2) = K_7 + K_7,$$

we obtain all the other possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(1, 15), (5, 12), (9, 9), \dots, (21, 0)\}.$$

The graph $K_7(3)$ can be decomposed into one copy of P_4 :

$$(x_4x_7x_1x_6),$$

and fifteen copies of S_4 :

$$\begin{aligned} &S(x_1; x_2, x_3, x_4, x_5), & S(x_1; x_2, x_3, x_4, x_5), \\ &S(x_1; x_2, x_3, x_4, x_5), & S(x_2; x_3, x_4, x_6, x_7), \\ &S(x_2; x_3, x_4, x_6, x_7), & S(x_2; x_3, x_5, x_6, x_7), \\ &S(x_3; x_4, x_5, x_6, x_7), & S(x_3; x_4, x_5, x_6, x_7), \\ &S(x_4; x_2, x_3, x_5, x_6), & S(x_5; x_2, x_3, x_4, x_7), \\ &S(x_5; x_2, x_4, x_6, x_7), & S(x_6; x_1, x_4, x_5, x_7), \\ &S(x_6; x_1, x_3, x_4, x_5), & S(x_7; x_1, x_4, x_5, x_6), \\ &S(x_7; x_1, x_3, x_4, x_6). \end{aligned}$$

By taking

$$K_7(3) = K_7(2) + K_7,$$

we obtain all the other possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 21), (4, 18), (8, 15), \dots, (28, 0)\}.$$

By Theorem 2.2, $K_7(4)$ is $\{0P_4, 21S_4\}$ -decomposable. By taking

$$K_7(4) = K_7(2) + K_7(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_7(\lambda) = \frac{\lambda}{4}K_7(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_7(\lambda) = K_7(\lambda - 1) + K_7 = \frac{\lambda - 1}{4}K_7(4) + K_7.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_7(\lambda) = K_7(\lambda - 2) + K_7(2) = \frac{\lambda - 2}{4}K_7(4) + K_7(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_7(\lambda) = K_7(\lambda - 3) + K_7(3) = \frac{\lambda - 3}{4}K_7(4) + K_7(3).$$

Therefore, $K_7(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 4. $n = 8$.

If $\lambda = 1$, then $(p, q) \in \{(0, 7), (4, 4), (8, 1)\}$. By Theorem 2.2, K_8 is $\{0P_4, 7S_4\}$ -decomposable. By Theorem 2.1, K_4 is $\{2P_4, 0S_4\}$ -decomposable. By Theorem 2.4, $K_{4,4}$ is $\{0P_4, 4S_4\}$ -decomposable, and by Theorem 2.3, $K_{4,3}$ is $\{4P_4, 0S_4\}$ -decomposable. By taking

$$K_8 = K_4 + K_4 + K_{4,4},$$

we obtain the decomposition when $(p, q) = (4, 4)$, and by taking

$$K_8 = K_5 + K_4 + K_{4,3},$$

we obtain the decomposition when $(p, q) = (8, 1)$.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 14), (4, 11), (8, 8), \dots, (16, 2)\}.$$

By taking

$$K_8(2) = K_8 + K_8,$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(0, 21), (4, 18), (8, 15), \dots, (28, 0)\}.$$

By Theorem 2.1, $K_8(3)$ is $\{28P_4, 0S_4\}$ -decomposable. By taking

$$K_8(3) = K_8(2) + K_8,$$

we obtain all the above possible decompositions.

If $\lambda \geq 4$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{3}$, then we write

$$K_8(\lambda) = \frac{\lambda}{3} K_8(3).$$

If $\lambda \equiv 1 \pmod{3}$, then we write

$$K_8(\lambda) = K_8(\lambda - 1) + K_8 = \frac{\lambda - 1}{3} K_8(3) + K_8.$$

If $\lambda \equiv 2 \pmod{3}$, then we write

$$K_8(\lambda) = K_8(\lambda - 2) + K_8(2) = \frac{\lambda - 2}{3} K_8(3) + K_8(2).$$

Therefore, $K_8(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 5. $n = 9$.

If $\lambda = 1$, then $(p, q) \in \{(0, 9), (4, 6), (8, 3), (12, 0)\}$. The graph $K_{1,8}$ is $\{0P_4, 2S_4\}$ -decomposable. By Theorem 2.1, K_9 is $\{12P_4, 0S_4\}$ -decomposable. By taking

$$K_9 = K_8 + K_{1,8},$$

we obtain all the other possible decompositions.

If $\lambda \geq 2$, then $K_9(\lambda)$ can be decomposed into λ copies of K_9 . Therefore, $K_9(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 6. $n = 10$.

If $\lambda = 1$, then $(p, q) \in \{(3, 9), (7, 6), (11, 3), (15, 0)\}$. The graph K_{10} can be decomposed into three copies of P_4 :

$$(x_4x_2x_3x_6), \quad (x_8x_6x_7x_5), \quad (x_3x_4x_6x_2),$$

and nine copies of S_4 :

$$\begin{aligned} &S(x_1; x_2, x_3, x_4, x_5), \quad S(x_1; x_6, x_7, x_8, x_9), \\ &S(x_5; x_2, x_3, x_4, x_6), \quad S(x_7; x_2, x_3, x_4, x_8), \\ &S(x_8; x_2, x_3, x_4, x_5), \quad S(x_9; x_2, x_3, x_4, x_{10}), \\ &S(x_9; x_5, x_6, x_7, x_8), \quad S(x_{10}; x_1, x_2, x_3, x_4), \\ &S(x_{10}; x_5, x_6, x_7, x_8). \end{aligned}$$

The graph K_4 is $\{2P_4, 0S_4\}$ -decomposable. By taking

$$K_{10} = K_6 + K_4 + K_{6,4},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 21), (6, 18), (10, 15), \dots, (30, 0)\}.$$

By Theorem 2.4, $K_{5,4}(2)$ is $\{0P_4, 10S_4\}$ -decomposable. By taking

$$K_{10}(2) = K_6(2) + K_5(2) + K_{5,4}(2),$$

we obtain the decomposition when $(p, q) = (2, 21)$. By taking

$$K_{10}(2) = K_{10} + K_{10},$$

we obtain all the other possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(1, 33), (5, 30), (9, 27), \dots, (45, 0)\}.$$

The graph $K_{10}(3)$ can be decomposed into one copy of P_4 :

$$(x_1x_5x_4x_2),$$

and thirty-three copies of S_4 :

$$\begin{aligned}
& S(x_1; x_2, x_3, x_4, x_5), & S(x_1; x_2, x_3, x_4, x_7), \\
& S(x_2; x_1, x_3, x_4, x_7), & S(x_3; x_2, x_4, x_5, x_6), \\
& S(x_3; x_2, x_4, x_5, x_6), & S(x_3; x_1, x_5, x_6, x_7), \\
& S(x_4; x_1, x_2, x_3, x_7), & S(x_5; x_1, x_2, x_6, x_7), \\
& S(x_5; x_2, x_4, x_6, x_7), & S(x_5; x_2, x_4, x_6, x_7), \\
& S(x_6; x_1, x_2, x_4, x_7), & S(x_6; x_1, x_2, x_4, x_7), \\
& S(x_6; x_1, x_2, x_4, x_7), & S(x_7; x_1, x_2, x_3, x_4), \\
& S(x_7; x_1, x_2, x_3, x_4), & S(x_8; x_4, x_5, x_6, x_7), \\
& S(x_8; x_4, x_5, x_6, x_7), & S(x_8; x_4, x_5, x_6, x_7), \\
& S(x_8; x_1, x_2, x_3, x_{10}), & S(x_8; x_1, x_2, x_3, x_{10}), \\
& S(x_8; x_1, x_2, x_3, x_{10}), & S(x_9; x_1, x_2, x_3, x_4), \\
& S(x_9; x_1, x_2, x_3, x_4), & S(x_9; x_1, x_2, x_3, x_4), \\
& S(x_9; x_5, x_6, x_7, x_8), & S(x_9; x_5, x_6, x_7, x_8), \\
& S(x_9; x_5, x_6, x_7, x_8), & S(x_{10}; x_1, x_2, x_3, x_4), \\
& S(x_{10}; x_1, x_2, x_3, x_4), & S(x_{10}; x_1, x_2, x_3, x_4), \\
& S(x_{10}; x_5, x_6, x_7, x_9), & S(x_{10}; x_5, x_6, x_7, x_9), \\
& S(x_{10}; x_5, x_6, x_7, x_9).
\end{aligned}$$

By taking

$$K_{10}(3) = K_{10}(2) + K_{10},$$

we obtain all the other possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 45), (4, 42), (8, 39), \dots, (60, 0)\}.$$

By Theorem 2.2, $K_{10}(4)$ is $\{0P_4, 45S_4\}$ -decomposable. By taking

$$K_{10}(4) = K_{10}(2) + K_{10}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{10}(\lambda) = \frac{\lambda}{4} K_{10}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{10}(\lambda) = K_{10}(\lambda - 1) + K_{10} = \frac{\lambda - 1}{4} K_{10}(4) + K_{10}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{10}(\lambda) = K_{10}(\lambda - 2) + K_{10}(2) = \frac{\lambda - 2}{4} K_{10}(4) + K_{10}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{10}(\lambda) = K_{10}(\lambda - 3) + K_{10}(3) = \frac{\lambda - 3}{4}K_{10}(4) + K_{10}(3).$$

Therefore, $K_{10}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 7. $n = 11$.

If $\lambda = 1$, then

$$(p, q) \in \{(1, 13), (5, 10), (9, 7), \dots, (17, 1)\}.$$

The graph K_{11} can be decomposed into one copy of P_4 :

$$(x_1x_{11}x_2x_{10}),$$

and thirteen copies of S_4 :

$$\begin{aligned} &S(x_1; x_2, x_3, x_4, x_5), & S(x_1; x_6, x_7, x_8, x_9), \\ &S(x_2; x_3, x_4, x_5, x_6), & S(x_4; x_3, x_5, x_6, x_7), \\ &S(x_5; x_3, x_6, x_7, x_8), & S(x_6; x_3, x_7, x_8, x_9), \\ &S(x_7; x_2, x_3, x_8, x_9), & S(x_8; x_2, x_3, x_4, x_9), \\ &S(x_9; x_2, x_3, x_4, x_5), & S(x_{10}; x_1, x_3, x_4, x_5), \\ &S(x_{10}; x_6, x_7, x_8, x_9), & S(x_{11}; x_3, x_4, x_5, x_6), \\ &S(x_{11}; x_7, x_8, x_9, x_{10}). \end{aligned}$$

By taking

$$K_{11} = K_7 + K_5 + K_{6,4},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 26), (6, 23), (10, 20), \dots, (34, 2)\}.$$

By taking

$$K_{11}(2) = K_{11} + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(3, 39), (7, 36), (11, 33), \dots, (55, 0)\}.$$

By Theorem 2.1, $K_{11}(3)$ is $\{55P_4, 0S_4\}$ -decomposable. By taking

$$K_{11}(3) = K_{11}(2) + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 55), (4, 52), (8, 49), \dots, (72, 1)\}.$$

By Theorem 2.2, $K_{11}(4)$ is $\{0P_4, 55S_4\}$ -decomposable. By taking

$$K_{11}(4) = K_{11}(3) + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 5$, then

$$(p, q) \in \{(1, 68), (5, 65), (9, 62), \dots, (89, 2)\}.$$

By taking

$$K_{11}(5) = K_{11}(4) + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 6$, then

$$(p, q) \in \{(2, 81), (6, 78), (10, 75), \dots, (110, 0)\}.$$

By Theorem 2.1, $K_{11}(6)$ is $\{110P_4, 0S_4\}$ -decomposable. By taking

$$K_{11}(6) = K_{11}(4) + K_{11}(2),$$

we obtain all the above possible decompositions.

If $\lambda = 7$, then

$$(p, q) \in \{(3, 94), (7, 91), (11, 88), \dots, (127, 1)\}.$$

By taking

$$K_{11}(7) = K_{11}(6) + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 8$, then

$$(p, q) \in \{(0, 110), (4, 107), (8, 104), \dots, (144, 2)\}.$$

By taking

$$K_{11}(8) = K_{11}(4) + K_{11}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 9$, then

$$(p, q) \in \{(1, 123), (5, 120), (9, 117), \dots, (165, 0)\}.$$

By Theorem 2.1, $K_{11}(9)$ is $\{165P_4, 0S_4\}$ -decomposable. By taking

$$K_{11}(9) = K_{11}(8) + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 10$, then

$$(p, q) \in \{(2, 136), (6, 133), (10, 130), \dots, (182, 1)\}.$$

By taking

$$K_{11}(10) = K_{11}(6) + K_{11}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 11$, then

$$(p, q) \in \{(3, 149), (7, 146), (11, 143), \dots, (199, 2)\}.$$

By taking

$$K_{11}(11) = K_{11}(10) + K_{11},$$

we obtain all the above possible decompositions.

If $\lambda = 12$, then

$$(p, q) \in \{(0, 165), (4, 162), (8, 159), \dots, (220, 0)\}.$$

By Theorem 2.1, $K_{11}(12)$ is $\{220P_4, 0S_4\}$ -decomposable. By taking

$$K_{11}(12) = K_{11}(8) + K_{11}(4),$$

we obtain all the above possible decompositions.

If $\lambda \geq 13$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{12}$, then we write

$$K_{11}(\lambda) = \frac{\lambda}{12} K_{11}(12).$$

If $\lambda \equiv 1 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 1) + K_{11} = \frac{\lambda - 1}{12} K_{11}(12) + K_{11}.$$

If $\lambda \equiv 2 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 2) + K_{11}(2) = \frac{\lambda - 2}{12} K_{11}(12) + K_{11}(2).$$

If $\lambda \equiv 3 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 3) + K_{11}(3) = \frac{\lambda - 3}{12} K_{11}(12) + K_{11}(3).$$

If $\lambda \equiv 4 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 4) + K_{11}(4) = \frac{\lambda - 4}{12} K_{11}(12) + K_{11}(4).$$

If $\lambda \equiv 5 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 5) + K_{11}(5) = \frac{\lambda - 5}{12} K_{11}(12) + K_{11}(5).$$

If $\lambda \equiv 6 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 6) + K_{11}(6) = \frac{\lambda - 6}{12} K_{11}(12) + K_{11}(6).$$

If $\lambda \equiv 7 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 7) + K_{11}(7) = \frac{\lambda - 7}{12} K_{11}(12) + K_{11}(7).$$

If $\lambda \equiv 8 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 8) + K_{11}(8) = \frac{\lambda - 8}{12}K_{11}(12) + K_{11}(8).$$

If $\lambda \equiv 9 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 9) + K_{11}(9) = \frac{\lambda - 9}{12}K_{11}(12) + K_{11}(9).$$

If $\lambda \equiv 10 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 10) + K_{11}(10) = \frac{\lambda - 10}{12}K_{11}(12) + K_{11}(10).$$

If $\lambda \equiv 11 \pmod{12}$, then we write

$$K_{11}(\lambda) = K_{11}(\lambda - 11) + K_{11}(11) = \frac{\lambda - 11}{12}K_{11}(12) + K_{11}(11).$$

Therefore, $K_{11}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 8. $n = 12$.

If $\lambda = 1$, then

$$(p, q) \in \{(2, 15), (6, 12), (10, 9), \dots, (22, 0)\}.$$

By Theorem 2.1, K_{12} is $\{22P_4, 0S_4\}$ -decomposable, and by Theorem 2.4, $K_{7,4}$ is $\{0P_4, 7S_4\}$ -decomposable. The graph $K_{1,4}$ is $\{0P_4, 1S_4\}$ -decomposable. By taking

$$K_{12} = K_8 + K_5 + K_{6,4} + K_{1,4},$$

we obtain the decompositions when $(p, q) \in \{(14, 6), (18, 3)\}$. By taking

$$K_{12} = K_8 + K_5 + K_{7,4},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 33), (4, 30), (8, 27), \dots, (44, 0)\}.$$

By Theorem 2.2, $K_{12}(2)$ is $\{0P_4, 33S_4\}$ -decomposable. By taking

$$K_{12}(2) = K_{12} + K_{12},$$

we obtain all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{2}$, then we write

$$K_{12}(\lambda) = \frac{\lambda}{2}K_{12}(2).$$

If $\lambda \equiv 1 \pmod{2}$, then we write

$$K_{12}(\lambda) = K_{12}(\lambda - 1) + K_{12} = \frac{\lambda - 1}{2}K_{12}(2) + K_{12}.$$

Therefore, $K_{12}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 9. $n = 13$.

If $\lambda = 1$, then

$$(p, q) \in \{(2, 18), (6, 15), (10, 12), \dots, (26, 0)\}.$$

By Theorem 2.1, K_{13} is $\{26P_4, 0S_4\}$ -decomposable. The graph $K_{1,12}$ is $\{0P_4, 3S_4\}$ - decomposable. By taking

$$K_{13} = K_{12} + K_{1,12},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 39), (4, 36), (8, 33), \dots, (52, 0)\}.$$

By Theorem 2.2, $K_{13}(2)$ is $\{0P_4, 39S_4\}$ -decomposable. By taking

$$K_{13}(2) = K_{13} + K_{13},$$

we obtain all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{2}$, then we write

$$K_{13}(\lambda) = \frac{\lambda}{2}K_{13}(2).$$

If $\lambda \equiv 1 \pmod{2}$, then we write

$$K_{13}(\lambda) = K_{13}(\lambda - 1) + K_{13} = \frac{\lambda - 1}{2}K_{13}(2) + K_{13}.$$

Therefore, $K_{13}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 10. $n = 14$.

If $\lambda = 1$, then

$$(p, q) \in \{(1, 22), (5, 19), (9, 16), \dots, (29, 1)\}.$$

The graph K_{14} can be decomposed into one copy of P_4 :

$$(x_5x_3x_2x_{13}),$$

and twenty-two copies of S_4 :

$$\begin{aligned} &S(x_1; x_2, x_3, x_7, x_{13}), & S(x_4; x_1, x_2, x_3, x_5), \\ &S(x_5; x_1, x_2, x_6, x_7), & S(x_6; x_1, x_2, x_3, x_4), \\ &S(x_7; x_2, x_3, x_4, x_6), & S(x_8; x_1, x_2, x_3, x_9), \\ &S(x_8; x_4, x_5, x_6, x_7), & S(x_9; x_1, x_2, x_3, x_{11}), \\ &S(x_9; x_4, x_5, x_6, x_7), & S(x_{10}; x_4, x_5, x_6, x_7), \\ &S(x_{10}; x_1, x_2, x_3, x_{11}), & S(x_{10}; x_8, x_9, x_{13}, x_{14}), \\ &S(x_{11}; x_5, x_6, x_7, x_8), & S(x_{11}; x_1, x_2, x_3, x_4), \\ &S(x_{12}; x_1, x_2, x_3, x_{13}), & S(x_{12}; x_4, x_5, x_6, x_7), \\ &S(x_{12}; x_8, x_9, x_{10}, x_{11}), & S(x_{13}; x_3, x_5, x_6, x_7), \\ &S(x_{13}; x_4, x_8, x_9, x_{11}), & S(x_{14}; x_2, x_3, x_4, x_{13}), \\ &S(x_{14}; x_5, x_6, x_7, x_8), & S(x_{14}; x_1, x_9, x_{11}, x_{12}). \end{aligned}$$

By taking

$$K_{14} = K_8 + K_6 + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 44), (6, 41), (10, 38), \dots, (58, 2)\}.$$

By taking

$$K_{14}(2) = K_{14} + K_{14},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(3, 66), (7, 63), (11, 60), \dots, (91, 0)\}.$$

By Theorem 2.1, $K_{14}(3)$ is $\{91P_4, 0S_4\}$ -decomposable. By taking

$$K_{14}(3) = K_{14}(2) + K_{14},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 91), (4, 88), (8, 85), \dots, (120, 1)\}.$$

By Theorem 2.2, $K_{14}(4)$ is $\{0P_4, 91S_4\}$ -decomposable. By taking

$$K_{14}(4) = K_{14}(3) + K_{14},$$

we obtain all the above possible decompositions.

If $\lambda = 5$, then

$$(p, q) \in \{(1, 113), (5, 110), (9, 107), \dots, (149, 2)\}.$$

By taking

$$K_{14}(5) = K_{14}(4) + K_{14},$$

we obtain all the above possible decompositions.

If $\lambda = 6$, then

$$(p, q) \in \{(2, 135), (6, 132), (10, 129), \dots, (182, 0)\}.$$

By Theorem 2.1, $K_{14}(6)$ is $\{182P_4, 0S_4\}$ -decomposable. By taking

$$K_{14}(6) = K_{14}(4) + K_{14}(2),$$

we obtain all the above possible decompositions.

If $\lambda = 7$, then

$$(p, q) \in \{(3, 157), (7, 154), (11, 151), \dots, (211, 1)\}.$$

By taking

$$K_{14}(7) = K_{14}(4) + K_{14}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 8$, then

$$(p, q) \in \{(0, 182), (4, 179), (8, 176), \dots, (240, 2)\}.$$

By taking

$$K_{14}(8) = K_{14}(4) + K_{14}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 9$, then

$$(p, q) \in \{(1, 204), (5, 201), (9, 198), \dots, (273, 0)\}.$$

By Theorem 2.1, $K_{14}(9)$ is $\{273P_4, 0S_4\}$ -decomposable. By taking

$$K_{14}(9) = K_{14}(8) + K_{14},$$

we obtain all the above possible decompositions.

If $\lambda = 10$, then

$$(p, q) \in \{(2, 226), (6, 223), (10, 220), \dots, (302, 1)\}.$$

By taking

$$K_{14}(10) = K_{14}(6) + K_{14}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 11$, then

$$(p, q) \in \{(3, 248), (7, 245), (11, 242), \dots, (331, 2)\}.$$

By taking

$$K_{14}(11) = K_{14}(8) + K_{14}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 12$, then

$$(p, q) \in \{(0, 273), (4, 270), (8, 267), \dots, (364, 0)\}.$$

By Theorem 2.1, $K_{14}(12)$ is $\{364P_4, 0S_4\}$ -decomposable. By taking

$$K_{14}(12) = K_{14}(8) + K_{14}(4),$$

we obtain all the above possible decompositions.

If $\lambda \geq 13$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{12}$, then we write

$$K_{14}(\lambda) = \frac{\lambda}{12} K_{14}(12).$$

If $\lambda \equiv 1 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 1) + K_{14} = \frac{\lambda - 1}{12} K_{14}(12) + K_{14}.$$

If $\lambda \equiv 2 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 2) + K_{14}(2) = \frac{\lambda - 2}{12}K_{14}(12) + K_{14}(2).$$

If $\lambda \equiv 3 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 3) + K_{14}(3) = \frac{\lambda - 3}{12}K_{14}(12) + K_{14}(3).$$

If $\lambda \equiv 4 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 4) + K_{14}(4) = \frac{\lambda - 4}{12}K_{14}(12) + K_{14}(4).$$

If $\lambda \equiv 5 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 5) + K_{14}(5) = \frac{\lambda - 5}{12}K_{14}(12) + K_{14}(5).$$

If $\lambda \equiv 6 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 6) + K_{14}(6) = \frac{\lambda - 6}{12}K_{14}(12) + K_{14}(6).$$

If $\lambda \equiv 7 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 7) + K_{14}(7) = \frac{\lambda - 7}{12}K_{14}(12) + K_{14}(7).$$

If $\lambda \equiv 8 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 8) + K_{14}(8) = \frac{\lambda - 8}{12}K_{14}(12) + K_{14}(8).$$

If $\lambda \equiv 9 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 9) + K_{14}(9) = \frac{\lambda - 9}{12}K_{14}(12) + K_{14}(9).$$

If $\lambda \equiv 10 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 10) + K_{14}(10) = \frac{\lambda - 10}{12}K_{14}(12) + K_{14}(10).$$

If $\lambda \equiv 11 \pmod{12}$, then we write

$$K_{14}(\lambda) = K_{14}(\lambda - 11) + K_{14}(11) = \frac{\lambda - 11}{12}K_{14}(12) + K_{14}(11).$$

Therefore, $K_{14}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 11. $n = 15$.

If $\lambda = 1$, then

$$(p, q) \in \{(3, 24), (7, 21), (11, 18), \dots, (35, 0)\}.$$

By taking

$$K_{15} = K_9 + K_7 + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 51), (6, 48), (10, 45), \dots, (70, 0)\}.$$

By taking

$$K_{15}(2) = K_9(2) + K_7(2) + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(1, 78), (5, 75), (9, 72), \dots, (105, 0)\}.$$

By taking

$$K_{15}(3) = K_9(3) + K_7(3) + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 105), (4, 102), (8, 99), \dots, (140, 0)\}.$$

By Theorem 2.2, $K_{15}(4)$ is $\{0P_4, 105S_4\}$ -decomposable. By taking

$$K_{15}(4) = K_{15}(2) + K_{15}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{15}(\lambda) = \frac{\lambda}{4} K_{15}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{15}(\lambda) = K_{15}(\lambda - 1) + K_{15} = \frac{\lambda - 1}{4} K_{15}(4) + K_{15}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{15}(\lambda) = K_{15}(\lambda - 2) + K_{15}(2) = \frac{\lambda - 2}{4} K_{15}(4) + K_{15}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{15}(\lambda) = K_{15}(\lambda - 3) + K_{15}(3) = \frac{\lambda - 3}{4} K_{15}(4) + K_{15}(3).$$

Therefore, $K_{15}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 12. $n = 16$.

If $\lambda = 1$, then

$$(p, q) \in \{(0, 30), (4, 27), (8, 24), \dots, (40, 0)\}.$$

By Theorem 2.2, K_{16} is $\{0P_4, 30S_4\}$ -decomposable, and by Theorem 2.4, $K_{2,8}$ is $\{0P_4, 4S_4\}$ -decomposable. By taking

$$K_{16} = K_8 + K_8 + K_{6,4} + K_{6,4} + K_{2,8},$$

we obtain the decomposition when $(p, q) = (4, 27)$. By Theorems 2.3 and 2.4, $K_{6,6}$ is both $\{12P_4, 0S_4\}$ -decomposable and $\{0P_4, 9S_4\}$ -decomposable. By taking

$$K_{16} = K_{10} + K_6 + K_{4,6} + K_{6,6},$$

we obtain all the above possible decompositions.

If $\lambda \geq 2$, then $K_{16}(\lambda)$ can be decomposed into λ copies of K_{16} . Therefore, $K_{16}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 13. $n = 17$.

If $\lambda = 1$, then

$$(p, q) \in \{(0, 34), (4, 31), (8, 28), \dots, (44, 1)\}.$$

By Theorem 2.4, $K_{2,8}$ is $\{0P_4, 4S_4\}$ -decomposable, and by Theorems 2.3 and 2.4, $K_{6,6}$ is both $\{12P_4, 0S_4\}$ -decomposable and $\{0P_4, 9S_4\}$ -decomposable. By taking

$$K_{17} = K_{11} + K_7 + K_{4,6} + K_{6,6},$$

we obtain the decomposition when $(p, q) = (44, 1)$, and by taking

$$K_{17} = K_9 + K_9 + K_{6,4} + K_{6,4} + K_{2,8},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 68), (4, 65), (8, 62), \dots, (88, 2)\}.$$

By taking

$$K_{17}(2) = K_{17} + K_{17},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(0, 102), (4, 99), (8, 96), \dots, (136, 0)\}.$$

By Theorem 2.1, $K_{17}(3)$ is $\{136P_4, 0S_4\}$ -decomposable. By taking

$$K_{17}(3) = K_{17}(2) + K_{17},$$

we obtain all the above possible decompositions.

If $\lambda \geq 4$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{3}$, then we write

$$K_{17}(\lambda) = \frac{\lambda}{3} K_{17}(3).$$

If $\lambda \equiv 1 \pmod{3}$, then we write

$$K_{17}(\lambda) = K_{17}(\lambda - 1) + K_{17} = \frac{\lambda - 1}{3} K_{17}(3) + K_{17}.$$

If $\lambda \equiv 2 \pmod{3}$, then we write

$$K_{17}(\lambda) = K_{17}(\lambda - 2) + K_{17}(2) = \frac{\lambda - 2}{3}K_{17}(3) + K_{17}(2).$$

Therefore, $K_{17}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 14. $n = 18$.

If $\lambda = 1$, then

$$(p, q) \in \{(3, 36), (7, 33), (11, 30), \dots, (51, 0)\}.$$

By Theorem 2.4, $K_{10,8}$ is $\{0P_4, 20S_4\}$ -decomposable. By taking

$$K_{18} = K_{10} + K_8 + K_{10,8},$$

we obtain the decomposition when $(p, q) = (3, 36)$, and by taking

$$K_{18} = K_{12} + K_6 + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 75), (6, 72), (10, 69), \dots, (102, 0)\}.$$

By taking

$$K_{18}(2) = K_{12}(2) + K_6(2) + K_{4,6} + K_{4,6} + K_{4,6} \\ + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(1, 114), (5, 111), (9, 108), \dots, (153, 0)\}.$$

By Theorem 2.4, $K_{4,8}$ is $\{0P_4, 8S_4\}$ -decomposable. By taking

$$K_{18}(3) = K_{10}(3) + K_8(3) + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} \\ + K_{6,4} + K_{6,4} + K_{4,8} + K_{4,8} + K_{4,8},$$

we obtain the decomposition when $(p, q) = (1, 114)$, and by taking

$$K_{18}(3) = K_{12}(3) + K_6(3) + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} \\ + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the other possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 153), (4, 150), (8, 147), \dots, (204, 0)\}.$$

By Theorem 2.2, $K_{18}(4)$ is $\{0P_4, 153S_4\}$ -decomposable. By taking

$$K_{18}(4) = K_{18}(2) + K_{18}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{18}(\lambda) = \frac{\lambda}{4}K_{18}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{18}(\lambda) = K_{18}(\lambda - 1) + K_{18} = \frac{\lambda - 1}{4}K_{18}(4) + K_{18}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{18}(\lambda) = K_{18}(\lambda - 2) + K_{18}(2) = \frac{\lambda - 2}{4}K_{18}(4) + K_{18}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{18}(\lambda) = K_{18}(\lambda - 3) + K_{18}(3) = \frac{\lambda - 3}{4}K_{18}(4) + K_{18}(3).$$

Therefore, $K_{18}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 15. $n = 19$.

If $\lambda = 1$, then

$$(p, q) \in \{(1, 42), (5, 39), (9, 36), \dots, (57, 0)\}.$$

By Theorem 2.4, $K_{4,8}$ is $\{0P_4, 8S_4\}$ -decomposable. By taking

$$K_{19} = K_{11} + K_9 + K_{6,4} + K_{6,4} + K_{4,8},$$

we obtain the decomposition when $(p, q) = (1, 42)$, and by taking

$$K_{19} = K_{13} + K_7 + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the other possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 84), (6, 81), (10, 78), \dots, (114, 0)\}.$$

By taking

$$K_{19}(2) = K_{19} + K_{19},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(3, 126), (7, 123), (11, 120), \dots, (171, 0)\}.$$

By taking

$$K_{19}(3) = K_{19}(2) + K_{19},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 171), (4, 168), (8, 165), \dots, (228, 0)\}.$$

By Theorem 2.2, $K_{19}(4)$ is $\{0P_4, 171S_4\}$ -decomposable. By taking

$$K_{19}(4) = K_{19}(2) + K_{19}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{19}(\lambda) = \frac{\lambda}{4}K_{19}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{19}(\lambda) = K_{19}(\lambda - 1) + K_{19} = \frac{\lambda - 1}{4}K_{19}(4) + K_{19}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{19}(\lambda) = K_{19}(\lambda - 2) + K_{19}(2) = \frac{\lambda - 2}{4}K_{19}(4) + K_{19}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{19}(\lambda) = K_{19}(\lambda - 3) + K_{19}(3) = \frac{\lambda - 3}{4}K_{19}(4) + K_{19}(3).$$

Therefore, $K_{19}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 16. $n = 20$.

If $\lambda = 1$, then

$$(p, q) \in \{(2, 46), (6, 43), (10, 40), \dots, (62, 1)\}.$$

By taking

$$K_{20} = K_{12} + K_8 + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 95), (4, 92), (8, 89), \dots, (124, 2)\}.$$

By Theorem 2.2, $K_{20}(2)$ is $\{0P_4, 95S_4\}$ -decomposable. By taking

$$K_{20}(2) = K_{20} + K_{20},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(2, 141), (6, 138), (10, 135), \dots, (190, 0)\}.$$

By Theorem 2.1, $K_{20}(3)$ is $\{190P_4, 0S_4\}$ -decomposable. By taking

$$K_{20}(3) = K_{20}(2) + K_{20},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 190), (4, 187), (8, 184), \dots, (252, 1)\}.$$

By Theorem 2.2, $K_{20}(4)$ is $\{0P_4, 190S_4\}$ -decomposable. By taking

$$K_{20}(4) = K_{20}(3) + K_{20},$$

we obtain all the above possible decompositions.

If $\lambda = 5$, then

$$(p, q) \in \{(2, 236), (6, 233), (10, 230), \dots, (314, 2)\}.$$

By taking

$$K_{20}(5) = K_{20}(3) + K_{20}(2),$$

we obtain all the above possible decompositions.

If $\lambda = 6$, then

$$(p, q) \in \{(0, 285), (4, 282), (8, 279), \dots, (380, 0)\}.$$

By Theorem 2.2, $K_{20}(6)$ is $\{0P_4, 285S_4\}$ -decomposable. By taking

$$K_{20}(6) = K_{20}(3) + K_{20}(3),$$

we obtain all the above possible decompositions.

If $\lambda \geq 7$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{6}$, then we write

$$K_{20}(\lambda) = \frac{\lambda}{6} K_{20}(6).$$

If $\lambda \equiv 1 \pmod{6}$, then we write

$$K_{20}(\lambda) = K_{20}(\lambda - 1) + K_{20} = \frac{\lambda - 1}{6} K_{20}(6) + K_{20}.$$

If $\lambda \equiv 2 \pmod{6}$, then we write

$$K_{20}(\lambda) = K_{20}(\lambda - 2) + K_{20}(2) = \frac{\lambda - 2}{6} K_{20}(6) + K_{20}(2).$$

If $\lambda \equiv 3 \pmod{6}$, then we write

$$K_{20}(\lambda) = K_{20}(\lambda - 3) + K_{20}(3) = \frac{\lambda - 3}{6} K_{20}(6) + K_{20}(3).$$

If $\lambda \equiv 4 \pmod{6}$, then we write

$$K_{20}(\lambda) = K_{20}(\lambda - 4) + K_{20}(4) = \frac{\lambda - 4}{6}K_{20}(6) + K_{20}(4).$$

If $\lambda \equiv 5 \pmod{6}$, then we write

$$K_{20}(\lambda) = K_{20}(\lambda - 5) + K_{20}(5) = \frac{\lambda - 5}{6}K_{20}(6) + K_{20}(5).$$

Therefore, $K_{20}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 17. $n = 21$.

If $\lambda = 1$, then

$$(p, q) \in \{(2, 51), (6, 48), (10, 45), \dots, (70, 0)\}.$$

By taking

$$K_{21} = K_{13} + K_9 + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 105), (4, 102), (8, 99), \dots, (140, 0)\}.$$

By Theorem 2.2, $K_{21}(2)$ is $\{0P_4, 105S_4\}$ -decomposable. By taking

$$K_{21}(2) = K_{21} + K_{21},$$

we obtain all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{2}$, then we write

$$K_{21}(\lambda) = \frac{\lambda}{2}K_{21}(2).$$

If $\lambda \equiv 1 \pmod{2}$, then we write

$$K_{21}(\lambda) = K_{21}(\lambda - 1) + K_{21} = \frac{\lambda - 1}{2}K_{21}(2) + K_{21}.$$

Therefore, $K_{21}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 18. $n = 22$.

If $\lambda = 1$, then

$$(p, q) \in \{(1, 57), (5, 54), (9, 51), \dots, (77, 0)\}.$$

By Theorem 2.4, $K_{14,8}$ is $\{0P_4, 28S_4\}$ -decomposable. By taking

$$K_{22} = K_{14} + K_8 + K_{14,8},$$

we obtain the decomposition when $(p, q) = (1, 57)$. By taking

$$K_{22} = K_{16} + K_6 + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 114), (6, 111), (10, 108), \dots, (154, 0)\}.$$

By taking

$$K_{22}(2) = K_{22} + K_{22},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(3, 171), (7, 168), (11, 165), \dots, (231, 0)\}.$$

By taking

$$K_{22}(3) = K_{22}(2) + K_{22},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 231), (4, 228), (8, 225), \dots, (308, 0)\}.$$

By Theorem 2.2, $K_{22}(4)$ is $\{0P_4, 231S_4\}$ -decomposable. By taking

$$K_{22}(4) = K_{22}(2) + K_{22}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{22}(\lambda) = \frac{\lambda}{4} K_{22}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{22}(\lambda) = K_{22}(\lambda - 1) + K_{22} = \frac{\lambda - 1}{4} K_{22}(4) + K_{22}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{22}(\lambda) = K_{22}(\lambda - 2) + K_{22}(2) = \frac{\lambda - 2}{4} K_{22}(4) + K_{22}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{22}(\lambda) = K_{22}(\lambda - 3) + K_{22}(3) = \frac{\lambda - 3}{4} K_{22}(4) + K_{22}(3).$$

Therefore, $K_{22}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 19. $n = 23$.

If $\lambda = 1$, then

$$(p, q) \in \{(3, 61), (7, 58), (11, 55), \dots, (83, 1)\}.$$

By taking

$$K_{23} = K_{17} + K_7 + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 125), (6, 122), (10, 119), \dots, (166, 2)\}.$$

By taking

$$\begin{aligned} K_{23}(2) = & K_{17}(2) + K_7(2) + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} \\ & + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6}, \end{aligned}$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(1, 189), (5, 186), (9, 183), \dots, (253, 0)\}.$$

By taking

$$\begin{aligned} K_{23}(3) = & K_{17}(3) + K_7(3) + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} \\ & + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} \\ & + K_{4,6} + K_{4,6}, \end{aligned}$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 253), (4, 250), (8, 247), \dots, (336, 1)\}.$$

By Theorem 2.2, $K_{23}(4)$ is $\{0P_4, 253S_4\}$ -decomposable. By taking

$$K_{23}(4) = K_{23}(3) + K_{23},$$

we obtain all the above possible decompositions.

If $\lambda = 5$, then

$$(p, q) \in \{(3, 314), (7, 311), (11, 308), \dots, (419, 2)\}.$$

By taking

$$K_{23}(5) = K_{23}(3) + K_{23}(2),$$

we obtain all the above possible decompositions.

If $\lambda = 6$, then

$$(p, q) \in \{(2, 378), (6, 375), (10, 372), \dots, (506, 0)\}.$$

By taking

$$K_{23}(6) = K_{23}(3) + K_{23}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 7$, then

$$(p, q) \in \{(1, 442), (5, 439), (9, 436), \dots, (589, 1)\}.$$

By taking

$$K_{23}(7) = K_{23}(4) + K_{23}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 8$, then

$$(p, q) \in \{(0, 506), (4, 503), (8, 500), \dots, (672, 2)\}.$$

By taking

$$K_{23}(8) = K_{23}(4) + K_{23}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 9$, then

$$(p, q) \in \{(3, 567), (7, 564), (11, 561), \dots, (759, 0)\}.$$

By taking

$$K_{23}(9) = K_{23}(6) + K_{23}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 10$, then

$$(p, q) \in \{(2, 631), (6, 628), (10, 625), \dots, (842, 1)\}.$$

By taking

$$K_{23}(10) = K_{23}(6) + K_{23}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 11$, then

$$(p, q) \in \{(1, 695), (5, 692), (9, 689), \dots, (925, 2)\}.$$

By taking

$$K_{23}(11) = K_{23}(8) + K_{23}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 12$, then

$$(p, q) \in \{(0, 759), (4, 756), (8, 753), \dots, (1012, 0)\}.$$

By Theorem 2.2, $K_{23}(12)$ is $\{0P_4, 759S_4\}$ -decomposable. By taking

$$K_{23}(12) = K_{23}(6) + K_{23}(6),$$

we obtain all the above possible decompositions.

If $\lambda \geq 13$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{12}$, then we write

$$K_{23}(\lambda) = \frac{\lambda}{12} K_{23}(12).$$

If $\lambda \equiv 1 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 1) + K_{23} = \frac{\lambda - 1}{12} K_{23}(12) + K_{23}.$$

If $\lambda \equiv 2 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 2) + K_{23}(2) = \frac{\lambda - 2}{12} K_{23}(12) + K_{23}(2).$$

If $\lambda \equiv 3 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 3) + K_{23}(3) = \frac{\lambda - 3}{12} K_{23}(12) + K_{23}(3).$$

If $\lambda \equiv 4 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 4) + K_{23}(4) = \frac{\lambda - 4}{12} K_{23}(12) + K_{23}(4).$$

If $\lambda \equiv 5 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 5) + K_{23}(5) = \frac{\lambda - 5}{12} K_{23}(12) + K_{23}(5).$$

If $\lambda \equiv 6 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 6) + K_{23}(6) = \frac{\lambda - 6}{12} K_{23}(12) + K_{23}(6).$$

If $\lambda \equiv 7 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 7) + K_{23}(7) = \frac{\lambda - 7}{12} K_{23}(12) + K_{23}(7).$$

If $\lambda \equiv 8 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 8) + K_{23}(8) = \frac{\lambda - 8}{12} K_{23}(12) + K_{23}(8).$$

If $\lambda \equiv 9 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 9) + K_{23}(9) = \frac{\lambda - 9}{12} K_{23}(12) + K_{23}(9).$$

If $\lambda \equiv 10 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 10) + K_{23}(10) = \frac{\lambda - 10}{12} K_{23}(12) + K_{23}(10).$$

If $\lambda \equiv 11 \pmod{12}$, then we write

$$K_{23}(\lambda) = K_{23}(\lambda - 11) + K_{23}(11) = \frac{\lambda - 11}{12} K_{23}(12) + K_{23}(11).$$

Therefore, $K_{23}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 20. $n = 24$.

If $\lambda = 1$, then

$$(p, q) \in \{(0, 69), (4, 66), (8, 63), \dots, (92, 0)\}.$$

By Theorem 2.2, K_{24} is $\{0P_4, 69S_4\}$ -decomposable. By taking

$$K_{24} = K_{12} + K_{12} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda \geq 2$, then $K_{24}(\lambda)$ can be decomposed into λ copies of K_{24} . Therefore, $K_{24}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 21. $n = 25$.

If $\lambda = 1$, then

$$(p, q) \in \{(0, 75), (4, 72), (8, 69), \dots, (100, 0)\}.$$

By Theorem 2.2, K_{25} is $\{0P_4, 75S_4\}$ -decomposable. By taking

$$K_{25} = K_{13} + K_{13} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda \geq 2$, then $K_{25}(\lambda)$ can be decomposed into λ copies of K_{25} . Therefore, $K_{25}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 22. $n = 26$.

If $\lambda = 1$, then

$$(p, q) \in \{(3, 79), (7, 76), (11, 73), \dots, (107, 1)\}.$$

By taking

$$K_{26} = K_{18} + K_8 + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 161), (6, 158), (10, 155), \dots, (214, 2)\}.$$

By taking

$$\begin{aligned} K_{26}(2) &= K_{18}(2) + K_8(2) + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} \\ &\quad + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} \\ &\quad + K_{6,4} + K_{6,4}, \end{aligned}$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(1, 243), (5, 240), (9, 237), \dots, (325, 0)\}.$$

By taking

$$\begin{aligned} K_{26}(3) &= K_{18}(3) + K_8(3) + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} \\ &\quad + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} \\ &\quad + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} \\ &\quad + K_{6,4} + K_{6,4}, \end{aligned}$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 325), (4, 322), (8, 319), \dots, (432, 1)\}.$$

By Theorem 2.2, $K_{26}(4)$ is $\{0P_4, 325S_4\}$ -decomposable. By taking

$$K_{26}(4) = K_{26}(3) + K_{26},$$

we obtain all the above possible decompositions.

If $\lambda = 5$, then

$$(p, q) \in \{(3, 404), (7, 401), (11, 398), \dots, (539, 2)\}.$$

By taking

$$K_{26}(5) = K_{26}(4) + K_{26},$$

we obtain all the above possible decompositions.

If $\lambda = 6$, then

$$(p, q) \in \{(2, 486), (6, 483), (10, 480), \dots, (650, 0)\}.$$

By taking

$$K_{26}(6) = K_{26}(3) + K_{26}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 7$, then

$$(p, q) \in \{(1, 568), (5, 565), (9, 562), \dots, (757, 1)\}.$$

By taking

$$K_{26}(7) = K_{26}(4) + K_{26}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 8$, then

$$(p, q) \in \{(0, 650), (4, 647), (8, 644), \dots, (864, 2)\}.$$

By taking

$$K_{26}(8) = K_{26}(4) + K_{26}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 9$, then

$$(p, q) \in \{(3, 729), (7, 726), (11, 723), \dots, (975, 0)\}.$$

By taking

$$K_{26}(9) = K_{26}(6) + K_{26}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 10$, then

$$(p, q) \in \{(2, 811), (6, 808), (10, 805), \dots, (1082, 1)\}.$$

By taking

$$K_{26}(10) = K_{26}(6) + K_{26}(4),$$

we obtain all the above possible decompositions.

If $\lambda = 11$, then

$$(p, q) \in \{(1, 893), (5, 890), (9, 887), \dots, (1189, 2)\}.$$

By taking

$$K_{26}(11) = K_{26}(8) + K_{26}(3),$$

we obtain all the above possible decompositions.

If $\lambda = 12$, then

$$(p, q) \in \{(0, 975), (4, 972), (8, 969), \dots, (1300, 0)\}.$$

By Theorem 2.1, $K_{26}(12)$ is $\{1300P_4, 0S_4\}$ -decomposable. By taking

$$K_{26}(12) = K_{26}(8) + K_{26}(4),$$

we obtain all the above possible decompositions.

If $\lambda \geq 13$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{12}$, then we write

$$K_{26}(\lambda) = \frac{\lambda}{12} K_{26}(12).$$

If $\lambda \equiv 1 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 1) + K_{26} = \frac{\lambda - 1}{12} K_{26}(12) + K_{26}.$$

If $\lambda \equiv 2 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 2) + K_{26}(2) = \frac{\lambda - 2}{12} K_{26}(12) + K_{26}(2).$$

If $\lambda \equiv 3 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 3) + K_{26}(3) = \frac{\lambda - 3}{12} K_{26}(12) + K_{26}(3).$$

If $\lambda \equiv 4 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 4) + K_{26}(4) = \frac{\lambda - 4}{12} K_{26}(12) + K_{26}(4).$$

If $\lambda \equiv 5 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 5) + K_{26}(5) = \frac{\lambda - 5}{12} K_{26}(12) + K_{26}(5).$$

If $\lambda \equiv 6 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 6) + K_{26}(6) = \frac{\lambda - 6}{12}K_{26}(12) + K_{26}(6).$$

If $\lambda \equiv 7 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 7) + K_{26}(7) = \frac{\lambda - 7}{12}K_{26}(12) + K_{26}(7).$$

If $\lambda \equiv 8 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 8) + K_{26}(8) = \frac{\lambda - 8}{12}K_{26}(12) + K_{26}(8).$$

If $\lambda \equiv 9 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 9) + K_{26}(9) = \frac{\lambda - 9}{12}K_{26}(12) + K_{26}(9).$$

If $\lambda \equiv 10 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 10) + K_{26}(10) = \frac{\lambda - 10}{12}K_{26}(12) + K_{26}(10).$$

If $\lambda \equiv 11 \pmod{12}$, then we write

$$K_{26}(\lambda) = K_{26}(\lambda - 11) + K_{26}(11) = \frac{\lambda - 11}{12}K_{26}(12) + K_{26}(11).$$

Therefore, $K_{26}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 23. $n = 27$.

If $\lambda = 1$, then

$$(p, q) \in \{(1, 87), (5, 84), (9, 81), \dots, (117, 0)\}.$$

By taking

$$K_{27} = K_{19} + K_9 + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4} + K_{6,4},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(2, 174), (6, 171), (10, 168), \dots, (234, 0)\}.$$

By taking

$$K_{27}(2) = K_{27} + K_{27},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(3, 261), (7, 258), (11, 255), \dots, (351, 0)\}.$$

By taking

$$K_{27}(3) = K_{27}(2) + K_{27},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(0, 351), (4, 348), (8, 345), \dots, (468, 0)\}.$$

By Theorem 2.2, $K_{27}(4)$ is $\{0P_4, 351S_4\}$ -decomposable. By taking

$$K_{27}(4) = K_{27}(2) + K_{27}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{27}(\lambda) = \frac{\lambda}{4} K_{27}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{27}(\lambda) = K_{27}(\lambda - 1) + K_{27} = \frac{\lambda - 1}{4} K_{27}(4) + K_{27}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{27}(\lambda) = K_{27}(\lambda - 2) + K_{27}(2) = \frac{\lambda - 2}{4} K_{27}(4) + K_{27}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{27}(\lambda) = K_{27}(\lambda - 3) + K_{27}(3) = \frac{\lambda - 3}{4} K_{27}(4) + K_{27}(3).$$

Therefore, $K_{27}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 24. $n = 28$.

If $\lambda = 1$, then

$$(p, q) \in \{(2, 93), (6, 90), (10, 87), \dots, (126, 0)\}.$$

By taking

$$K_{28} = K_{16} + K_{12} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(0, 189), (4, 186), (8, 183), \dots, (252, 0)\}.$$

By Theorem 2.2, $K_{28}(2)$ is $\{0P_4, 189S_4\}$ -decomposable. By taking

$$K_{28}(2) = K_{28} + K_{28},$$

we obtain all the above possible decompositions.

If $\lambda \geq 3$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{2}$, then we write

$$K_{28}(\lambda) = \frac{\lambda}{2}K_{28}(2).$$

If $\lambda \equiv 1 \pmod{2}$, then we write

$$K_{28}(\lambda) = K_{28}(\lambda - 1) + K_{28} = \frac{\lambda - 1}{2}K_{28}(2) + K_{28}.$$

Therefore, $K_{28}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Case 25. $n = 30$.

If $\lambda = 1$, then

$$(p, q) \in \{(145, 0), (141, 3), (137, 6), \dots, (1, 108)\}.$$

By Theorem 2.4, $K_{22,8}$ is $\{0P_4, 44S_4\}$ -decomposable. By taking

$$K_{30} = K_{22} + K_8 + K_{22,8},$$

we obtain the decomposition when $(p, q) = (1, 108)$, and by taking

$$K_{30} = K_{24} + K_6 + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6} + K_{4,6},$$

we obtain all the above possible decompositions.

If $\lambda = 2$, then

$$(p, q) \in \{(290, 0), (286, 3), (282, 6), \dots, (2, 216)\}.$$

By taking

$$K_{30}(2) = K_{30} + K_{30},$$

we obtain all the above possible decompositions.

If $\lambda = 3$, then

$$(p, q) \in \{(435, 0), (431, 3), (427, 6), \dots, (3, 324)\}.$$

By taking

$$K_{30}(3) = K_{30}(2) + K_{30},$$

we obtain all the above possible decompositions.

If $\lambda = 4$, then

$$(p, q) \in \{(580, 0), (576, 3), (572, 6), \dots, (0, 435)\}.$$

By Theorem 2.2, $K_{30}(4)$ is $\{0P_4, 435S_4\}$ -decomposable. By taking

$$K_{30}(4) = K_{30}(2) + K_{30}(2),$$

we obtain all the above possible decompositions.

If $\lambda \geq 5$, then the proof is divided into the following cases.

If $\lambda \equiv 0 \pmod{4}$, then we write

$$K_{30}(\lambda) = \frac{\lambda}{4}K_{30}(4).$$

If $\lambda \equiv 1 \pmod{4}$, then we write

$$K_{30}(\lambda) = K_{30}(\lambda - 1) + K_{30} = \frac{\lambda - 1}{4}K_{30}(4) + K_{30}.$$

If $\lambda \equiv 2 \pmod{4}$, then we write

$$K_{30}(\lambda) = K_{30}(\lambda - 2) + K_{30}(2) = \frac{\lambda - 2}{4}K_{30}(4) + K_{30}(2).$$

If $\lambda \equiv 3 \pmod{4}$, then we write

$$K_{30}(\lambda) = K_{30}(\lambda - 3) + K_{30}(3) = \frac{\lambda - 3}{4}K_{30}(4) + K_{30}(3).$$

Therefore, $K_{30}(\lambda)$ is fully $\{P_4, S_4\}$ -decomposable.

Now we prove the result for odd $n > 27$ and even $n > 30$. Let $n = 2r$ or $n = 2r + 1$, where $r \geq 2$. We prove the result by mathematical induction on n , splitting the proof into two cases as follows.

Case A. $n \equiv 0 \pmod{2}$.

Let $n = 2r$, where $r \geq 3$. If $3 \leq r \leq 15$, then the result follows from Cases 2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, and 25. Now, for some $t > 15$, assume that there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{2r}(\lambda)$ for all r with $3 \leq r < t$. We write

$$\begin{aligned} K_{2t}(\lambda) &= K_{2(t-12)}(\lambda) + K_{24}(\lambda) + K_{2(t-12),24}(\lambda) \\ &= K_{2(t-12)}(\lambda) + K_{24}(\lambda) + (t-12)K_{2,24}(\lambda) \\ &= K_{2(t-12)}(\lambda) + K_{24}(\lambda) + 2(t-12)K_{2,12}(\lambda). \end{aligned}$$

By the induction hypothesis, there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{2(t-12)}(\lambda)$. By Case 20, there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{24}(\lambda)$. By Lemma 3.2, $K_{2,12}$ is fully $\{P_4, S_4\}$ -decomposable, and hence $K_{2,12}(\lambda)$ is decomposable by taking λ copies. Therefore, a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{2t}(\lambda)$ exists.

Case B. $n \equiv 1 \pmod{2}$.

Let $n = 2r + 1$, where $r \geq 2$. If $2 \leq r \leq 13$, then the result follows from Cases 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, and 23. Now, for some $t > 13$, assume that there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{2r+1}(\lambda)$ for all r with $2 \leq r < t$. We write

$$\begin{aligned} K_{2t+1}(\lambda) &= K_{2(t-12)+1}(\lambda) + K_{25}(\lambda) + K_{2(t-12),24}(\lambda) \\ &= K_{2(t-12)+1}(\lambda) + K_{25}(\lambda) + (t-12)K_{2,24}(\lambda) \\ &= K_{2(t-12)+1}(\lambda) + K_{25}(\lambda) + 2(t-12)K_{2,12}(\lambda). \end{aligned}$$

By the induction hypothesis, there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{2(t-12)+1}(\lambda)$. By Case 21, there exists a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{25}(\lambda)$. By Lemma 3.2, $K_{2,12}$ is fully $\{P_4, S_4\}$ -decomposable, and hence $K_{2,12}(\lambda)$ is decomposable by taking λ copies. Therefore, a $\{P_4, S_4\}_{\{p,q\}}$ -decomposition of $K_{2t+1}(\lambda)$ exists, and the result follows by mathematical induction. \square

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